

**Zeitschrift:** Archives des sciences [1948-1980]  
**Herausgeber:** Société de Physique et d'Histoire Naturelle de Genève  
**Band:** 29 (1976)  
**Heft:** 2

**Artikel:** Details of hydrogen-burning thermonuclear reactions  
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**DOI:** <https://doi.org/10.5169/seals-739677>

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# DETAILS OF HYDROGEN-BURNING THERMONUCLEAR REACTIONS

BY

P. BOUVIER and L. WEIBEL

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## ABSTRACT

In connection with the triple  $pp$  chain, we study here the temperature dependance of the rate of energy production in the range where the  $PP\ II$  chain is predominant. Moreover, we examine the approach towards equilibrium of the complete  $CNO$  cycle, composed of the  $CN$  and the two  $NO$  cycles, showing that the presence of the second  $NO$  cycle entails a quasi-equilibrium for the  $CN$  cycle.

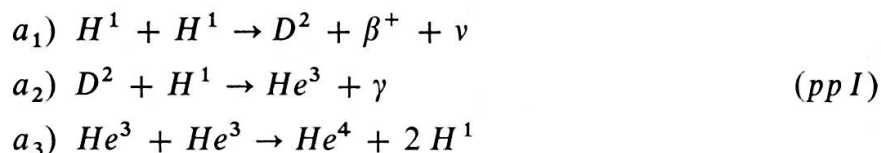
## RÉSUMÉ

Relativement à la triple chaîne  $pp$ , on examine ici la dépendance du débit d'énergie à l'égard de la température, dans l'intervalle où la chaîne  $PP\ II$  domine. Par ailleurs, nous abordons l'étude de l'approche à l'équilibre du cycle complet  $CNO$  formé du cycle  $CN$  et des deux cycles  $NO$ , en montrant que la présence du deuxième cycle  $NO$  entraîne un quasi-équilibre pour le cycle  $CN$ .

## 1. ENERGY PRODUCTION IN THE PROTON-PROTON CHAIN

The two most important ways by which hydrogen is converted into helium within stars are the so called proton proton chain and the  $CNO$  cycle.

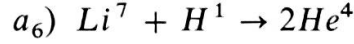
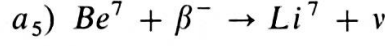
Now the  $pp$  chain operates over the temperature range  $8 \leq T_6 \leq 30$  (where  $T_6 = T \times 10^{-6} \text{ }^\circ K$ ), yielding  $26.21 \text{ MeV}$  in form of radiation energy; it consists in fact in a three-fold chain of reactions; the 1<sup>st</sup> being



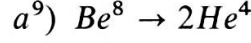
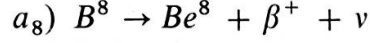
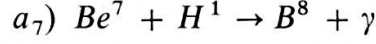
When  $He^4$  becomes sufficiently abundant, we may have instead of  $a_3$ ),



and whether  $Be^7$  will capture either a free electron or a proton, we will finally obtain  $He^4$  by the completion of the  $pp-II$  or the  $pp-III$  chain respectively, *ViZ.*



or



When the complete  $pp$  chain proceeds in equilibrium, the rate of helium formation is given by the expression (see Clayton, 1968)

$$\frac{d\text{He}^4}{dt} = \left( \frac{d\text{He}^4}{dt} \right)_I \varphi(\alpha_1)$$

where  $(d\text{He}^4/dt)_I$  is the rate of formation by the  $ppI$  chain only and

$$\varphi(\alpha_1) = 1 - \alpha_1 + \alpha_1 \left( 1 + \frac{2}{\alpha_1} \right)^{\frac{1}{2}}$$

where

$$\alpha_1 \left( T, \frac{\text{He}^4}{\text{H}} \right) = \frac{\lambda_{34}^2}{\lambda_{33} \lambda_{pp}} \left( \frac{\text{He}^4}{\text{H}} \right)^2$$

the  $\lambda$ 's being the reaction rates per pair of interacting nuclei. [ $\text{cm}^3 \text{sec}^{-1}$ ].

