Theory of thin curved shells not subjected to bending

IVa 1

Theory of Thin Curved Shells not Subjected to Bending.

Einführung in die allgemeine Theorie der biegungsfreien Schalen.

Etude des voiles minces courbes ne subissant pas de flexion.

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1) *Review of the general equations for statical equilibrium in rectilinear coordinates.*

Let $z = f(x,y)$ be the equation for the surface in rectilinear coordinates not necessarily rectangular. The conditions of stresses at a point m of the surface are determined by knowledge of the stresses n_1, $-n_2$, $-\Theta$ acting on the elements mm_1 and mm_2 respectively, parallel to planes zox and zoy. The stress n_1 acts on mm_2 parallel to plane zox, and n_2 stresses mm_1 parallel to plane zoy and Θ acts at the same time on mm_1 parallel to zox and on mm_2 parallel to zoy (fig. 1). Let α_1, o, γ_1 and o, β_2, γ_2 be the coefficients governing the directions of the tangents to elements mm_1 and mm_2, in other words, the projections of the unit vector upon ox, oy, oz, for each of the tangents.

Let us assume that the surface is loaded in some way, and that Xdxdy, Ydxdy and Zdxdy are the components parallel to ox, oy, oz of this load for the element $mm_1 m'm_2$, defined by parallels mm_1 and $m'm_2$ to plane zox and by parallels mm_2 and m_1m' to plane zoy. The investigation of the conditions of equilibrium of these elements leads to the following equations:

$$(1) \quad \frac{\partial \nu_1}{\partial x} + \frac{\partial \Theta}{\partial y} = X$$

$$(2) \quad \frac{\partial \Theta}{\partial x} + \frac{\partial \nu_2}{\partial y} = Y$$

$$(3) \quad r\nu_1 + 2s\Theta + t\nu_2 = \zeta$$

taking: $p = \frac{\partial f}{\partial x} \qquad q = \frac{\partial f}{\partial y} \qquad r = \frac{\partial^2 f}{\partial x^2} \qquad s = \frac{\partial^2 f}{\partial x \partial y} \qquad t = \frac{\partial^2 f}{\partial y^2}$

$$\zeta = Z - pX - qY$$

$$\nu_1 = n_1 \frac{\alpha_1}{\beta_2} \qquad \nu_2 = n_2 \frac{\alpha_2}{\beta_1}.$$

2) *Geometrical interpretation of the quantities contained in the general equations for equilibrium.*

The term ζ contained on the right side of equation (3) is nothing more than the projection of vector (X, Y, Z) on oz, this vector being projected parallel to the tangential plane of the surface. In order to interpret the quantities $v_1 - v_2$ and Θ, which are the unknown quantities of the equations of equilibrium, we must give a general definition of what we shall call "reduced stress". By definition, the reduced stress acting on an element of the surface is the projection on the xy plane of the elastic force which acts on this element, divided by the length of the projection of this element.

It can easily be realized that the distribution of reduced stresses around a given point follows the same laws as the real stresses and in particular the theory of *Mohr* can be applied. The quantities v_1, v_2, Θ are, in fact, the reduced stresses in relation to elements which are projected along parallels to axis of x and y. It will be noticed that the shear stresses Θ are maintained in projection, whereas this is not the case for the other stresses n_1, n_2.

3) *Geometrical interpretation of the general equations for equilibrium.*

Equations (1) and (2) evidently express the conditions of equilibrium, in projection on the tangential plane. Equation (3), on the contrary, expresses the equilibrium of forces normally applied on the surface. For a geometrical translation, let us take the origin 0 of the trihedron oxyz on the surface itself and direct ox and oy along the directions of two optional elements. We are then able to complete the definition of trihedron oxyz by taking arbitrarily the direction oz. Equation (3) determines a linear relation between the stresses acting on the optional elements ox and oy and the projection ζ on oz, parallel to the tangential plane, of the density of the stress applied. If we change the direction oz without any modification to ox and oy, each term of the preceding linear relation is only multiplied by the same coefficient.

