Theorie of statically indeterminate systems

Autor(en): **Melan, E.**

Objekttyp: **Article**

Zeitschrift: **IABSE congress report = Rapport du congrès AIPC = IVBH Kongressbericht**

Band (Jahr): **2 (1936)**

PDF erstellt am: **21.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-3152

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Theorie of Statically Indeterminate Systems.

Theorie statisch unbestimmter Systeme.

Théorie des systèmes hyperstatiques.

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A) Ideal-Plastic Material.

A) Idealplastischer Baustoff.

A) Matériau idéalement plastique.

The object of statics as applied to steel structures is to determine the internal stresses and deformations of systems composed of slender members.

Three groups of equations are available for solving these problems, viz., (1) those describing the conditions of equilibrium, (2) those dealing with the geometrical conditions, and (3) those expressing the relation between internal stresses and deformations.

For statically determined systems, the first group of equations alone is sufficient for determining all internal stresses, whereas the two other groups are only necessary for calculating the deformations. In the case of statically undetermined structures, however, it is not possible to arrive at the internal stresses with the aid of only the first group of equations. The two other groups of equations must be used as well, and, in addition, the relations between deformations and stresses must be known.

The usual text-book theories assume full validity of Hooke's law, which says that the deformation of a bar is proportional to the axial stress and, further, that a change in the angle of contingency for a curved bar is proportional to the bending moment with regard to a particular point.

It has been stated elsewhere that the validity of Hooke's law is limited — a fact proved by experience — and that the linear relation between stress and deformation is only true within certain limits. These limits are by no means fixed. Actually, they are dependent on the momentary deformations already existing, as well as being a function of the rate at which the load changes. It is also essential to know whether the stress increases or decreases outside the range for which Hooke's law is applicable, and this of course does not make it easy to analyse the actual conditions obtaining.

To obtain rehults which really correspond with the actual conditions, Hooke's law must be replaced, outside the range for which it is valid, by fresh assumptions which are more in accordance with test results.

For statically determined structures this, as already indicated, only affects the calculation of the deformations, whereas, for statically undetermined structures, the substitution of Hooke's law by some other relation leads to new internal stresses. In order to obtain really practical and useful results, it will be necessary more or less to idealise the assumptions in terms of the actual conditions. The usual thing is to assume that the speed at which the change in load takes place has no influence, or, in other words, that the change of stresses is gradual (as usually assumed in statics) provided no "elasto-kinematic" considerations are included. Diagram (1a) may be considered as the basis of this theory, whereas, if we are satisfied with a less accurate reproduction of the real facts, the stress-strain diagram (1b) indicates the flow of the stresses and deformations. Over and above this, however, the theory admits of a further simplification as shown in diagram (1c), which illustrates the case for the "ideal-plastic" material. It is on such assumption that most of the following examinations are based, and this assumption will be dealt with in detail in the second part of this paper. It should be noted that it coincides with the theory postulated by Messrs. *Haar-Karman, v. Mises,* and *Hencky*[1] for stresses in the

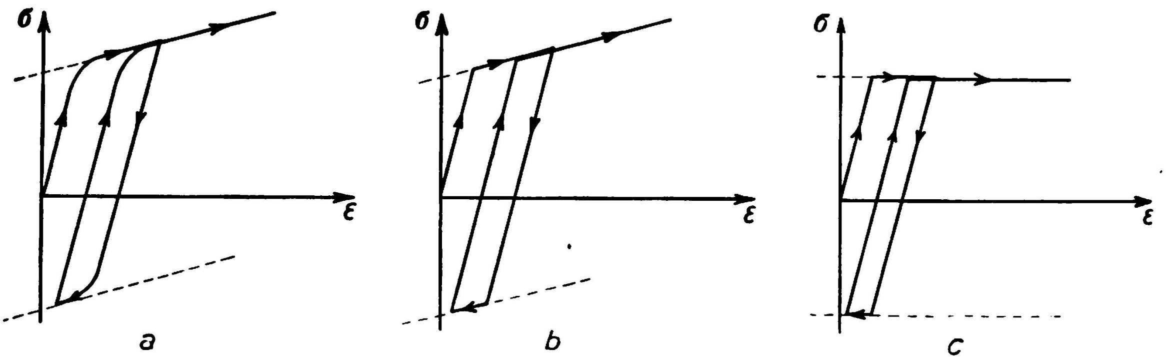

Fig. 1.

direction of the three axes, provided the necessary application and specialisation is made to suit the stress conditions in one direction only.

The results of axial stress and deformation tests can be applied as they stand for tensile members of a lattice girder. For compressive members, the load limit will as a rule be defined by the buckling load. If diagram (1c) should be applied for negative stresses as well, then we must assume that, under the influence of the buckling load, any shortening of the axis of a member is possible (e. g., shortening due to deflection of the axis), and that the buckling load always retains its critical value, which can never be exceeded. When the member is unstressed, we again have linear proportion between axial stress and deformation.

The theory outlined above is less important when applied to the ideal-plastic lattice girder than it is for system comprising members stiff against bending. The general investigations actually show that systems composed of an ideal-plastic material possess this remarkable property — that stress-peaks which would occur in a perfectly elastic material are reduced and distributed to places of lower stresses. Expressed differently, this means that there must be a certain

[1] *A. Haar* and *Th. v. Karman*: "Theory of Stresses in Plastic and Sandy Materials". Nachr. d. kgl. Ges. der Wiss., Göttingen 1909.

v. Mises: "Mechanics of Solids in the Plasto-deformable State", ibid. 1913.

H. Hencky: "Theorie of Plastic Deformations and the Stresses which they subsequently set up in the Material". Ztschr. für angew. Math. und Mech., 1924.

reserve of material which, however, in a carefully calculated and suitably dimensioned lattice girder, is less than it would be in a solid continuous beam over several spans with constant cross-section. The main reason why the fundamental investigations have nevertheless been carried out on lattice girders is that, in contrast to systems composed of members stiff against bending, the analytical investigation leads to equations having a limited number of variables, and this means that the complexity of the problem is greatly simplified for the less mathematical reader. In principle, the generalisation of the problem for an infinite number of variables is possible, so that the results obtained from lattice girders can be employed to solve problems for systems having members stiff against bending.

When investigating systems with members resistant to bending and made of ideal-plastic material, it is usual to assume, with sufficient accuracy, the same coordination between moment and radius of curvature as was done in Fig. 1c for stress and deformation. It should be noted, however, that these two assumptions do not fully tally with each other. Information on this point and on other questions connected with it is provided by a interesting paper by *Eisenmann* in "Stahlbau", 1933. Another interesting paper to which we may refer at this juncture is one entitled "The Behaviour of statically undetermined Steel Structures beyond the Elastic Limit", published by *W. Prager* in the "Bauingenieur", 1933. In this paper, the author only gives one definite example to show that, in statically undetermined systems, the deformations are affected much more than the moments are by the assumption of a special law of elasticity, and that the simplified assumption, previously mentioned, of the relationship between internal moment and alteration in curvature can be adopted without objection.

Attention is further drawn to two important facts which, as the result of substituting Hooke's law by a non-linear equation, is not stressed sufficiently in many of the published papers.

Due to the non-linearity of the equations, the validity of the law of superposition is lost, or, expressed more accurately, it remains valid only for the internal stresses of statically determined systems. With the exception of this latter case, it is therefore impossible to plot influence lines. It is also impossible to define the stress conditions in statically undetermined systems unless the previous stressing or the "previous history" is known. It is precisely to this particular point that special attention should be paid in all cases where a single loading is involved, as in the papers of Messrs. *J. Fritsche, Schaim, Kazinczy* and *Girkmann*[2], all of which deal with special systems of ideal-plastic material.

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[2] *J. Fritsche*: „Die Tragfähigkeit von Balken aus Stahl mit Berücksichtigung des plastischen Verformungsvermögens". Bautechnik, 1930.

G. H. Schaim: „Der Durchlaufträger unter Berücksichtigung der Plastizität". Stahlbau, 1930.

G. v. Kazinczy: „Statisch unbestimmte Tragwerke unter Berücksichtigung der Plastizität". Stahlbau, 1931, and „Bemessung unvollkommen eingespannter Stahl-I-Deckenträger unter Berücksichtigung der plastischen Formänderungen". Proc. of International Association for Bridge and Structural Engineering, Vol. 2, 1933—34.

K. Girkmann: „Bemessung von Rahmentragwerken unter Zugrundelegung eines idealplastischen Stahles". Sitzungsber. der Akademie der Wiss., Vienna 1931, and „Über die Auswirkung der Selbsthilfe des Baustahles in rahmenartigen Stabwerken", Stahlbau, 1932.