The rate of energy liberation per unit volume,  $\rho\epsilon$  [ $\text{erg sec}^{-1} \text{cm}^{-3}$ ], will not include the energy carried off by the neutrinos which amounts respectively, for each of the neutrinos appearing in  $a_1$ ,  $a_5$ ,  $a_8$ ), to 1.9, 4.0 and 28 per cent of the total energy connected to the mass defect  $4m_{\text{H}} - m_{\text{He}}$ .

Denoting by  $F_{ppi}$  the fraction of  $\alpha$  particles produced by the chain  $i$  ( $i = I, II, III$ ), we may write

$$\rho\epsilon = \frac{d\text{He}^4}{dt} (4m_{\text{H}} - m_{\text{He}}) c^2 (0.981 F_{ppI} + 0.960 F_{ppII} + 0.720 F_{ppIII})$$

whence, if  $\epsilon = \epsilon_I$  when  $F_{ppI} = 1$  ( $ppI$  acting alone),

$$\epsilon = \frac{\epsilon_I}{0.981} \varphi(\alpha_1) [0.981 F_{ppI} + 0.960 F_{ppII} + 0.720 F_{ppIII}]$$

There is no difficulty in showing that, under equilibrium conditions (Clayton, 1968)

$$\frac{F_{ppI}}{F_{ppII} + F_{ppIII}} \equiv \frac{F_{ppI}}{1 - F_{ppI}} = \frac{1}{4} \left[ \left( 1 + \frac{2}{\alpha_1} \right)^{\frac{1}{2}} - 1 \right]$$

whence

$$F_{ppI} = \left[ \left( 1 + \frac{2}{\alpha_1} \right)^{\frac{1}{2}} - 1 \right] \left[ \left( 1 + \frac{2}{\alpha_1} \right)^{\frac{1}{2}} + 3 \right]^{-1}$$

Furthermore, in terms of the branching ratio

$$\alpha_2 = \frac{F_{ppIII}}{F_{ppII} + F_{ppIII}}$$

of the  $ppIII$  chain at  $Be^7$ , we have

$$F_{ppIII} = \alpha_2 (1 - F_{ppI}) \quad F_{ppII} = (1 - \alpha_2) (1 - F_{ppI})$$

so that, after some easy algebraic manipulation, we can write  $\epsilon$  in the form

$$\epsilon = \epsilon_I \psi(\alpha_1, \alpha_2)$$

where

$$\psi(\alpha_1, \alpha_2) = 1 + (0.962 - 0.5 \alpha_2) [\varphi(\alpha_1) - 1]$$

which is (improved value) the expression given by Reeves (1965), obtained here in a slightly different way, through the explicit introduction of the  $F_{ppi}$ .