We can take advantage of the indetermination of the direction of elements ox and oy to simplify the equation (3). If these elements in particular are directed according to two conjugated directions of the surface, that is, according to two directions conjugated in relation to the indicator, the coefficient of Θ becomes null and equation (3) is reduced to a linear relation between the longitudinal stresses v_1 and v_2. We can wonder whether it is not possible to direct the elements ox and oy in such a manner that they are eliminated from equation (3), leaving only one stress. We can at once realize that this is not possible if the surface is convex, that is, if the principal radii of curvature are in the same direction, and that on the contrary it is possible if the surface is not convex.

Let us consider the latter hypothesis and discriminate between two cases, according to whether the stress which remains in the equation (3) is an axial stress or is the shearing stress Θ. The first case is not possible unless the surface is developable, i. e. if we can consider it as being the envelope of a family of tangential planes relating to a parameter. If we consider the element ox with respect to the direction of the linear generatrix which passes through 0, the equation (3) is reduced to:

$$r v_1 = \zeta \qquad (4)$$

The second case applies to surfaces of opposite curvatures. If we consider ox and oy with respect to the asymptotical directions, the equation (3) is reduced to:

$$2s\Theta = \zeta \tag{5}$$

Equations (4) and (5) can be immediately interpreted. Let us first examine the equation (4). It is obvious that the only stresses acting on an infinitely small element of the surface and admitting a component which is not located in the tangential plane to the surface, are the stresses projected along v_1 and equation (4) simply expresses the identity between the projections of the stresses n_1 on oz parallelly to the tangential plane and the projection of the applied load, in the same conditions.

Let us now examine equation (5): it suffices here to consider an elementary quadrangle, two consecutive sides of which are formed by asymptotical arcs transecting at 0. The longitudinal stresses n_1 and n_2 applied to this quadrangle admit a resultant in the tangential plane owing to the fact that this resultant is the geometrical summation of the resultant of the stresses n_1 and of the resultant of the stresses n_2 and that each of these two latter resultants is necessarily in the osculatory plane of an asymptotical arc, such osculatory plane concurring with the tangential plane, being given the very definition of asymptotical lines. Therefore, component ζ of the stresses applied to the surface, outside the tangential plane, depends only on shear Θ, to which it is, in fact, proportional. The coefficient of proportion, the value of which is $2s$, admits a very simple geometrical significance: it is the quotient of twice the distance from the vertex opposed to 0, in the quadrangle, to the tangential plane at 0, this distance being evaluated parallelly to the direction oz, by the product of the lengths of the asymptotical arcs which form the sides of the quadrangle.

4) *Classification of thin shells with respect to their mechanical properties.*

The above considerations lead to a classification of the thin shells into three groups. The first group covers developable surfaces, such as cylinders, cones; the second group comprises the convex surfaces, such as spheres, elliptical paraboloid, ellipsoids, polar-symmetrical hyperboloids and, generally speaking, all surfaces of double curvature, which are generated by a curve the concavity of which is directed downwards and which rest on a curved directrix, the concavity of which is also directed downwards. In the third group, we find the surfaces characterized by opposed curvatures, such as hyperbolical paraboloid, hyperboloids, conoids, all undevelopable ruled surfaces and, generally speaking, all surfaces which can be generated from a curve, the concavity of which is directed upwards and which rests on a directrix whose concavity is directed downwards.

This classification has been suggested to us by the geometrical interpretation of equation (3). Shells of the first group are those for which equation (3) can take the form (4); shells of the second group are those for which equation (3) can take the form:

$$r v_1 + t v_2 = \zeta \tag{6}$$

r and t being preceded by the same sign; shells of the third group are those for which equation (3) can take the form (5).

It should be noted that for shells of the third group, the equation (3) can also take the form (6), but r and t are then of opposite signs. It should also be noted that for shells of the second group, equation (3) can also take the form (5), where Θ represents the shear on the asymptotical lines but equation (5) is then no longer an equation with real terms, as s and t are purely imaginary expressions.