In all these investigations, it is tacitly assumed that all external stresses increase proportionally from nil to their final value. The same remark applies to investigations dealing with repeated loading (stressing), where it is assumed that the load varies in the same way between two limiting values. Mention should be made here of the fundamental work of *M. Grüning* „Die Tragfähigkeit statisch unbestimmter Tragwerke aus Stahl mit beliebig häufig wiederholter Belastung“ (The Loading Capacity of Statically Undetermined Steel Structures subject to any number of repeated Loadings), Berlin 1926. These investigations show that, with repeated loading oscillating between two fixed values, the system finally undergoes deformations which are also limited. *Grüning's* investigations are based on the less specialised case of Fig. 1a. Alongside these special researches should be ranked the special work of Mr. *J. Fritsche* on continuous beams under repeated loading, based on ideal-plastic material („Bauingenieur“, 1932).

Mr. *Grüning's* results have been extended — for ideal-plastic material only — by Mr. *H. Bleich* (cf. „Bauingenieur“, 1932). In this paper, Mr. *Bleich* shows that, even under variable loads, i. e., not a mere oscillation of the external stresses between definite limits, it is possible, under certain conditions, and by suitable dimensioning, to make the structure work like one of perfectly elastic material, provided that a sufficient number of changes in load have taken place. This hypothesis, dealt with in detail in the second part of this paper, forms the basis for calculating statically undetermined systems of ideal-plastic material. The method built up on this basis, and for which the term „*Traglastverfahren*“ ("Theory of plastic equilibrium") has come into use, will be explained in further detail in the paper to be read by Mr. *F. Bleich*, unless the reader chooses to consult other sources, say, Mr. *F. Bleich's* book „Stahlhochbauten“, Berlin 1932.

The above rather incomplete summary of the published literature dealing with the theory of statically undetermined systems, must be supplemented by two papers which deal with the theory of these systems from a more general viewpoint. These are: Mr. *J. Fritsche's* paper in the *Zeitschrift für angewandte Mathematik und Physik,* 1931, and a paper by Mr. *Hohenemser* in *Ingenieurarchiv,* 1931, and both writers take the potential energy of such systems as their starting point. On the other hand, the writer employs in his investigations now to be dealt with, and which aim at a general proof of Mr. *Bleich's* theory, the conditions of equilibrium in conjunction with the law of elasticity for ideal-plastic material. It is well to state here that the undetermined condition of the deformations, which, in my opinion, has not previously been observed, and which necessitate certain reservations as regards the validity of Mr. *Bleich's* theorem, is due to the special assumption of the stress-strain law beyond the range for which *Hooke's* law is valid. If the diagram is replaced by Fig. 1a or 1b, the undefined solutions will disappear and also the considerations which they involve. This simplification is gained, of course, at the expense of a new material constant, viz., the inclination of the straight lines at the boundary in Fig. 1a and 1b.

1) The proof adduced by Mr. *H. Bleich* in support of his theory of statically undetermined systems of ideal-plastic material is not very simple, and is worked

out for no more than twice statically undetermined systems. An extensions of the idea he develops does not seem possible for higher statically undetermined structures on account of the time required for calculation.

In what follows, a proof of general validity for any systems of multiple static undetermined structures will be established. This proof is comparatively easy to demonstrate, provided a clear conception has been reached as to the fundamental properties of ideal-plastic material. In addition, the more general discussions preceding the demonstration of the theory will be calculated to give a better idea of the nature of such structures. Only a few simple principles of the theory of linear equation systems are required, and these can be found in any algebra text-book. To avoid having to deal with an infinite number of variables, the following considerations will be confined to lattice girders. The transition to an infinite number of variables, such as would be involved when dealing with systems composed of members stiff against bending, is similar to that used in the theory of integral equations, and presents no real difficulties.

It will be remembered that the linear deformation of a bar of ideal-plastic material is expressed by the relation:

$$\Delta_S = v + \rho \cdot S \tag{1}$$

Here ρ is a constant depending on the length of bar, the modulus of elasticity, and the cross-section of the bar, v is the so-called "permanent" deformation, and S the axial stress. If S comes within the interval $T' < S < T$, where T' and T are the yield points under compression and tension, then v is constant. Only when $S = T'$ or $S = T$ is it possible for v to vary; the value v can only either increase or decrease if $S = T$ or $S = T'$ respectively. The bar is said to "yield" or "flow". Values like $S < T'$ and $S > T$ are excluded. The relation just outlined is illustrated in Fig. 2, the time being plotted on the abscissae. In the upper diagram, the ordinates represent the stress in the bar, and in the lower diagram, the deformation (elongation). In the time intervals $t_1 < t < t_2$, $t_3 < t < t_4$ and $t_5 < t$, the bar is perfectly elastic, whereas it flows in the other intervals. This assumes, of course, that dv/dt is always finite, and that the permanent deformations corresponding to a finite rate of flow do not change suddenly.

2) In a statically undetermined lattice girder two types of members may be differentiated: (1) those which are absolutely necessary and which, if cut through, cause the whole system to become movable, and (2) those which are not necessary, i. e., one or more of which may be cut through without rendering the system movable. According to definition there must, in a v-times statically undetermined system, be at least one group of maximum v members which, if cut through, would leave still one immovable system, which is defined as a statically determined basic system in the statically undetermined system we are considering.

We shall now consider this particular lattice girder at a given moment, where the external load has a definite value. Due to previous loadings, the several members have already undergone permanent deformations v, so that the stresses S occur in these members. If the system were perfectly elastic, then, according to the usual theory of statically undetermined systems, the stresses B would have been set up in the members. S, B and v are then linked up by the following system of equations:

$$S_1 + \sum q_{ik} v_k = B_i \qquad (i = 1, 2, \ldots r) \tag{2}$$

The significance of q_{ik} will be readily appreciated; it represents the stress in the bar i when, for the unloaded system (all B' s = 0) all permanent deformations v are nil with the exception of member k. In this member, $v_k = -1$.

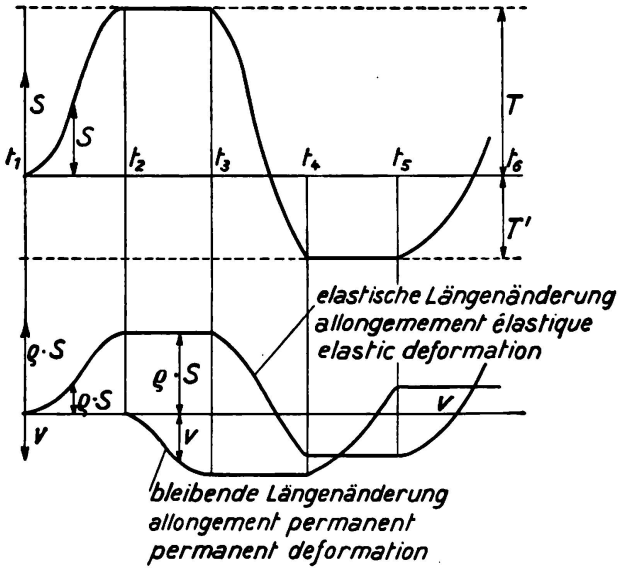

Fig. 2.

Inserting these values in equation (2) we immediately get:

$$S_i = q_{ik}$$

This tells us that, for necessary bars, $q_{ik} = 0$, and since, as will be shown, $q_{ik} = q_{ki}$, the summation in equation (2) applies only for all unnecessary members 1, 2, . . . r. The matrix $(q_{11} \ldots q_{rr})$ is symmetrical, and hence $q_{ik} = q_{ki}$; this follows from the theorem of axial stresses in members of a lattice girder, which is analogous to *Maxwell*'s theorem of the reciprocity of deformation. This theorem, which can be easily proved by solving the equations for elasticity, says that the axial force in a member i produced by a permanent deformation $v_k = -1$ of member k, hence q_{ik}, is equal to the axial force in member k produced by a deformation $v = -1$ of member i, so that $q_{ik} = q_{ki}$. This theorem naturally only applies to statically undetermined systems and only to the members in it which are not necessary, as otherwise it becomes trivial, as otherwise $q_{ik} = q_{ki} = 0$.

Writing $\sigma_1 = B_i - S_i$, we obtain from (2) the following type of equation:

$$\sigma_i = \sum q_{ik} v_k \qquad (i = 1, 2, \ldots r) \tag{3}$$

The system of the values σ_1 is termed a system of coercive forces; the $-\sigma_1$ are the stresses in members of an unloaded system which, owing to the previous loadings, have already undergone the permanent deformations v_k.

