# New Hamiltonians for high-order equations of the calculus of variations : a generalization of the Dorfman approach

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# NEW HAMILTONIANS FOR HIGH-ORDER EQUATIONS OF THE CALCULUS OF VARIATIONS: A GENERALIZATION OF THE DORFMAN APPROACH

BY

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ABSTRACT

Through generalized Hamiltonians, which have a direct economic interpretation, we show that we can derive in a straightforward way high-order variational equations, as well as the Ostrogradski equation. The latter is of considerable importance, since it lies at the very heart of mathematical physics: in particular, it leads to the Laplace and to the Schrödinger equations.

RÉSUMÉ

L'objet de cet article est de montrer comment on peut définir de nouveaux Hamiltoniens afin de pouvoir dériver très simplement les équations d'Euler-Poisson et d'Ostrogradski appartenant au calcul des variations. L'équation d'Ostrogradski est centrale à la physique mathématique, car elle conduit aux équations de Laplace et de Schrödinger.

In a remarkable paper, Robert DORFMAN [1] had shown how the maximum principle could be derived from straightforward economic reasoning, specifically from capital theory. He had shown, in particular, that differentiating a modified Hamiltonian both with respect to the state variable and the control variable led directly to the maximum principle.

In this note we would like to point out that it is possible to extend this method to obtain important building blocks of the calculus of variations, namely the Euler-Poisson and the Ostrogradski equations; the latter leads to central results in physics. We will do so by introducing Hamiltonians of increasing generality, all of which have direct economic interpretations.

## *I. From the Dorfman Hamiltonian to the Euler equation*

Using Dorfman's notation, recall the problem of finding a control variable $x \equiv x(t)$ which maximizes the functional $\int_0^T u(k, x, t)\, dt$ where $k \equiv k(t)$ is a state variable, under

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the constraint $\dot{k} = f(k, x, t)$. One procedure, suggested by Dorfman, is to form a modified Hamiltonian $H^* = u(k, x, t) + \frac{d}{dt}(\lambda k)$; equating to zero the partial derivatives of H* with respect to x and k will lead to the simplest version of Pontryagin's maximum principle.

Remember also the link existing between the maximum principle and the calculus of variations, which enables to get in one line the Euler equation. Set the constraint $\dot{k} = f(k, x, t)$ equal to x, as a particular case. We then have to find $\dot{k}(t)$ (or k(t)) such that $\int_0^T u(k, \dot{k}, t)dt$ is maximized, which is the basic problem of the calculus of variations.

Form $H^* = u(k, x, t) + \frac{d}{dt}(\lambda k) = u(k, \dot{k}, t) + \dot{\lambda} k + \lambda \dot{k}$. $H^*_k = 0$ implies $u_k + \dot{\lambda} = 0$; $H^*_{\dot{k}} = 0$ implies $u_{\dot{k}} + \lambda = 0$ and $\dot{u}_{\dot{k}} + \dot{\lambda} = 0$. Hence we must have $u_k - \frac{d}{dt} u_{\dot{k}} = 0$, which is the Euler equation[1].

## *II. Generalizations*

This reasoning can be generalized. Let us first introduce some definitions, which will then be applied to concrete calculus of variations problems.

- *price of a variable*: we define the *price* of the $j^{th}$ variable of a functional as the partial derivative of the functional with respect to this variable: that is the rate of increase of the functional with respect to the variable at any time.
- *value of a variable*: the value of a variable is defined as the price of the variable times the variable itself.
- *value of a system*: we define the value of a system as the total effects of the set of variables. We shall separate short-term (or direct) effects from long-term (or indirect) ones. The short-term effect is the integrand of the functional. The long-term effect is the rate of increase of the value of all variables in the system.

The following table summarizes these definitions.

| Concept | Definition |
|---|---|
| • price of a variable | • partial derivative of the maximum value of the functional with respect to the variable |
| • value of a variable | • product between the price of the variable and the variable itself |
| • short-term effects of a set of variables | • integrand of the functional |
| • long-term effects of a set of variables | • sum of the time derivatives of the value of each variable = derivative of the sum of the values of the variables |
| • total effects of a set of variables = value of a system of variables | • short-term plus long-term effects of a set of variables = integrand of the functional plus the derivative of the sum of the values of the variables |

[1] A deep account of the relationship between the classical calculus of variations and optimal control theory is given in L. PONTRYAGIN et alii [4], chapter 5. A simpler presentation will be found in I.M. GELFAND and S.V. FOMIN [3], appendix II. The reader unfamiliar with the calculus of variations will find an excellent introduction in L. ELSGOLC [2].

In order to maximise (or minimise) a functional, we will then introduce a generalized Hamiltonian which will simply be the equal to the total effects of a set of variables, and we will maximise (or minimize) this Hamiltonian with respect to each variable.