Shells of the first group are characterized by the fact that the normal component of the stress on the rectilineal generatrices is, at each point, proportional to the normal component of the density of the applied load. Shells of the second group are characterized by the fact that the purely imaginary shear stress acting on the imaginary elements of asymptotical lines is, at each point, proportional to the normal component of the density of the applied load. Shells of the third group are characterized by the fact that the shear stress acting on the elements of asymptotical lines is, at each point, proportional to the normal component of the density of the applied load.

The following difference should also be noted between shells of the second and of the third group. If we consider at a given point, the longitudinal stresses acting on two conjugated elements, then the normal component of the applied load, which can itself be considered as the bulging produced by these longitudinal stresses, is a linear form of these stresses. The related coefficients are of the same sign for surfaces of the second group, and of opposite signs for shells of the third group. It therefore follows that the carrying capacity of a shell of the second group can be considered as a result of the action of longitudinal stresses of same direction, acting on two conjugated elements, and that the carrying capacity of a shell of the third group can, in a similar way, be considered as produced by longitudinal stresses of opposite senses, acting on two conjugated elements.

As regards shells of the second group, the conjugated elements can always be chosen so that the coefficients of the corresponding stresses are equal, in the linear form which represents the normal component of the density of the applied load. Such elements will be called canonical. It can then be said that in shells of the second group, the normal component of the density of the applied load is proportional to the summation of the longitudinal stresses acting according to the directions of canonical elements.

These differences in properties just mentioned above and which distinguishing the three groups of shell from one another, are of the utmost importance as regards the kinds of supports which can be considered for the periphery of such shells, in order to achieve their equilibrium, and as regards the actual method of calculation for the stresses in the shells as functions of the conditions on the periphery.

5) *Shells of the first group.*

The study of shells of the first group is a generalization of the study of cylinders. Equation (4) gives, for each point of the shell, the normal component relating to the rectilinear generatrix passing through this point, of the stress acting on an element of this generatrix. Consequently, if we draw on the surface a family of geodesic lines intersecting the rectilinear generatrices at a constant angle, we shall know the value of the longitudinal stress which acts parallelly to these geodesic lines on the elements of the rectilinear generatrices. Equation (2) gives then, by immediate integration, the value of shear on the generatrices and geodesic lines and a second integration from formula (1) gives the longitudinal stresses acting on the elements of the geodesic lines parallelly to the generatrices.

Such a determination of the stresses is not complete unless we assume to be given, on a given curve intersecting only once each generatrix, the values of the stresses acting on the elements of the said curve. In the same way, we can also assume to be given, on two curves, each of them intersecting each generatrix only once a relation between the components of the stress acting on any element of the two curves.

6) *Shells of the second group.*

Let us consider a thin shell of the second group. We have seen that the normal component of the density of the applied load is, at each point, proportional to the summation of the longitudinal stresses acting on canonical elements. We shall now suppose that these longitudinal stresses are equal. Their value is therefore clearly determined, at each point, by the value of the normal component of the density of the load. Thus equation (3) is fulfilled. Equations (1) and (2), which express equilibrium in the tangential plane, are then fulfilled only if the tangential component of the density of the load has a determined value, which can be obtained precisely by writing the conditions of equilibrium parallelly to the tangential plane. We shall call "fundamental system of loads" every system of loads corresponding to the preceding conditions, that is, such that the longitudinal stresses acting on two conjugated elements, symmetrical to the principal directions, be equal. It then becomes obvious that any system of loads can be considered as the summation of a fundamental system and of a system which would be exclusively composed of tangential loads, which we shall call "complementary system" to the fundamental system of loads.

We are thus induced to study the complementary systems, i. e. the systems in which the applied load is tangential to the surface. In such systems, the longitudinal stresses on canonical elements are equal and therefore the stress on any element now depends only on two parameters, for instance the components of the stress which acts on one of the two preceding canonical elements. It is obvious that these two parameters can be arbitrarily chosen. It will be easily understood that we can determine two conjugated imaginary functions φ and ψ in such a manner that, when choosing as parameters two quantities which we shall call S_φ and S_ψ, the elastic forces acting on any element of the surface consist of linear forms from the differential expressions $S_\varphi\, d\psi$ and $S_\psi\, d\varphi$. The equations for equilibrium in the tangential plane then show that the partial derivative of S_φ with respect to φ and the partial derivative of S_ψ with respect to ψ are linear functions of S_φ and S_ψ. By elimination of one of the two parameters, S_ψ for example, between these two relations, we get an equation to linear partial derivatives of the second order, with imaginary characteristics, which the retained parameter S_φ must fulfil.