Multiplying each of the equations of (3) in turn by v_i, and adding, we obtain the "quadratic form"

$$J = \sum_i \sigma_i v_i = \sum_i \sum_k q_{ik} v_i v_k \tag{4}$$

In adducing the proof, it is necessary that J be always essentially positive in character, that is to say, which ever way the values of v_k are chosen, J should never be negative. The simplest way of demonstrating this is to start from the principle of virtual deformations which, for a lattice girder, may be written

$$\sum S \Delta_s = \sum P \delta$$

The summation on the right applies for all external forces, and that on the left for all members of the system. The forces S and P represent a system of equilibrium. The deformation Δs can be arbitrary, compatible only with the geometrical relation, which means that, for au ν-times statically undetermined system, only up to ν-values can be selected arbitrarily. A compatible system of Δs values is formed by the values $v + \rho S$, and also by the values ρB, which are deformations of ideal-plastic and perfectly elastic material respectively. The difference $v + \rho (S - B) = v - \rho\sigma$ will certainly also give a compatible system of deformations. The quantities δ finally indicate the displacement of the points of attack of the external forces P in the direction of these forces when all members of the system have undergone the deformation Δs.

If, therefore, we write $\Delta s = v - \rho\sigma$ and apply the principle of virtual deformations (1) for the ideal-plastic and (2) for the perfectly elastic system, noting, however, hat the same external loads P apply to the forces S and B, whereas the δ are identical in both cases, corresponding to the values $v - \rho\sigma$, we then obtain, by subtracting the two equations:

$$\sum (v - \rho\sigma)\,\sigma = 0$$

and hence

$$J = \sum v\sigma = \sum \rho\sigma^2 \geqq 0 \tag{4a}$$

thus proving our assertion that J cannot assume negative values, since $\rho = s/EF$ is never negative and $\sum \rho\sigma^2$ is for positive terms only. The value J can only disappear when all $\sigma = 0$; but it must not be inferred from this that v must be nil too under all circumstances.

Assuming the forces in the members for the ideal-plastic system have, at a given moment, attained the values S and that, at this particular moment, the forces in the members 1, 2 .... μ have reached the yield point where $S_1 = T_1$, $S_2 = T_2 \ldots . S_\mu = T_\mu$, then if the external force alters, S will become $S + \Delta S$, v will become $v + \Delta V$, and B will become $B + \Delta B$. In accordance with the system of equations (2)

$$(S_i + \Delta S_i) + \sum q_{ik} (v_k + \Delta v_k) = (B_i + \Delta B_i)$$

and, subtracting equation (2) from this gives us

$$\Delta S_i + \sum q_{ik} \Delta v_k = \Delta B_i \quad (i = 1, 2, \ldots \mu) \tag{5}$$

or

$$\Delta \sigma_i = \sum q_{ik} \Delta v_k \tag{6}$$

where $\Delta\sigma_i$ indicates the change in the coercive forces. The sum $\Sigma q_{ik} \Delta v_k$ is applicable to no more than the first μ-members, since, according to the definition of an ideal-plastic material, the only members which can yield are those for which $S = T'$ or $S = T$. This assumes, of course, that all ΔB are chosen so small that no other members than the first μ-members have reached the yield point. As long as one and the same members yield, we speak of a certain "phase", and a new phase starts if other members start or cease to yield. For the phase immediately following the particular moment we are considering, equation (5) applies, where, however, in accordance with the properties of the ideal-plastic material, the following alternative should be noted for the first μ-members.

If $\Delta S_i = 0$, then Δv_i has not the opposite sign to $S_i = T_i$ or $S_i = T'_i$;
or if $\Delta v_i = 0$, then ΔS_i has not the same sign as $S_i = T_i$ or $S_i = T'_i$.
$(i = 1, 2 \ldots . \mu)$ (5a)

In this connection only the first μ-equations of (5) are of interest, since S_j $(\mu < j \leqq r)$ can be determined without difficulty from the remaining equations, provided the values Δv_k have been worked out from the first μ-equations. If, as we assumed, one and the same phase occurs — and this can always be attained where the value ΔB is small enough, the expression $S_j + \Delta S_j \leqq T_j$ or T'_j certainly applies, and we need not trouble about these equations at present.

3) Applying the principle of virtual deformation, it can easily be shown that the alternative equation (5a) is just sufficient to determine the ΔS of equation (5), viz., from

$$\Delta S_i + \Sigma q_{ik} \Delta v_k = \Delta B_i \qquad (i = 1, 2, \ldots r)$$

provided that they exist at all. Unless the ΔS were unequivocal, then values $S' = S + \Delta S'$ and also $S'' = S + \Delta S''$ would be possible for the external loads $P = P + \Delta P$, and alterations in the length of a member would be $v' = \rho S'$ and $v'' + \rho S''$. If the principle of virtual deformations is applied in its four possible combinations, we obtain:

$$\Sigma S''(v'' + \rho S'') = \Sigma \overline{P} \delta'' \qquad \Sigma S'(v'' + \rho S'') = \Sigma \overline{P} \delta''$$

$$\Sigma S''(v' + \rho S') = \Sigma \overline{P} \delta' \qquad \Sigma S'(v' + \rho S') = \Sigma \overline{P} \delta'$$

Herein, δ'' and δ' are displacements of the points of intersection which are coordinated to the values $(v'' + \rho S'')$ and $(v' + \rho S')$ respectively.

Subtracting from the difference of the two equations on the right the difference of the two equations on the left, we get:

$$\Sigma (S'' - S')(v'' - v') + \rho (S'' - S')^2 = 0$$

or

$$\Sigma (\Delta S'' - \Delta S')(\Delta v'' - \Delta v') + \Sigma (\Delta S'' - \Delta S')^2 \rho = 0. \qquad (7)$$

The summation $\Sigma (\Delta S'' - \Delta S')^2 \rho$ cannot be negative on account of the quadratic terms, and the same applies to $\Sigma (\Delta S'' - \Delta S')(\Delta v'' - \Delta v')$. If in both cases a member is yielding, it can only be for $\Delta S'' = \Delta S' = 0$, while if a member is not yielding in either case, then $\Delta v'' = \Delta v' = 0$, which means that the corresponding summations disappear in the above equation. If a member yields in one case and not in the other, then $\Delta S'' = 0$, and $\Delta v''$ has not the opposite sign to $S = T$, while $\Delta v' = 0$ und $\Delta S'$ has not the same sign as $S = T$. In this case, the particular term is reduced to $- \Delta S' \Delta v''$, wherein $\Delta S'$ and $\Delta v''$ will never have the same sign, so that $= \Delta S' \Delta v''$ will not be negative. If S'' and S' are different, then (7) consists only of non-negative terms, and can only disappear if $S'' = S'$. A very definite phase can only follow if the values ΔB (or, more precisely, if dB/dt in the interval of time dt immediately following the moment under consideration) are given. It should be noted, however, that these arguments only prove the singularity of the axial forces in members, and not the permanent deformations. As a matter of fact, there may, under certain conditions, be multiple solutions for Δv.

In the phase considered above, at the beginning of which the first λ members have reached the yield point, only the first members ought actually to yield. What we have to do, then, is to solve the system of equations

$$\sum_{1}^{\lambda} q_{ik} \Delta v_k = \Delta B_i = \Delta \sigma_i \quad (i = 1, 2, \ldots \lambda) \tag{8a}$$

for the yielding members only, and

$$\Delta S_j + \sum_{1}^{\lambda} q_{jk} \Delta v_k = \Delta B_j \quad (\lambda < j \leq r) \tag{8b}$$

for the non-yielding members.

According to a well-known theorem in algebra, the equations (8a) supply singular solutions only when the matrix $(q_{11} \ldots q_{\lambda\lambda})$ is not singular, in other words, if the determinant

$$\begin{vmatrix} q_{11} \cdots\cdots q_{1\lambda} \\ \cdots\cdots\cdots\cdots \\ q_{\lambda 1} \cdots\cdots q_{\lambda\lambda} \end{vmatrix}$$

does not disappear. Sufficient, but not necessary to establish this, is the fact that $(q_{11} \ldots q_{\mu\mu})$ is not singular. In this case, if all $\Delta B = 0$, the only possible solutions are that all $\Delta v = 0$, and this at the same time is the necessary and sufficient condition for the disappearance of the quadratic form

$$J = \sum \sum q_{ik} \Delta v_i \Delta v_k.$$

But if $(q_{11} \ldots q_{\lambda\lambda})$ is singular, and if this matrix has, say, the range $\vartheta < \lambda$, which means that not only the determinant disappears, but all sub-determinants with more than ϑ rows, while at least one determinant of ϑ lines can be different from nil, the Δv are no longer of singular value. These solutions can be written introducing an arbitrary constant c_p:

$$\Delta v_i = \Delta v_i^* + \sum_{p=\vartheta+1}^{\lambda} c_p \mathfrak{v}_{ip} \tag{9}$$

where the Δv_i^* are the solutions of the system

$$\sum q_{ik} \Delta v_k^* = \Delta B_i = \Delta \sigma_i \qquad (i = 1, 2, \ldots \vartheta)$$

while

$$\Delta v_k^* = 0 \text{ for } k = \vartheta + 1, \vartheta + 2, \ldots \lambda$$

As there is at least one non-singular matrix $(q_{11} \ldots q\ )$, the solutions Δv_*^i are definitely determined. The $\mathfrak{v}_{ip}$ are the solutions of the following homogeneous system of equations:

$$\sum q_{ik} \mathfrak{v}_{kp} = 0 \qquad (i = 1, 2, \ldots \lambda)$$

which has as solutions $\lambda - \vartheta$ different "fundamental systems" $\mathfrak{v}_{kp}$ $(p = \vartheta + 1, \vartheta + 2, \ldots \lambda)$. It can be shown that these also satisfy the remaining equations

$$\sum q_{ik} \mathfrak{v}_{kp} = 0 \qquad (i = \lambda + 1, \lambda + 2 \ldots r).$$

This follows from the fact that the rectangular matrix

$$\begin{matrix} q_{11} \cdots q_{1\lambda} \\ \cdots\cdots \\ \cdots\cdots \\ q_{r1} \cdots q_{r\lambda} \end{matrix} \qquad (r > \lambda)$$

can only have the range ϑ. This means that all sub-determinants formed from these elements with more than ϑ lines and rows can only have the range ϑ where the corresponding quadratic term $J = \Sigma\Sigma q_{ik} v_i v_k$ is essentially positive and the main sub-determinant of the above matrix is also of range ϑ only.