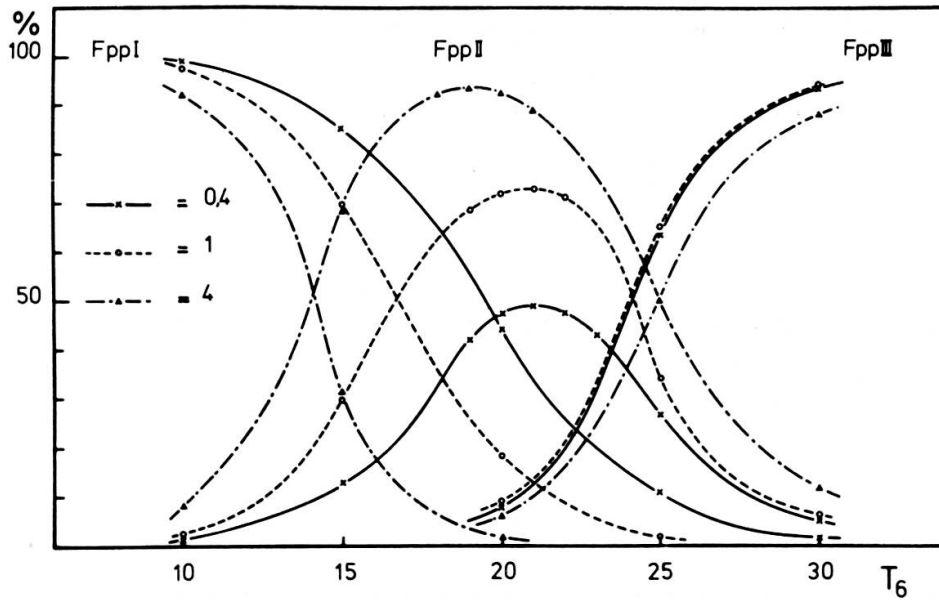


FIG. 1. — Illustrates the dependence of the fractions  $F_{ppi}$  on temperature, at given  $Y/X = 4 \text{ He}^4/H$  ratio.

## 2. PREDOMINANCE OF $pp II$ CHAIN

Fig. 1 reveals that, when enough  $He^4$  is present, there is a temperature interval in which the  $pp II$  chain prevails distinctly over the 2 other chains ( $17 \lesssim T_6 \lesssim 22$ ); since  $pp III$  has not started significantly yet, we have  $\alpha_2 = 0$  and consequently,

$$\epsilon \approx \epsilon_I \varphi(\alpha_1)$$

$\epsilon_i$  is proportional to the function  $\tau^2 e^{-\tau}$  characteristic of a non resonant reaction, where  $\tau = BT_6^{-1/3}$ ,  $B$  being a constant depending on the chemical parameters ( $A, Z$ ) of the interacting nuclei.

On the other hand, it is customary to express the  $\epsilon(T)$  dependence as a power law, so that

$$\epsilon \div T^v \div \tau^{-3v} \div \tau^2 e^{-\tau} \varphi(\alpha_1)$$

By logarithmic derivation, we obtain

$$3v = \left(1 - \frac{\varphi'_\tau}{\varphi}\right) \tau - 2$$

The explicit form for  $\alpha_1$  is, to a high degree of accuracy,

$$\alpha_1 = A \exp(-100 T_6^{-1/3})$$

where  $A$  is practically independent of  $T$ ; consequently,

$$\varphi'_\tau = \varphi'_{\alpha_1} \frac{\partial \alpha_1}{\partial \tau} - = \frac{100}{B} \alpha_1 \varphi'_{\alpha_1} \simeq -3\alpha_1 \varphi'_{\alpha_1}$$

since the value of  $B$  in the present  $pp$  reaction amounts to 33.7.

Therefore

$$v = \frac{\tau - 2}{3} + \alpha_1 \frac{\varphi'_{\alpha_1}}{\varphi} \tau$$

the first term pertains to the  $pp$  I chain only, the second term being the contribution of the  $pp$  II chain. Table 1 summarizes the values obtained for  $v$  at different  $Y/X$  and  $T_6$  values.

TABLE 1

$T_6 \backslash Y/X$	18	20	22
0.4	3.62 + 1.03	3.50 + 1.32	3.34 + 1.32
0.63	3.62 + 1.32	3.50 + 1.37	3.34 + 1.08
1	3.62 + 1.42	3.50 + 1.21	3.34 + 0.73
2	3.62 + 1.23	3.50 + 0.65	3.34 + 0.27
4	3.62 + 0.65	3.50 + 0.23	3.34 + 0.08

$v$  is increased by the  $pp$  II influence, but not considerably;  $pp$  II is less sensitive to the temperature as the helium abundance is high ( $Y/X$  large).

### 3. THE CNO TRICYCLE

When  $T_6$  reaches 12, hydrogen may also burn by the CNO cycle, which around  $T_6 = 18$ , for standard compositions of population I stars, soon becomes the dominant reaction as  $T_6$  goes on rising. The CNO tricycle, sketched in fig. 2, is made up of the basic CN cycle (hexagon) and of two additional NO cycles, the first of which, connected to the CN cycle joins  $N^{15}$  to  $N^{14}$  by passing  $O^{16}$  whereas the second NO cycle, added to the first one, joins  $O^{17}$  to  $N^{15}$  through  $O^{18}$ .