In order to arrive at a solution for such an equation, we can assume a given the value of S_φ and of one of its derivatives on an optional curve of the surface. provided however that, the equation having imaginary characteristics, certain conditions of analycity be fulfilled. If we observe that S_φ of the curve and one of its derivatives may be assumed, the values S_φ and S_ψ of this curve can be obtained and in consequence there of the stresses acting on any element of the curve. After making certain reservations for analycity, we now see that the stresses in the shell can be determined, provided that the stresses acting on the elements of a curve be known.

The foregoing reserves for analycity are not merely formal reserves. They correspond to a physical reality which is the following. We know that in all problems where a function used to verify an equation with imaginary characteristics is determined by means of the values it assumed — the same applies to one of its derivatives —, on a given curve, the solution is not a continuous function of the given values; in other words, by slightly varying the given values, results of any desired difference can be obtained from this function, on points arbitrarily chosen. It follows that the states of equilibrium of a convex shell. corresponding to given values of the stresses acting on a curve, are not stable with respect to the values of the stresses on this curve.

In order to arrive at stable solutions, it is necessary to consider, for the limits, conditions which are different from those we have taken. Instead of assuming the values of the stresses on a curve to be given, we shall assume on a closed curve, a given relation between the components of the stresses acting on the elements of the curve. The problem then becomes clearly determined and its solution will be a continuous function of the given values. The corresponding equilibrium will be stable. Let us suppose, for example, that we wish the normal component of the stress along the given curve to be null. The relative indetermination of the parameters S_φ and S^ψ allows us to determine them in such a manner that S_φ represents, along the given curve, the value of the normal component of the stress acting on the elements of this curve. The theory of integral equations then allows us to determine the function S_φ by a method similar to that used by Fredholm and his successors in solving problems of the same type, relating to equations with imaginary characteristics.

7) *Shells of the third order.*

Let us consider a thin shell of the third order. The value of the normal component of the density of the load determines first of all at each point of the shell the shear stresses on the asymptotical elements. Let us assume that the stresses in the shell are reduced to these shearing stresses. For this purpose it is necessary and sufficient, for the tangential component of the load which is applied to an elementary quadrangle of asymptotical arcs to balance the projection of the resultant of the tangential stresses applied to the elements of the quadrangle on the tangential plane. We shall call "fundamental system of loads", any system of loads corresponding to the preceding conditions, i. e. such that the stresses acting on the elements of asymptotics are reduced to shear stresses. It is quite obvious that any system of loads can be considered as a superposition of a fundamental system of loads and of a system which we shall again call "complementary system", exclusively composed of tangential loads.

We thus return to the study of the action of complementary systems. For this purpose, we again observe that the elastic force acting on an element of shell can be translated into a linear form of differential expressions such as $S_\varphi d\psi$ and $S_\psi d\varphi$, φ and ψ now being two real functions and S_φ and S_ψ two real parameters. The equations for equilibrium according to the tangential plane allow then of expressing the partial derivatives of S_φ with relation to φ and of S_ψ with relation to ψ, in linear functions of S_φ and S_ψ. The elimination of S_ψ between these equations leads to an equation in S_φ, linear to the partial derivatives of the second order, with real characteristics. The characteristics of such an equation to the partial derivatives are precisely the asymptotical lines.