There are again two possibilities to consider: either it is impossible to chose for the general solution the arbitrary values for c_p in such a way that $\Sigma c_p \mathfrak{v}_{ip}$ has the signs prescribed by the alternative (5a). If, then, $\Delta v_i = \Delta v_i^* + \Sigma c_p \mathfrak{v}_{ip}$ has the required signs, $\Delta v_i = \Delta v_i^* + \Sigma (c_p + \delta c_p) \mathfrak{v}_{ip}$ will also possess the same signs, provided δc has been selected sufficiently small. In this case, the Δv_i are not of a singular value, but lie between certain finite limits. If, however $\Sigma c_p \mathfrak{v}_{ip}$ alone fulfils the required condition with regard to sign, then $\Sigma k c_p \mathfrak{v}_{ip}$ with any positive value of k has the same sign, and if k becomes infinite, the solution $\Delta v_i = \Delta v_i^* + k \Sigma c_p \mathfrak{v}_{ip}$ also assumes infinite values.

The following relations also provide additional support in proof of the theorem already postulated. If the quantities z_i and w_i are connected by the equation:

$$z_i = \sum_{k=1}^{r} q_{ik} w_k \qquad (i = 1, 2, \ldots r),$$

then

$$\sum_{k=1}^{r} z_k \mathfrak{v}_{kp} = 0. \tag{10}$$

The correctness of the above equation will be evident by inserting in it the values $z_k = \sum_i q_{ki} w_i$. This gives us:

$$\sum_k z_k \mathfrak{v}_{kp} = \sum_k \sum_i q_{ki} w_i \mathfrak{v}_{kp},$$

and, by interchanging the order of summation, $\sum_i w_i \sum_k q_{ik} \mathfrak{v}_{kp}$. Since, according to definition $\sum_k q_{ik} \mathfrak{v}_{kp} = 0$, then also $\sum_k z_k \mathfrak{v}_{kp} = 0$.

The following observation is added for the sake of completeness. If we employ the principle of virtual displacements with the deformations $\Delta B \rho$ which are regarded as compatible with the geometrical relations of the lattice girder, and use (1) the axial force ΔS due to the loading ΔP, and (2) the axial forces ΔB of the members, and if we then subtract the two equations, we then get for $\sigma = \Delta S - \Delta B$ the equation:

$$\sum \Delta\sigma \, \Delta B \rho = 0.$$

Hence it follows that $\Delta\sigma$ and ΔB can never have the same sign for all members, because, if they had, as ρ would always be positive, the summation would also be positive and could not disappear as required. This is the very typical expression for what in practice is more sub-consciously termed "self-adaptation" of the material of ideal-plastic nature, where a reduction in stresses at certain places sets up an increase in stresses at other places.

4) The results obtained up to now may become clearer to the engineer if we study the lattice truss in a particular phase, introducing, however, certain alterations.

With the particular lattice truss we have been studying, we imagine a new system being formed by having cut through such members as were not necessarily required. At these cuts we will assume the incorporation of such mechanisms as only allow, for the two faces of the gap, movements in a particular direction. With some of the cut members it should only be possible to have movements increasing the gap. This could be achieved, say, by cutting the members at right angles to the axis and making the two ends simply abut without being joined. For other members, the cuts will be assumed to be such that only a movement closing the gap can occur, which could be done by replacing the particular cut member by a rope.

If such a systems is charged with the loads ΔP, the new loads in the members are ΔS, which, together with the loads ΔB previously obtained, are connected by the equation:

$$\Delta S_i + \sum q_{ik} \, \Delta v_k = \Delta B_i \quad (i = 1, 2, \ldots r).$$

The Δv represent the displacements between the faces of the cuts in the first μ-members. For these μ-members the equations are as under:

$$\Delta S_i + \sum_1^{\mu} q_{ik} \, \Delta v_k = \Delta B_i \quad (i = 1, 2, \ldots \mu),$$

with the alternatives for

$\Delta B = 0$, in which case $\Delta v \gtrless 0$

or $\Delta S \gtrless 0$, in which case $\Delta v = 0$,

where the sign of inequality is governed by the particular kind of mechanism above explained. These are the same equations with the same alternatives as for the system of ideal-plastic material previously considered and for which the equations (5) and (5a) were developed.

We may therefore say that the original lattice girder of ideal-plastic material, at the beginning of the particular phase, may be regarded as being replaced by another girder in which the members which would have reached the yield point in the original girder are regarded as replaced by such mechanisms as above described. If this new girder is loaded with the increment of the external forces corresponding with the particular phase, then the stresses ΔS thus set up in the members are identical to the increments of the stresses in the original girder for the same phase. If, under these conditions, the cut members should not have the singular matrix $(q_{11} \ldots q_{\mu\mu})$, this means that the new girder (irrespective of the auxiliary mechanisms) is immovable, since the homogeneous system of equation

$$\sum q_{ik}\,\mathfrak{v}_{kp} = 0$$

does not in this case admit of any other solution than that all $\mathfrak{v} = 0$. We also find that the system

$$\Delta S_i + \sum q_{ik}\,\Delta v_k = \Delta B_i$$

must always give, for any values of ΔB, singularly determined solutions for ΔS and Δv. But if the matrix $(q_{11} \ldots q_{\mu\mu})$ has a singular value and at the same time range ϑ, then two cases are possible. Either the new girder becomes "self-locked" through these mechanisms, and the solutions of the homogeneous equations do not possess the signs proper to the machanisms. The system is immovable and suits any values of ΔB, although the values Δv are not singular, but lie between two finite limits. Or the new girder system may be movable despite the mechanisms, and may give solutions of the homogeneous system of equations with the proper signs. In this case, should solutions be possible, the ΔB can no longer be arbitrarily chosen, while the Δv may even assume infinite values. Simple kinematic consideration will make matters clear in specific cases. All these facts are explained by the simple example shown in Fig. 3. The members d_1 and d_2 are assumed to have reached the yield point, in which case the truss is as shown in Fig. 3a. This new system is then immovable, whatever the nature of the mechanisms incorporated at the cuts. The matrix $\begin{matrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{matrix}$ is by no means singular, and which ever way the load varies in the succeeding phase, equilibrium will be maintained. But if the members d_1 und d_2 are stressed up to the yield limit (Fig. 3b), then the matrix $\begin{matrix} q_{11} & q_{13} \\ q_{31} & q_{33} \end{matrix}$ is singular in value, because the girder system is movable. When both members are stressed to the limits of compression T′ or tension T, the corresponding mechanisms (in this case ropes) prevent movability and, with any change in loading, equilibrium remains established because the system is self-locked. If, however, one member reaches the limit of compression and the other the limit of tension (Fig. 3c) and if a change in load occurs which causes both members actually to yield, then the values ΔB must be conditional and such as to make equilibrium and the solution of the equations possible. At the same time, however, the Δv may assume infinite values.

5) After these preliminary remarks, we now proceed to prove the theorem just postulated. It says that, for a statically undetermined system of ideal-plastic nature subjected to variable loading the permanent deformations produced by sufficiently repeated changes in loading tend towards limited values. This assumes that the stresses in members of a perfectly elastic material lie within a previously determined interval $B_i^{max} \geqq B_i \geqq B_i^{min}$ and that such a system of coercive forces $\bar{\sigma}$ can be indicated where, for all members,

$$-\bar{\sigma}_i + B_i^{max} \leqq T_i \text{ and } -\bar{\sigma}_i + B_i^{min} \geqq T'_i$$

If we put $B_i^{max} - T_i = t_i$ and $B_i^{min} - T'_i = t'_i$ then a system of coercive forces $\bar{\sigma}$ must exist for which

$$t'_i \leqq \bar{\sigma}_i \leqq t_i.$$

Supposing we had indicated a system of $\bar{\sigma}$ values, and now form the values $\bar{B}_i^{max} = \bar{\sigma}_i + T_i$ or $\bar{B}_i^{min} = \bar{\sigma}_i + T_i$, then obviously the theorem, if corr?ct at all, must also be applicable to variable loads B_i, for which

$$\bar{B}_i^{max} \geqq B_i \geqq \bar{B}_i^{min},$$

where $\bar{B}_i^{max} \geqq B_i^{max}$ and $B_i^{min} \geqq \bar{B}_i^{min}$ (cf. Fig. 4).