As a first example, we will consider functionals depending on high-order derivatives; we will then turn towards multiple integrals, i.e. functionals involving functions of severable variables and their partial derivatives up to the $n^{th}$ order.

*1. Functionals depending on high-order derivatives*

Suppose that we want to find y(x) such that

$$\int_a^b F(x, y, y', \dots, y^{(j)}, \dots, y^{(n)})\, dx$$

is maximized ($y^{(j)}$ denotes the $j^{th}$ derivative of the function $y(x)$; by convention we set $y^{(0)} \equiv y(x)$).

In this problem we define a system of $n+1$ prices, each corresponding to the $n+1$ variables $y^{(0)}, \dots, y^{(n)}$:

$$\lambda j(x) = \frac{\partial}{\partial y^{(j)}} \int_a^b F(x, y, y', \dots, y^{(n)})\, dx, j = 0, \dots, n$$

The *value* of each variable, i.e. the product of the unit value of the variable and the magnitude of the variable, is equal to $\lambda j(x) \cdot y^{(j)}$.

The short-term effects of the set of variables $y, \dots, y^{(n)}$ is just $F(x, y, y', \dots, y^{(n)}$, and the long term effect is $\frac{d}{dx}\,[\sum_{j=0}^{n-1} \lambda j\,(x) y^{(j)}]$.

Hence the value of the system of variables for any x is equal to: $H^n = F(x, y, y', \dots, y^{(n)}) + \frac{d}{dx}\,[\sum_{j=0}^{n-1} \lambda j\,(x) y^{(j)}]$

$$= (x, y, y', \dots, y^{(n)}) + \sum_{j=0}^{n-1} [\lambda\, y_j^{(j+1)} + \lambda\, 'y_j^{(j)}]$$

$$= F(x, y, y', \dots, y^{(n)}) + \lambda_0 y' + \lambda'_0 y + \lambda_1 y'' + \lambda'_1 y' + \dots$$

$$+ \lambda_{j-1} y^{(j)} + \lambda'_{j-1} y^{(j-1)} + \lambda_j y^{(j+1)} + \lambda'_j y^{(j)} + \dots$$

$$+ \lambda_{n-1} y^{(n)} + \lambda'_{n-1} y^{(n-1)}$$

Each of the components of $\frac{d}{dx}(\lambda_j y^{(j)})$ is equal to $\lambda_j y^{(j+1)} + \lambda'_j y^{(j)}$ : it is therefore made up of the rate of increase of $y^{(j)}$ multiplied by its price, and of the price increase of $y^{(j)}$ multiplied by $y^{(j)}$ (this latter part corresponds to the "capital gain" in the Dorfman interpretation of his modified Hamiltonian).

Let us now derive the total value of the system $H^n$ partially with respect to each variable $y, y', \dots, y^{(n)}$ and set each derivative equal to 0. We then have the following system of $n+1$ differential equations:

$$H^n_y = F_y + \lambda'_0 = 0$$

$$H^n_{y'} = F_{y'} + \lambda_0 + \lambda'_1 = 0$$

$$\dots$$

(2) $$H^n_{y(j)} = F_{y(j)} + \lambda_{j-1} + \lambda'_1 = 0$$

$$\dots$$

$$H^n_{y(n-1)} = F_{y(n-1)} + \lambda_{n-2} + \lambda'_{n-1} = 0$$

$$H_{y(n)} = F_{y(n)} + \lambda_{n-1} = 0$$

Differentiate each equation $H^n_{y(j)} = 0 \ (j = 0, ..., n)$ j times with respect to $x$. We get:

$$F_y + \lambda'_0 = 0$$

$$\frac{d}{dx} F_{y'} + \lambda'_0 + \lambda''_1 = 0$$

(3) $$\dots$$

$$\frac{d^{(j)}}{dx^{(j)}} F_{y(j)} + \lambda^{(j)}_{j-1} + \lambda_j^{(j+1)} = 0$$

$$\dots$$

$$\dots$$

$$\frac{d^{(n-1)}}{dx^{(n-1)}} F_{y(n-1)} + \lambda^{(n-1)}_{n-2} + \lambda^{(n)}_{n-1} = 0$$

$$\frac{d^{(n)}}{dx^{(n)}} F_{y(n)} + \lambda^{(n)}_{n-1} = 0$$

We can now write, from system (3):

$$\lambda^{(n)}_{n-1} = - \frac{d^{(n)}}{dx^{(n)}} F_{y(n)}$$

$$\lambda^{(n-1)}_{n-2} = - \frac{d^{(n-1)}}{dx^{(n-1)}} F_{y(n-1)} + \frac{d^{(n)}}{dx^{(n)}} F_{y(n)}$$