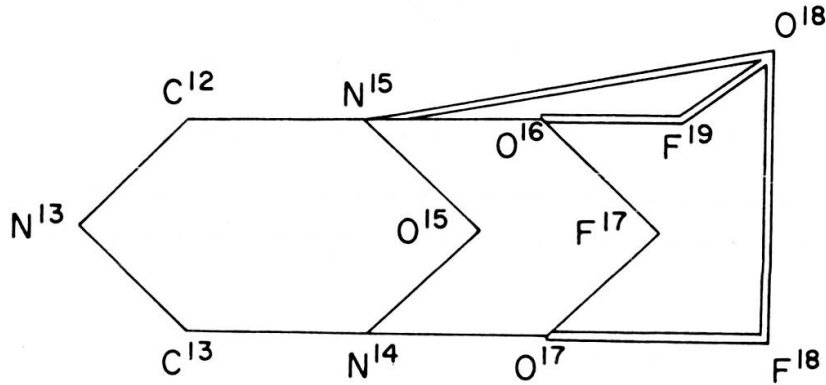


FIG. 2.

All reactions considered here are induced by protons; we neglect any  $\alpha$ -induced reaction, corresponding to simultaneous hydrogen and helium burning (Caughlan and Fowler, 1962).

The branching ratio of the  $N^{15} (p, \gamma) O^{16}$  reaction is approximately  $\gamma_1 = 8 \times 10^{-4}$  according to Fowler et al. (1975; quoted as *FCZ II*), so that it takes about  $10^3$  complete CN cycles to go through before a significant number of CN nuclei will have switched over to the NO cycles. In general, the lifetimes of the different nuclear species are fairly time-independent during a steady hydrogen burning phase, so that we may regard these lifetimes as constant at a given temperature; the branching ratios will then also be constant and moreover weakly dependent on temperature changes.

In the physical conditions  $T_6 = 25$ ,  $\rho X_H = 28$  which we had adopted in a preliminary study to test the assumption of constant nuclear lifetimes during a particular nuclear burning stage, the lifetimes of the relevant nuclei are, in years,

$$\begin{aligned} \tau_{12} &= 8.98 \times 10^2 & \tau_{16} &= 4.51 \times 10^6 \\ \tau_{13} &= 2.58 \times 10^2 & \tau_{19} &= 1.98 \times 10^5 \\ \tau_{14} &= 8.38 \times 10^4 \end{aligned}$$

Beyond  $F^{19}$ ,  $p$ -induced reactions could lead either to  $O^{16}$  or to  $Ne^{20}$ ; both such reactions are extremely slow, the second being still slower than the first which could close the chain at  $O^{16}$ , thus completing a CNOF quadricycle.

We further label  $\tau_{A,B}$  the lifetimes of nucleus  $A$  with respect to the transmutation  $A \rightarrow B$  through  $p$ -capture; always according to the *FCZ II* reaction rates, we have, in years,

$$\tau_{15,12} = 3.3 \quad \text{and} \quad \tau_{15,16} = 4.1 \times 10^3$$

For the isotopes  $O^{17}$ ,  $O^{18}$ , the values are sensitive to experimental uncertainties; according to whether a resonance in the compound nucleus contributes by a vanishing, partial or dominant amount to the  $p$ -capture cross-section, *FCZ II* give three possible values (denoted  $L$ ,  $I$ ,  $H$  respectively). Retaining here only the intermediate values, viz. in years,

$$\begin{aligned} \tau_{17,14} &= 2.78 \times 10^6 & \tau_{18,15} &= 12,4 \\ \tau_{17,18} &= 3.71 \times 10^6 & \tau_{18,19} &= 7.4 \times 10^5 \end{aligned}$$

whence

$$\gamma_2 = \frac{\tau_{17,14}}{\tau_{17,14} + \tau_{17,18}} = 0.43 \quad \gamma_3 = \frac{\tau_{18,15}}{\tau_{18,15} + \tau_{18,19}} = 1.68 \cdot 10^{-5}$$

In spite of the uncertainty in these values, they do show us that the frequency of occurrence of both *ON* cycles is about the same, and that of the complete quadricycle is comparatively much less.