Whe shall now deal with a lattice girder after it has undergone a series of changes in load, and shall assume that the coercive forces $\sigma^{(\varphi)}$ obtain after

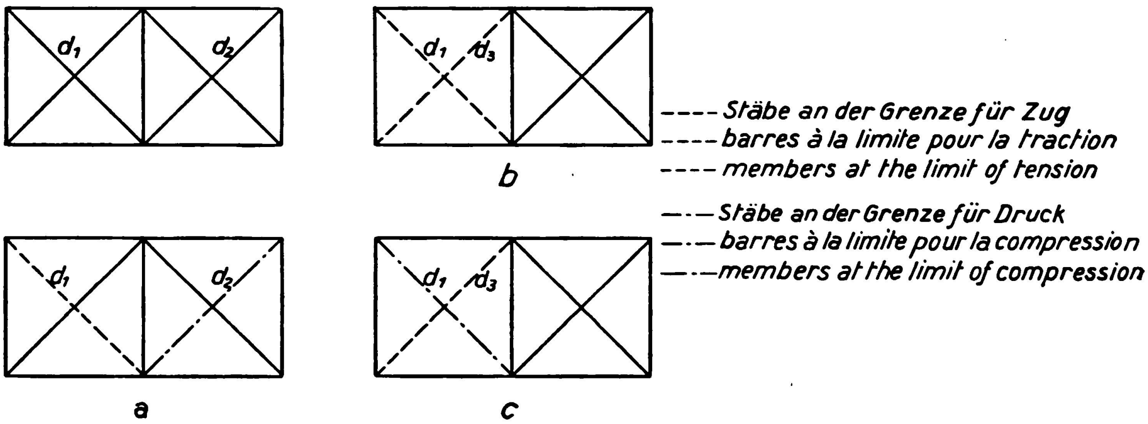

Fig. 3.

the expiration of φ phases. At the beginning of the (φ + 1st) phase, members for which $\sigma^{(\varphi)} > \bar{\sigma}$ can only yield, if at all, when the stress in the merber is $S = T'$. Values $S = T$ are, however, excluded; for even if $B = \bar{B}^{max}$, the stress in the member $S = \bar{B}^{max} - \sigma$ is smaller than $T = \bar{B}^{max} - \bar{\sigma}$.. Consequently, a member for which $\sigma^{(\varphi)} > \bar{\sigma}$, can only undergo negative permanent deformations $\Delta v^{(\varphi+1)}$ during the (φ + 1st) phase. For a member which yields during this phase, the condition $\sigma > \bar{\sigma}$ must obtain during the whole of the phase. Similarly, $\sigma^{(\varphi+1)} > \bar{\sigma}$ must apply for the end of the (φ + 1st) phase or the beginning of the (φ + 2nd) phase. Conversely, $\Delta v^{(\varphi+1)} > 0$ for members for which $\sigma^{(\varphi)} < \bar{\sigma}$ and also $\sigma^{(\varphi+1)} \leqq \bar{\sigma}$. Summarizing, we thus find for a member yielding during the (φ + 1st) phase

$$\left.\begin{array}{l} z^{(\varphi)} = \bar{\sigma} - \sigma^{(\varphi)} > 0, \text{ and also } \Delta v^{(\varphi+1)} > 0 \text{ and further} \\ z^{(\varphi+1)} = z^{(\varphi)} - \Delta\sigma^{(\varphi+1)} \geqq 0 \\ \text{and where} \\ z^{(\varphi)} = \bar{\sigma} - \sigma^{(\varphi)} < 0, \text{ then } \Delta v^{(\varphi+1)} < 0; \text{ and further} \\ z^{(\varphi+1)} = z^{(\varphi)} - \Delta\sigma^{(\varphi+1)} \leqq 0. \end{array}\right\} \quad (11)$$

Obviously no fresh changes $\Delta\sigma$ of the coercive forces can occur in subsequent phases, provided $z^{(\varphi)} = \bar{\sigma} - \sigma^{(\varphi)} = 0$ for all the members. Our proof therefore falls into two parts: we show (1) that the condition $\bar{\sigma} - \sigma^{(\varphi)} = z^{(\varphi)} = 0$ will actually obtain for all members, and (2) that the permanent deformations $\bar{v} = \Sigma\Delta v$ which have developed up to that point can only assume finite values.

Between the quantities $\bar{\sigma}$ and the permanent deformations $\bar{v}$ due to $\bar{\sigma}$ there obviously exists the system of equations

$$\bar{\sigma}_i = \sum q_{ik} \bar{v}_k \qquad (i = 1, 2, \ldots r),$$

whereas, for the coervice forces $\sigma^{(\varphi)}$ with the deformation $v^{(\varphi)}$ at the beginning of the (φ + 1st) phase, we have the equations

$$\sigma_i^{(\varphi)} = \sum q_{ik} \bar{v}_k^{(\varphi)} \qquad (i = 1, 2, \ldots r).$$

The above summations apply for all members, although it is possible, (as for the following sums) that some of the quantities $\bar{v}_k$ or v become nil in particular cases. By also writing $w^{(\varphi)} = \bar{v} - v^{(\varphi)}$ and $z^{(\varphi)} = \bar{\sigma} - \sigma^{(\varphi)}$, we get

$$z_i^{(\varphi)} = \sum q_{ik} w_k^{(\varphi)},$$

and, similarly, for the end of the (φ + 1st) phase,

$$z_i^{(\varphi+1)} = \sum q_{ik} w_k^{(\varphi+1)}$$

whence we immediately confirm that

$$z_i^{(\varphi+1)} = z_i^{(\varphi)} - \Delta\sigma_i^{(\varphi+1)} \text{ and } w_i^{(\varphi+1)} = w_i^{(\varphi)} - \Delta v_i^{(\varphi+1)}. \qquad (12)$$

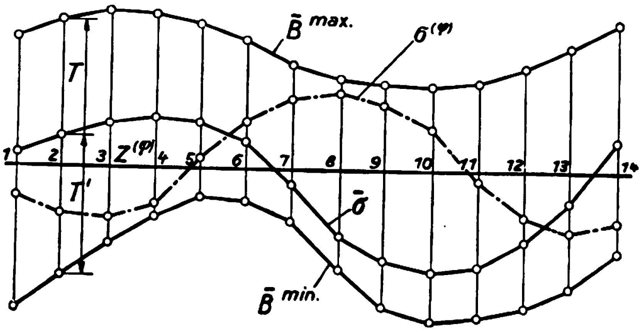

Fig. 4.

Taking the difference between the two quadratic forms

$$\Delta K^{(\varphi+1)} = K^{(\varphi+1)} - K^{(\varphi)} = \sum (z^{(\varphi+1)} w^{(\varphi+1)} - z^{(\varphi)} w^{(\varphi)})$$

and inserting $z^{(\varphi+1)}$ and $w^{(\varphi+1)}$ as per equation (12), we get:

$$\Delta K^{(\varphi+1)} = - \sum z^{(\varphi)} \Delta v^{(\varphi+1)} - \sum \Delta\sigma^{(\varphi+1)} w^{(\varphi)} + \sum \Delta v^{(\varphi+1)} \Delta\sigma^{(\varphi+1)}.$$

The first two sums are equal, as will be seen when introducing into the first the term $z^{(\varphi)} = \Sigma q_{ik} w^{(\varphi)}$ and into the second the term $\Delta\sigma^{(\varphi+1)} = \Sigma q_{ik} \Delta v_k^{(\varphi+1)}$, since in both cases we obtain the bilinear form

$$\sum\sum q_{ik} \Delta v_i^{(\varphi+1)} w_i^{(\varphi)},$$

and hence

$$\Delta K^{(\varphi+1)} = \sum(-2 z^{(\varphi)} + \Delta\sigma^{(\varphi+1)}) \Delta v^{(\varphi+1)} = -\sum(z^{(\varphi+1)} + z^{(\varphi)}) \Delta v^{(\varphi+1)}.$$

According to previous deductions, $z^{(\varphi+1)}$ never has the opposite sign to, and $z^{(\varphi)}$ has always the same sign as, $\Delta v^{(\varphi+1)}$, so that $\Delta K^{(\varphi+1)}$ can never become positive either. Consequently, the expression for K can never increase from phase to phase. K decreases if the permanent deformations change, but K remains constant if the system of all v-values remains unaltered during a phase. Since K, if it changes, can only diminish, but never be negative as per equation (4) or (4a), it must eventually become nil. If, however, K has reached nil, then z will certainly also be nil, and no fresh changes in $\Delta\sigma$ can take place from this moment.