$$\lambda^{(n-2)}_{n-3} = - \frac{d^{(n-2)}}{dx^{(n-2)}} F_{y(n-2)} + \frac{d^{(n-1)}}{dx^{(n-1)}} - \frac{d^{(n)}}{dx^{(n)}} F_{y(n)}$$

$$\dots$$

(4) $$\lambda^{(n-i+1)}_{n-i} = (-1) \frac{d^{(n-i+1)}}{dx^{(n-i+1)}} F_{y(n-i+1)} + (-1)^2 \frac{d^{(n-i+2)}}{dx^{(n-i+2)}} F_{y(n-i+2)}$$

$$+ \dots + (-1)^i \frac{d^{(n)}}{dx^{(n)}} F_{y(n)} = \sum_{j=1}^{i} (-1)^j \frac{d^{(n-i+j)}}{dx^{(n-i+j)}} F_{y(n-i+j)}$$

Hence, the variable $\lambda'_0$ appearing in the first equation of system (3), and which corresponds to $i = n$ in (4), is equal to

$$\lambda'_0 = \sum_{j=1}^{n} (-1)^j \frac{d^{(j)}}{dx^{(j)}} F_{y(j)}$$

We then get, from the first equation of (3), the Euler-Poisson equation which reads:

$$F_y + \sum_{j=1}^{n} (-1)^j \frac{d^{(j)}}{dx^{(j)}} F_{y(j)} = 0$$

*2. Functionals defined by multiple integrals*

This kind of economic reasoning can be further extended to functionals of the form

$$\int_R \dots \int F[x_1, \dots, x_n; z; z_{x_1}, \dots, z_{x_n}]\, dx_1, \dots, dx_n.$$

which have a considerable importance in physics, and which can be encountered in economics as well (for instance in dynamic location problems, or in stochastic optimization).

In order to get the Ostrogradski formula which sets a first order condition for $z(x_1, \dots, x_n)$ to be an extremal of this functional, let us form a general Dorfman-type Hamiltonian

$$H^G = F[x_1, \dots, x_n; z; z_{x_1}, \dots, z_{x_n}] + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} [\lambda\, (x_1, \dots, x_n) \cdot z(x_1, \dots, x_n)]$$

where $\lambda\, (x_1, \dots, x_n)$ is defined by

$$\lambda\, (x_1, \dots, x_n) = \frac{\partial}{\partial z} \int_R \dots \int F(x_1, \dots, x_n; z; z_{x_1}, \dots, z_{x_n})\, dx_1, \dots, d_{x_n}.$$

As before, $\lambda$ can be interpreted as the price of $z$. This Hamiltonian can conveniently be written as

$$H^G = F[x_1, \dots, x_n; z; z_{x_1}, \dots, z_{x_n}] + z \sum_{i=1}^{n} \lambda_{x_i} + \lambda \sum_{i=1}^{n} z_{x_i}$$

Differentiate $H^G$ with respect to $z$ and $z_{x_1}, \dots, z_{x_n}$, and set the $n+1$ resulting partial derivatives to zero. We have:

$$(5) \qquad \begin{aligned} H^G_z &= F_z + \lambda_{x_1} + \dots + \lambda_{x_n} = 0 \\ H^G_{z_{x_i}} &= F_{z_{x_i}} + \lambda = 0 \quad , i=1, \dots, n \end{aligned}$$

Differentiate each equation $H^G_{z_{x_i}} = 0$ with respect to $x_i$ $(i=1, \dots, n)$.

We get:

$$(6) \qquad \begin{aligned} H^G_z &= F_z + \lambda_{x_1} + \dots + \lambda_{x_n} = 0 \\ \frac{\partial}{\partial x_i} H^G_{z_{x_i}} &= \frac{\partial}{\partial x_i} F_{z_{x_i}} + \lambda_{x_i} = 0 \quad , i=1, \dots, n \end{aligned}$$

System (6) entails the Ostrogradski equation:

$$(7) \qquad F_z - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_{z_{x_i}} = 0$$

from which we can obtain the Laplace and the Schrödinger equations.

It is possible to generalize this method to functionals involving functions of n variables and their partial derivatives up to the m[th] order. We are then led to a generalization of both the Euler-Poisson and the Ostrogradski equations. For simplicity, we will treat the case $n = m = 2$. We have to find z that maximizes the functional

$$\iint_D F(x_1, x_2; z; z_{x_1}, z_{x_2}; z_{x_1x_1}, z_{x_1x_2}, z_{x_2x_2})dx_1dx_2,$$

denoted $\iint_D F( \ldots )dx_1dx_2$ for short.