#### 4. APPROACH TOWARD EQUILIBRIUM IN THE *NO* CYCLES

According to the  $\tau_{12}$  value given above, we see that the *CN* cycle achieves equilibrium in about  $10^3$  years and only later will the *NO* cycles become significantly operative. The stellar environment adopted here, namely  $\rho X_H = 28$ ,  $T_6 = 25$  corresponds closely to the central conditions of a  $3 M_\odot$  main sequence star (Iben, 1965); the fact that  $T_6$  lies well below 100 prevents us from having to consider  $\alpha$ -induced reactions and the complications of the rather extensive reaction network for the hot *CNO-Ne* cycle (Audouze et al., 1973).

The time variations of chemical abundances during the approach to equilibrium of the *NO* cycles are the following (see fig. 2):

$$\begin{aligned} \frac{dN^{14}}{dt} &= (1-\gamma_2) \frac{O^{17}}{\tau_{17}} + (1-\gamma_1) \frac{N^{15}}{\tau_{15}} - \frac{N^{14}}{\tau_{14}} \\ \frac{dN^{15}}{dt} &= \frac{N^{14}}{\tau_{14}} - \frac{N^{15}}{\tau_{15}} + (1-\gamma_3) \frac{O^{18}}{\tau_{18}} \\ \frac{dO^{16}}{dt} &= \gamma_1 \frac{N^{15}}{\tau_{15}} - \frac{O^{16}}{\tau_{16}} + \frac{F^{19}}{\tau_{19}} \\ \frac{dO^{17}}{dt} &= \frac{O^{16}}{\tau_{16}} - \frac{O^{17}}{\tau_{17}} \end{aligned} \tag{1}$$

where we have neglected the short-lived positron emitters. The running of the  $NO$  cycles alters somewhat the  $CN$  equilibrium abundances reached after  $10^3$  years; we now have

$$\frac{C_e^{12}}{\tau_{12}} = \frac{C_e^{13}}{\tau_{13}} = \frac{N_e^{15}}{\tau_{15}} = \frac{N^{14}}{\tau_{14}} + \gamma_2 \frac{O^{17}}{\tau_{17}}$$

where  $N^{14}$  and  $O^{17}$  will change on a longer time scale ( $\sim 10^5$  y); the  $CN$  cycle is therefore in a quasistatic equilibrium.

Dropping further the  $F^{19}$  channel ( $F^{19} = O, \gamma_3 = O$ ) and noticing that  $O^{18}$  reaches its equilibrium value in less than 20 years:

$$\frac{O_e^{18}}{\tau_{18}} = \frac{O_e^{18}}{\tau_{18,15}} = \gamma_2 \frac{O^{17}}{\tau_{17}}$$

we obtain

$$\frac{dN^{15}}{dt} = \frac{N^{14}}{\tau_{14}} - \frac{N^{15}}{\tau_{15}} + \gamma_2 \frac{O^{17}}{\tau_{17}}$$

Since  $\frac{1}{\tau_{15}} \approx \frac{1}{\tau_{15,12}}$ , and  $\tau_{15,12}$  is of order  $3\frac{1}{2}$  years, while  $\tau_{14}$  and  $\tau_{17}$  are much

larger, we may consider that in a few years time,  $N^{15}$  has attained its equilibrium value and we are left with the three equations

$$\begin{aligned} \frac{dN^{14}}{dt} &= (1 - \gamma_2 \gamma_1) \frac{O^{17}}{\tau_{17}} - \gamma_1 \frac{N^{14}}{\tau_{14}} \\ \frac{dO^{16}}{dt} &= \gamma_1 \frac{N^{14}}{\tau_{14}} + \gamma_1 \gamma_2 \frac{O^{17}}{\tau_{17}} - \frac{O^{16}}{\tau_{16}} \\ \frac{dO^{17}}{dt} &= \frac{O^{16}}{\tau_{16}} - \frac{O^{17}}{\tau_{17}} \end{aligned} \quad (2)$$

which entail  $N^{14} + O^{16} + O^{17} = N_{CNO}$  (const.). Letting  $\gamma_2 = O$  brings us back to the case of the  $CNO$ -bicycle.