We still have to show that the permanent deformations $\bar{v} = \Sigma\Delta v$ are finite. So long as the terms $\Delta z^{(\varphi)} = \sigma^{(\varphi)} - \sigma^{(\varphi+1)}$ are finite, the Δv will also be finite, with the exception of the case where a matrix of singular value occurs and for which the prime solutions $\sum_{\varphi} c_p \mathfrak{v}_{ip}$ satisfy the conditions laid down as to sign. Only in this case could infinite values of Δv occur for finite values of $\Delta z^{(\varphi)}$. In view, however, of our assumption as to the existence of a system of coercive forces $\bar{\sigma}$, this case is only possible where all $z = 0$. The values z_i must always have the same sign as Δv_i, in the present case the same sign as $\sum_p c_p \mathfrak{v}_{ip}$. Now $\sum_p \sum_i c_p \mathfrak{v}_{ip} z_i = \sum_p c_p \sum_i z_i \mathfrak{v}_{ip} = 0$, because $\sum_i z_i \mathfrak{v}_{ip}$ is already nil as per equation (10). Accordingly, z_i and $\Sigma c_p \mathfrak{v}_{ip}$ cannot have the same sign for all values if i, as otherwise $\sum_p \sum_i c_p \mathfrak{v}_{ip} z_i$ would certainly be positive and not nil. Generally speaking, then, it is not possible for the case to arise where the quantities Δv would increase beyond all limits. Only where all z are nil, i. e., the state of the coercive forces $\bar{\sigma}$ is already attained, can Δv assume infinite values, when the load B has attained the basic values $\bar{B}^{max}$ or $\bar{B}^{min}$ at μ places in a v-times statically undetermined system whose matrix $(q_{11} \ldots q_{\mu\mu})$ is singular and for which the corresponding new system is not self-locked. To exclude this possibility, it would be necessary to stipulate that the values B could go as near as possible to the limits $\bar{B}^{max}$ or $\bar{B}^{min}$ without actually attaining these limits, or to attain these limits only during an infinitely short interval of time dt, during which, for a finite rate of yield, no finite permanent deformation could develop.

In reality, such permanent deformation going beyond all limits will not obtain, as the stiffening of the material will prevent them.

In the introduction, it was mentioned that the foregoing proof can also be demonstrated in a similar way for systems with members stiff against bending. It is also highly probable that the above deductions made for lattice girders are applicable to solid structures of plastic material, provided a suitable definition is found for the properties of the ideal-plastic material.

B) Material with Linear Range of Hardening.

B) Baustoff mit linearem Verfestigungsbereich.

B) Matériau à zone de solidification linéaire.

In the paper on the theory of statically indeterminate systems consisting of an ideal-plastic structural material, the author called attention to the fact that the selection of the appropriate law of elasticity is of the utmost importance. In the following he has therefore set himself the task of considering what formulation must be given to *H. Bleich*'s theorem when a structural material with a linear range of hardening is employed. It thus becomes a question of generalising *M. Grüning*'s researches for any desired change of loading. Here, as in the previous paper mentioned, a new foundation will be applied to the theory which, in spite of its general character as against the line of reasoning hitherto brought forward, yet affords a considerable simplification of the latter.

The following investigations thus deal with statically indeterminate structures subjected to any variable loading and consisting of such material which follows

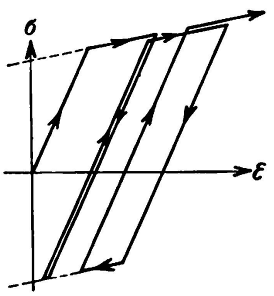

Fig. 1.

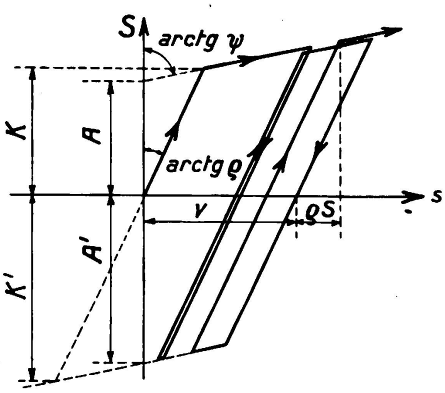

Fig. 2.

the stress diagram is shown in Fig. 1. With a view to simplifying calculation a somewhat more extended idealisation has been introduced in contrast to the assumptions made by *M. Grüning*.

If we, as the two authors mentioned have done, confine our investigations to latticed work, so that we can proceed with a finite number of variables, the forces S in the members and the deformations s of the latter are inter-related through the same connection as tension and elongation, i. e. as in Fig. 2. Accordingly, the deformations s occurring for a definitive force S in a member is given by the relation

$$s = v + \rho S \tag{1}$$

whereby we call v the "permanent" and ρS the "elastic" deformations. $\rho = l/EF$ is a constant, naturally always positive, depending on the length of bar l, its cross section F and modulus of elasticity E. As is clear from Fig. 2, v remains unchanged as long as S lies within the interval

$$cv + K' < S < cv + K, \tag{2}$$

where by, as can easily be confirmed, the meaning of the constants is

$$c = \frac{1}{\psi - \rho}, \quad K' = A' \frac{\psi}{\psi - \rho} \quad \text{and} \quad K = A \frac{\psi}{\psi - \rho}$$

c being certainly positive, K′ negative and K positive.

Should S attain one of these ultimate values, v will only remain constant during the following time interval if the absolute value of S does not increase, i. e. if the increment ΔS is not preceded by the same sign as $S = cv + K'$ or $S = cv + K$ respectively. If, however, the absolute value of S increases, ΔS is then preceded by the same sign as S and Δv will also increase in its absolute value, so that the increment Δv can be expressed by the equation

$$\Delta S = c \Delta v \tag{2a}$$

where Δv, since c is positive, must always be preceded by the same sign as ΔS and incidentally S.

If *Hooke*'s Law can be applied unrestrictedly, we can now, in what follows, regard the material as being perfectly elastic. In this case the forces B in the members would ensue for a certain loading, just as they do when following the usual theory of statically indeterminate structures. These B forces are related to the forces S in the members, which are found in our structure for the same loading, in terms of the equation.

$$S_i + \Sigma q_{ik} v_k = B_i \tag{3}$$

The significance of q_{ik} is easy to recognise. In the unstressed, perfectly elastic system (where all $B = 0$), q_{ik} represents the force in member i, in case of all $v =$ zero and for only a single member k has been permanently deformed to the extent $v_k = -1$. It can be shown in a very simple manner that $q_{ik} = q_{ki}$, for by a theorem similar to *Maxwell's* the force S_i in a member is identical with q_{ki}. Hereby, the force S_i is produced by $v_k = -1$ or in other words q_{ik} is equal to the force S_k in member K which is produced by a deformation $v_i = -1$ of the member i.

Thus the matrix

$$\begin{matrix} q_{11} \cdots q_{1r} \\ \cdots\cdots \\ q_{r1} \cdots q_{rr} \end{matrix}$$

is symmetrical; it has, however, another pronounced property, namely that the "quadratic form"

$$J = \Sigma\Sigma q_{ik} \, v_i \, v_k \geqq 0$$

is "semi-definitely positive", i. e., however v_i is chosen, the term for J cannot be negative under any circumstances whatever. The proof of this, which is not difficult to find on the principle of virtual deformations, may be drawn from the paper mentioned under (3).

Should the external forces change, the forces B in the member of a perfectly elastic system will alter by ΔB, the forces S in the members of an elastic-plastic system by ΔS and the permanent deformations v by Δv.

As long as ΔB keep within sufficiently small limits, the Δv deformations will occur only in identical members. We then speak of a definitive "Phase". A new phase thus begins when the permanent deformations begin to change in other members not hitherto involved, or when Δv disappears for certain members.