We introduce a Dorfman-type Hamiltonian as follows: let

$$\lambda^0 = \frac{\partial}{\partial z} \iint_D F( \ldots )dx_1dx_2$$

and $\lambda^j = \dfrac{\partial}{\partial z_{x_j}} \iint F( \ldots )dx_1dx_2 \quad , j = 1,2$

define the marginal contributions to the functional (or the prices) of $z$, $z_{x_1}$ and $z_{x_2}$, respectively. A general Hamiltonian can then be written as follows:

$$(8) \qquad \begin{aligned} H^{2,2} &= F(x_1, x_2; z; z_{x_1}, z_{x_2}; z_{x_1x_1}, z_{x_1x_2}, z_{x_2x_2}) + \\ &\quad + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} [\lambda^0 z] + \sum_{j=1}^{2} \sum_{i=1}^{2} \frac{\partial}{\partial x_i} [\lambda^j z_{x_j}] \\ &= F( \ldots ) + \lambda^0_{x_1} z + \lambda^0 z_{x_1} + \lambda^0_{x_2} z + \lambda^0 z_{x_2} + \\ &\quad + \lambda^1_{x_1} z_{x_1} + \lambda^1 z_{x_1x_1} + \lambda^1_{x_2} z_{x_1} + \lambda^1 z_{x_1x_2} + \\ &\quad + \lambda^2_{x_1} z_{x_2} + \lambda^2 z_{x_1x_2} + \lambda^2_{x_2} z_{x_2} + \lambda^2 z_{x_2x_2} \end{aligned}$$

(the first superscript in $H^{2,2}$ refers to the number of variables of $z$; the second one indicates the highest order of the derivatives in the functional).

Differentiating $H^{2,2}$ with respect to $z$, $z_{x_1}$, $z_{x_2}$, $z_{x_1x_1}$, $z_{x_1x_2}$, $z_{x_2x_2}$, we get:

$$(9a) \qquad H^{2,2}_z = F_z + \lambda^0_{x_1} + \lambda^0_{x_2} = 0$$

$$(9b) \qquad H^{2,2}_{z_{x_1}} = F_{z_{x_1}} + \lambda^0 + \lambda^1_{x_1} + \lambda^0_{x_2} = 0$$

$$(9c) \qquad H^{2,2}_{z_{x_2}} = F_{z_{x_2}} + \lambda^0 + \lambda^2_{x_1} + \lambda^2_{x_2} = 0$$

(9d) $$H^{2,2}_{z_{x_1x_1}} = F_{z_{x_1x_1}} + \lambda^1 = 0$$

(9e) $$H^{2,2}_{z_{x_1x_2}} = F_{z_{x_1x_2}} + \lambda^1 + \lambda^2 = 0$$

(9f) $$H^{2,2}_{z_{x_2x_2}} = F_{z_{x_2x_2}} + \lambda^2 = 0$$

Differentiate (9b) and (9c) with respect to $x_1$ and $x_2$, respectively; differentiate twice (9d), (9e) and (9f) with respect to $x_1$, $x_1$ and $x_2$, and $x_2$, respectively. We then have the following system:

(10) $$F_z + \lambda^0_{x_1} + \lambda^0_{x_2} = 0$$

$$\frac{\partial}{\partial x_1} F_{z_{x_1}} + \lambda^0_{x_1} + \lambda^1_{x_1x_1} + \lambda^1_{x_1x_2} = 0$$

$$\frac{\partial}{\partial x_i} F_{z_{x_2}} + \lambda^0_{x_2} + \lambda^2_{x_1x_2} + \lambda^2_{x_2x_2} = 0$$

$$\frac{\partial^2}{\partial x_1^2} F_{z_{x_1x_1}} + \lambda^1_{x_1x_1} = 0$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} F_{z_{x_1x_2}} + \lambda^1_{x_1x_2} + \lambda^2_{x_1x_2} = 0$$

$$\frac{\partial^2}{\partial x_2^2} F_{z_{x_2x_2}} + \lambda^2_{x_2x_2} = 0$$

System (10) yields the fourth-order partial differential equation

$$F_z - \frac{\partial}{\partial x_1} F_{z_{x_1}} - \frac{\partial}{\partial x_2} F_{z_{x_2}} + \frac{\partial^2}{\partial x_1^2} F_{z_{x_1x_1}} +$$

(11) $$+ \frac{\partial^2}{\partial x_1 \partial x_2} F_{z_{x_1x_2}} + \frac{\partial^2}{\partial x_2^2} F_{z_{x_2x_2}} = 0$$

which is a general Ostrogradski equation.

We have thus shown that the Dorfman's modified Hamiltonian approach is perfectly general: all the fundamental equations of the calculus of variations can be derived from extensions of that approach.

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