## 5. TEMPORAL BEHAVIOUR OF THE CHEMICAL COMPOSITION

Labeling  $N_i$  ( $i=1, 2, 3$ ) the fractional abundances of the 3 isotopes involved in equations (2), it is easy to find, by suitable elimination, a  $2^d$ -order differential equation for each of the  $N_i$ , of the form

$$\frac{d^2 N_i}{dt^2} + A \frac{dN_i}{dt} + BN_i - C_i = 0 \quad (3)$$



where

$$N_1 = N^{14}, \quad N_2 = O^{16}, \quad N_3 = O^{17},$$

$$C_1 = \frac{1 - \gamma_1 \gamma_2}{\tau_{16} \tau_{17}} N_{CNO}, \quad C_2 = \frac{\gamma_1}{\tau_{14} \tau_{17}} N_{CNO}, \quad C_3 = \frac{\gamma_1}{\tau_{14} \tau_{16}} N_{CNO},$$

$$A = \frac{\gamma_1}{\tau_{14}} + \frac{1}{\tau_{16}} + \frac{1}{\tau_{17}} \quad N_{CNO} B = \sum_{i=1}^3 C_i$$

The general solution of (3) has the form

$$N_i(t) = K_i e^{r_1 t} + L_i e^{r_2 t} + \frac{C_i}{B} \quad (4)$$

where  $K_i, L_i$  are constants,  $\frac{C_i}{B}$  is the equilibrium value of  $N_i$  and

$$r_1 = -\frac{A}{2} + \frac{1}{2} \sqrt{A^2 - 4B}, \quad r_2 = -\frac{A}{2} - \frac{1}{2} \sqrt{A^2 - 4B}$$

The initial conditions to which  $K_i, L_i$  are connected are taken at a time  $t = 0$  chosen as the starting time for the  $NO$  cycles display, well after the  $CN$  cycle has attained equilibrium. In the particular stellar environment adopted here and also for other temperatures between 20 and 80 million degrees, it turns out that  $A^2 - 4B$  is always positive (sometimes only weakly); therefore  $r_1, r_2$  and consequently  $K_i, L_i$  are real and we get a superposition of exponential variations for  $N_i(t)$ . Some of the abundances will increase from  $N_i(0)$  to  $C_i/B$  while others shall decrease, since at all times we have

$$\sum_{i=1}^3 N_i(t) = N_{CNO}$$

But the accuracy of the  $\tau$ -values is often poor and, for other similar situations, we could possibly be faced with the case  $A^2 - 4B < 0$ ; this happens, for example, when the lifetimes  $\tau_{14}/\gamma, \tau_{16}, \tau_{17}$  become very close to each other and the approach to equilibrium would then have an oscillatory character.

Letting  $\frac{1}{2} \sqrt{4B - A^2} = \omega$ , the solution (4) becomes explicitly

$$N_i(t) = \left( N_i(0) - \frac{C_i}{B} \right) \exp \left( -\frac{A}{2} t \right) \left[ \cos \omega t + \frac{A}{2\omega} \sin \omega t \right] + \frac{C_i}{B}$$

having assumed  $\left( \frac{dN_i}{dt} \right)_{t=0} = 0$ .

The condition that  $N_i(t)$  must always remain positive is warranted by the fact that the period  $2\pi/\omega$  is distinctly larger than the damping time  $2/A$ . After a quarter of period, for instance, when  $t = \pi/2\omega$ , the amplitude of the oscillation is already reduced by  $e^{-\pi u/2} \cdot u$  (where  $u = A/2\omega$ ) and this factor is much less than unity

because, from the definitions of  $A$  and  $B$ ,  $A^2 - 2B$  is always  $> 0$ , so

$$u = \frac{A}{\sqrt{4B - A^2}} > 1.$$

We conclude that  $N_i(t)$  is indeed always  $\geq 0$ , whatever the sign of  $N_i(0) - C_i/B$ .

We are indebted to Professor W. A. Fowler for having sent us in advance of publication, a preprint of his revised thermonuclear reaction rates.

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