We now assume that at the end of a certain phase (φ) the values $S^{(\varphi)}$, $B^{(\varphi)}$ and $v^{(\varphi)}$ are given and that the first μ members of the lattice-work structure have just reached the ultimate values $S^{(\varphi)} = cv^{(\varphi)} + K$, and $S^{(\varphi)} = cv^{(\varphi)} + K'$ respectively. If $B^{(\varphi)}$ now alters by $\Delta B^{(\varphi+1)}$ in the following $(\varphi + 1)^{th}$ phase, $S^{(\varphi)}$ will generally change for all members by $\Delta S^{(\varphi+1)}$; new permanent deformations, however, can only have occurred, if at all, in the first μ members. Thus the equation system

$$S^{(\varphi)} + \Delta S^{(\varphi+1)} + \sum q_{ik} [v_k^{(\varphi)} + \Delta v_k^{(\varphi+1)}] = B_i^{(\varphi)} + \Delta B_i^{(\varphi+1)} \tag{4}$$

applies, and when the equations (3) are deducted herefrom, the result obtained is

$$\Delta S_i^{(\varphi+1)} + \sum q_{ik} \Delta v_k^{(\varphi+1)} = \Delta B_i^{(\varphi+1)} \quad (i = 1.2 - r) \tag{5}$$

Here it is sufficient to consider the first μ equations, which, however, contain 2μ unknowns, namely the values ΔS and Δv. It can now be demonstrated that the alternative existing for the first members, namely either:

ΔS preceded by a sign opposite to that of S, then
$\Delta v = 0$, (6)

or ΔS preceded by the same sign as S, then Δv, in accordance with equation (2a) $\Delta S = c\Delta v$, thus preceded by the same sign as ΔS and also S is just necessary and sufficient for the obtaining of definite results from the equation system (5). The proof can also be brought in the same manner as for ideal-plastic constructional material (see paper quoted under (3).

Now if new deformations really occur in the first members from the μ members mentioned during the $(\varphi + 1)$ phase, the equation it involves the solving of the system equation

$$c_i \Delta v_i^{(\varphi+1)} + \sum q_{ik} \Delta v_k^{(\varphi+1)} = \Delta B_i^{(\varphi+1)} \quad (i = 1, 2 \ldots \lambda) \tag{7}$$

deduced from (5) by introducing $\Delta S_i = c_i \Delta v_i$. Here of course Δv_i $(i = 1 \cdot 2 \ldots \lambda)$ be preceded by the same sign as $S_i^{(\varphi)} = c_i v_i^{(\varphi)} + K_i$ and $S_i^{(\varphi)} = c_i v_i^{(\varphi)} + K'_i$ respectively, at the end phase (φ). In this connection the quadratic form

$$J = \Sigma\Sigma q_{ik} v_i v_k + \Sigma c_i v_i^2$$

belonging to the matrix

$$\begin{pmatrix} q_{11} + c_1 \cdots\cdots & q_1{}^{\lambda} \\ \cdots\cdots\cdots\cdots & \cdots \\ q_{\lambda 1} \cdots\cdots & q_{\lambda\lambda} + c_\lambda \end{pmatrix}$$

ist most certainly "positively definite", i.e., however the v are chosen, J has a positive value and can only vanish when all the v are zero. For even if the first sum of the v having another value than zero should vanish, J will in any case become positive in view of the second sum. Thus, on a well-known algebraic theorem concerning linear systems of equation, we have for all systems of ΔB_i only a single system of solution for Δv_k. In this case, too, the only solution to "all$\Delta B_i = 0$" is in particular all $\Delta v_i = 0$. When the Δv have been deduced from the equations (7), $\Delta S_i^{(\varphi+1)} = c_i \Delta v_i^{(\varphi+1)}$ yields the forces in the members $i = 1 \cdot 2 \ldots \lambda$. For the remaining members

$$\Delta S_j^{(\varphi+1)} = \Delta B_j^{(\varphi+1)} - \sum q_{ik} \Delta v_k^{(\varphi+1)} \quad (j = \lambda + 1, \lambda + 2, \ldots \mu)$$

holds good, where by for the members $\lambda + 1$, $\lambda + 2, \ldots \mu$, $\Delta S_j^{(\varphi+1)}$ must not be preceded by the same sign as $S_j^{(\varphi)}$. For the members $\mu + 1$, $\mu + 2 \ldots r$ the equation

$$K'_i + c_i \left(v_i^{(\varphi)} + \Delta v_i^{(\varphi+1)}\right) \leqq S_i^{(\varphi)} + \Delta S_i^{(\varphi+1)} \leqq c_i \left(v_i^{(\varphi)} + v_i^{(\varphi+1)} + K_i\right.$$

must finally apply. We would add that the values $s - \sigma_i^{(\varphi)}$, which we shall call "coercive forces", and which are given by

$$\sigma_i^{(\varphi)} = -\Sigma q_{ik} v_k^{(\varphi)}$$

represent the forces in the members of a perfectly elastic structure when the individual members have undergone their permanent longitudinal deformations $v_k^{(\varphi)}$. Naturally the expression for $\Delta\sigma_i^{(\varphi+1)} = \Sigma q_{ik} \Delta v_k^{(\varphi+1)}$ also holds good.

We shall now demonstrate that for our system the theorem analogous to *H. Bleich's* theorem for systems of an idealplastic material runs as follows:

In a latticed framework in which the connection described in Fig. 2 exists between force in member and longitudinal deformation, certain finite ultimate values $\bar{v}$ of the permanent longitudinal deformation will occur for any variable loading after an adequate number of load in repetition; no further change takes place in these values on further change of loading if the condition

$$B_i^{max} - B_i^{min} \leqq Ki - K'_i.$$

is satisfied for every member. Here B^{max} and B^{min} signify the maximum and minimum values of the forces in members of a perfectly elastic material. The system will thus behave after the permanent longitudinal deformations $\bar{v}$ have been attained, like a structure composed of a perfectly elastic material.

This theorem can be proved in a manner similar to that shown by the author in the treatise mentioned under (3) for an ideal-plastic structural material. If we substitute the sign of une quality for that of equality, the theorem must also remain valid for the ultimate case

$$\bar{B}_i^{max} - \bar{B}_i^{min} = K_i - K'_i;$$

it is then bound to hold good for the smaller values $B_i^{max} - B_i^{min}$ as well.

In Fig. 3 the forces $\bar{B}'^{max}$ and $\bar{B}^{min}$ in the members have been applied at the points 1, 2, ... r. The condition at the end of the φth. phrase is also shown — at that moment the values $B_i^{(\varphi)}$ may be present. In accordance with the permanent longitudinal deformations $v_i^{(\varphi)}$ just existant (now apparent) the system of coercive forces $\sigma_i^{(\varphi)}$ may be present. The actual forces in the structure under consideration are now given by $S_i^{(\varphi)} B_i^{(\varphi)} - \sigma_i^{(\varphi)}$. In the following phase $(\varphi + 1)$ only those bars can undergo further permanent longitudinal deformations $v_i^{(\varphi)} + \Delta V_i^{(\varphi+1)} = v_i^{(\varphi+1)}$ in which $B_i^{(\varphi)}$ coincides with a limit of the interval $\sigma_i^{(\varphi)} + c_i v_i^{(\varphi)} + K_i$ or $\sigma_i^{(\varphi)} + c_i v_i^{(\varphi)} + K'_i$. In this case the extent of this interval is according to assumptions.

$$[\sigma_i^{(\varphi)} + c_i v_i^{(\varphi)} + K_i] - [\sigma_i^{(\varphi)} + c_i v_i^{(\varphi)} + K'_i] = K_i - K'_i = B'^{max}_i - B_i^{min}.$$

This is for instance only possible in the specially illustrated case of the member j for its upper limit, when $B_j^{max} - [\sigma_j^{(\varphi)} + c_j v_j^{(\varphi)} + K_j] > 0$, while the lower limit cannot be attained by $B_j^{(\varphi)}$. Thus in the $(\varphi + 1)^{th}$ phase for member j only positive $\Delta v_j^{(\varphi+1)}$ are possible.

Let $\bar{\sigma}$ be that system of coercive forces belonging to the aforementioned $\bar{v}$ and eventually appearing after a sufficient number of loading repetitions have taken place. Now Fig. 3 at once gives

$$\bar{\sigma}_i + c_i v_i = \bar{B}^{max} - K_i = \bar{B}^{min} - K'_i = \bar{D}_i$$

for each member i. Since $\bar{\sigma}_i = \sum q_{ik} \bar{v}_k$, the equation system

$$c_i \bar{v}_i + \sum q_{ik} \bar{v}_k = \bar{D}_i$$

is obtained for $\bar{v}$ and from foregoing elucidations it is certain that this equation system must contain definite solutions $\bar{v}$ for any values of $\bar{D}$. Such a system of

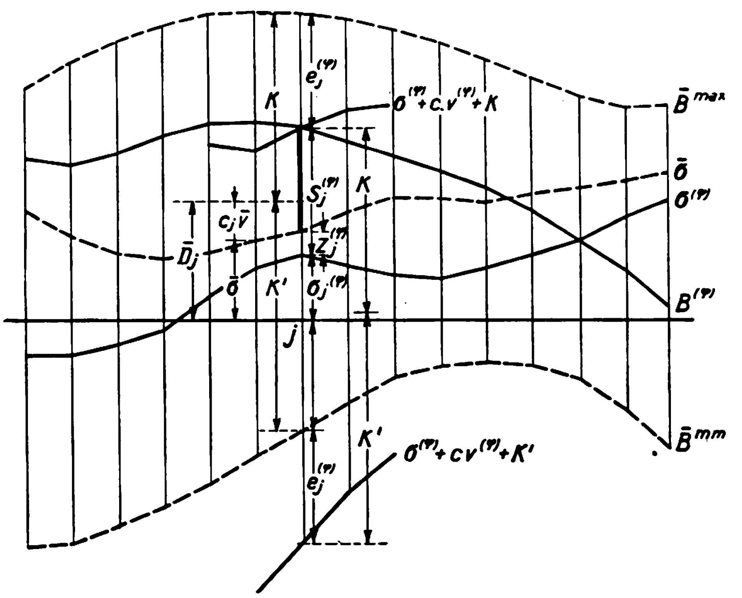

Fig. 3.

$\bar{v}_i$ and $\bar{\sigma}_i$ is therefore bound to exist. Finally, if we consider the value $e_j^{(\varphi)}$, which according to the diagram is calculated

$$e_j^{(\varphi)} = \bar{D}_j - [\sigma_j^{(\varphi)} + c_j v_j^{(\varphi)}]$$

and resolves positively for the member j, we can determine the following alternative:

either $\Delta v_i^{(\varphi+1)} > 0$, in which case $e_i^{(\varphi)} > 0$ and at the end of the $(\varphi + 1)^{th}$ phase $e_i^{(\varphi+1)} \geqq 0$,

or $\Delta v_i^{(\varphi+1)} < 0$, in which case $e_i^{(\varphi)} < 0$ and at the end of the $(\varphi + 1)^{th}$ phase $e_i^{(\varphi+1)} \leqq 0$.

Our theorem is obviously proved when we show that after a sufficient number of loading repetitions, possibly at the expiration of ε phases, $e^{(\varphi+\varepsilon)} = \bar{e} = \Theta$ must vanish for all the members. For if this is correct for all the members, the force S_i can no longer exceed the limits of the interval

$$\bar{\sigma}_i + c_i \bar{v}_i + K'_i \leqq S_i \leqq \bar{\sigma}_i + c_i \bar{v}_i + K_i$$

which then coincide with B_i^{max} and B_i^{min}. To show that as φ increases $e_i^{(\varphi)}$ approaches zero for all the members, we write

$$\bar{v}_i - v_i^{(\varphi)} = w_i^{(\varphi)}$$

and also $$\bar{\sigma}_i - \sigma_i^{(\varphi)} = z_i^{(\varphi)}$$

so that $$\Delta w_i^{(\varphi+1)} = w_i^{(\varphi+1)} - w_i^{(\varphi)} = -\Delta v_i^{(\varphi+1)}$$

and $$\Delta z_i^{(\varphi+1)} = z_i^{(\varphi+1)} - z_i^{(\varphi)} = -\Delta \sigma_i^{(\varphi+1)}$$

now operate and, in consequence of the equation systems

$$\sum q_{ik} \bar{v}_k = \bar{\sigma}_i \text{ and } \sum q_{ik} v_k^{(\varphi)} = \sigma_i^{(\varphi)}$$

the systems

$$\sum q_{ik} w_k^{(\varphi)} = z_i^{(\varphi)} \text{ and } \sum q_{ik} \Delta w^{(\varphi+1)} = \Delta z^{(\varphi+1)}$$

also exist. If we consider the quadratic form

$$J^{(\varphi)} = \Sigma(z_i^{(\varphi)} + c_i w_i^{(\varphi)})\, w_i^{(\varphi)} = \Sigma\Sigma q_{ik} w_i^{(\varphi)} w_k^{(\varphi)} + \Sigma c_i w_i^{(\varphi)2}$$

which, as has already been demonstrated, is always positive for any values of w and can only vanish for all $w = 0$, and further $J^{(\varphi+1)}$ at the end of the $(\varphi + 1)^{th}$ phase

$$J^{(\varphi+1)} = \sum (z_i^{(\varphi+1)} + c_i w_i^{(\varphi+1)})\, w_i^{(\varphi+1)}$$

then the increment of J during the $(\varphi + 1)^{th}$ phase becomes

$$\begin{aligned} \Delta J^{(\varphi+1)} &= \sum [z_i^{(\varphi+1)} + c_i w_i^{(\varphi+1)} w_i^{(\varphi+1)}] - [z_i^{(\varphi)} + c_i w_i^{(\varphi)}]\, w_i^{(\varphi)} \\ &= \sum z_i^{(\varphi)} \Delta w_i^{(\varphi+1)} + \Delta z_i^{(\varphi+1)} w_i^{(\varphi)} + \Delta z_i^{(\varphi+1)} \Delta w_i^{(\varphi+1)} \\ &\quad + c_i (2 w_i^{(\varphi)} + \Delta w_i^{(\varphi+1)} + \Delta w_i^{(\varphi+1)\,2}) \end{aligned}$$

Now, however, $\Sigma z_i^{(\varphi)} \Delta w_i^{(\varphi+1)} = \Sigma \Delta z_i^{(\varphi+1)} w_i^{(\varphi)}$, as can easily be confirmed by introducing $z_i^{(\varphi)} = \Sigma q_{ik} w_k^{(\varphi)}$ and $\Delta z^{(\varphi+1)} = \Sigma q_{ik} \Delta w_k^{(\varphi+1)}$ upon which the bilinear form $\Sigma\Sigma_{ik} w_i^{(\varphi)} \Delta w_k^{(\varphi+1)}$ arises for both cases, so that we now get

$$\Delta J^{(\varphi+1)} = \sum [2 z_i^{(\varphi)} + \Delta z_i^{(\varphi+1)} + c_i (2 w_i^{(\varphi)} + w_i^{(\varphi+1)})]\, \Delta w_i^{(\varphi+1)}$$

Finally we have

$$z_i^{(\varphi)} = \bar{\sigma}_i - \sigma_i^{(\varphi)} = (\bar{D}_i - c_i \bar{v}_i) - (\bar{D}_i - c_i v_i^{(\varphi)} - e_i^{(\varphi)}) = e_i^{(\varphi)} - c_i w_i^{(\varphi)}$$

and also

$$z_i^{(\varphi+1)} = e_i^{(\varphi+1)} - c_i w_i^{(\varphi+1)}$$

and from this

$$\Delta J^{(\varphi+1)} = -\sum (e_i^{(\varphi+1)} + e_i^{(\varphi)})\, \Delta v_i^{(\varphi+1)}$$

But as $e_i^{(\varphi+1)}$ and $e_i^{(\varphi)}$ must be preceded by the same sign as $\Delta v_i^{(\varphi+1)}$, it follows that $\Delta J^{(\varphi+1)}$ must always be negative. As a matter of fact the term for J, which is always positive, will decrease from phase to phase and only remain constant if all the members retain the same amount of longitudinal deformation during a phase. Thus, J must become zero after expiration of a sufficient number of phases. This makes $w^{(\varphi+\varepsilon)} = 0$ absolutely necessary, however, and in consequence $v^{(\varphi+\varepsilon)} - \bar{v} = 0$, which also gives $\bar{\sigma} - \sigma^{(\varphi+\varepsilon)} = 0$. There thus actually arises a system of coercive forces in which the permanent longitudinal deformation cannot change any more and from this point on the structure behaves as if it were composed of perfectly elastic material.

In comparison with structures of ideal-plastic material, it becomes clear — a significant fact — that the investigation of systems of coercive force is of no importance whatever; on the contrary, in this case it is quite sufficient if the condition $B^{max} - B^{min} \leqq K - K'$ is satisfied for each member. Moreover, owing to the term $c_i\, w_i^2$ occurring in the quadratic form, it is impossible for semi-definite terms to appear which, with respect to the singularity of their solutions, necessitate separate investigation as in the case of ideal-plastic material. On the contrary, it is certain that to every and finite value of $\bar{\sigma} - \sigma^{(\varphi)}$, only finite and singular values of $\bar{v} - v^{(\varphi)}$ can belong. It is naturally another question whether the total deformations finally occurring have not already attained inadmissible values and do not perhaps lie beyond the ultimate rupture point elongation. This question can just as pertinently be asked in the case of ideal-plastic material. A general answer is difficult to give.

Summary.

The first part of this paper gives a brief survey of the published literature and results in connection with the subject discussed. The second part discusses the essential properties of ideal-elastic lattice girders and gives a general proof of Mr. *H. Bleich's* theorem. This theorem, which forms the basis for calculating such systems and, in government regulations, is referred to as the ,,Traglastverfahren" or "the theory of plastic equilibrium", is broadly based on the fact that a certain positive quadratic form can always only diminish, i. e., become nil once only. Beyond the known results, certain limitations have been set forth for the validity of Mr. *H. Bleich*'s theorem, which consist in excluding certain cases of loading as soon as a system has become fully elastic, unless such cases of loading are limited to an infinitely short interval of time where the rate of yield is finite in value.