In order to obtain a solution from the preceding equation which would be valid in an area D limited by a contour C, we shall divide this contour into to two series of arcs Γ and Γ' in such a way that from any point of D two asymptotical lines are drawn intersecting Γ only once; we shall then divide Γ into to two series of arcs Γ_1 and Γ_2 in such a manner that any broken line of asymptotical arcs joining any point from Γ_1 to a point on Γ' has its intermediate vertices on Γ_2 or on Γ' and that there is no broken line of asymptotical arcs having its ends on Γ_1 and its intermediate vertices on Γ_2. We shall obtain a single solution, valid in D, when assuming a given the value on Γ_1 of the stress acting on the elements of Γ_1 and on Γ_2, a relation between the components of the stress acting on the elements of Γ_2. The value of this solution will be given by the *Riemann* formula, successively applied to different fractional zones of zone D. No condition of analycity is here necessary and the solution arrived at is always a continuous function of the data. On the other hand, there is generally no corresponding solution to a relation between the components of the stresses acting on the different elements of the closed curve C.

When the thin shell taken into consideration is a straight-line surface, the equation to the partial derivatives of the second order can be reduced to a linear equation with partial derivatives of the first order containing only one derivative, the integration of which is immediate, as it can be considered as a linear differential equation. In the case when the thin shell consists of a straight-line surface of the second order, the determination of parameters S_φ and S_ψ is reduced to the solving of two quadratic equations.

8) *Choice of supports for thin shells of the three groups.*

The choice of the system of supports for a thin shell depends essentially on the group to which the surface belongs. We shall distinguish between two classes of supports: single supports with which the reactions depend on one parameter, and double supports, with which the reactions depend on two parameters. The components of the stresses transmitted by the shell to a simple support therefore fulfil a relation which is known a priori, whereas the components of the stresses transmitted by the shell to a double support can assume independent values. However, certain parts of the shell, on the marginal zones, shall be left without support; in such a case the contour is said to be free.

We propose to find out how the free edges, single supports and double supports must be distributed on the contour of a thin shell, so that the latter is subjected to definite and stable equilibrium conditions.

Let us consider at first the case of a thin shell of the first group. We can assume the presence of a free edge on every part of the contour which comprises no rectilinear generatrix, intersected once at the most by any generatrix. If the free edge meets all the generatrices, the distribution of stresses in the shell is determined and therefore the other edges must be equipped with double supports. The resulting system of equilibrium is stable. If, on the contrary, we consider two edges, each of them only once intersected by all the generatrices and provided with double supports, we shall again obtain a state of stable equilibrium, on condition that the rest of the contour exclusively composed of generatrices, be arranged as double supports.

Should we now consider à shell of the second group, such a shell cannot admit of free edges, as the resulting equilibrium is not stable. The whole of the periphery can, however, be arranged for single supports and the resulting equilibrium is well defined and stable.

Let us finally consider a shell of the third group and divide its contour into three series of arcs Γ_1 Γ_2 Γ', defined as indicated in 7. We can assume a free edge according to Γ_1, single supports according to Γ_2 and double supports according to Γ'. The resulting state of equilibrium is well defined and it is a stable.

9) *Geometrical properties of the shells of the third group and their geometrical calculation.*

Shells of the third group show remarkable geometrical properties which allow for an accurate graphic calculation.

Let us first interpret geometrically the parameters S_φ and S_ψ and the functions φ and ψ introduced at 7. S_φ and S_ψ are the longitudinal stresses acting on the asymptotical lines for a complementary system of loads. φ and ψ are the curvilinear coordinates of the surface for which the lines of coordinates are the asymptotical lines.

Let us replace these thin shells by a skew reticular system, the meshes of which consist of rectilinear skew quadrangles formed by chords of asymptotical lines. The system thus obtained works as does the given surface and the assimilation of the two systems to one another is legitimate if the meshes are sufficiently small. The loads applied to the reticular system must be applied to the vertices of this system according to the tangential plane to the surface.

If we apply a single force F to any vertex of the reticular system, such a force can be decomposed between two of the bars passing at this point and corresponding to two different asymptotical lines. Force F is thus transferred to two other knots of the surface, where we operate in the same manner, and so on. If we suppose that the contour of the surface is divided into three series of arcs Γ_1 Γ_2 Γ' according to the foregoing conditions and if we conveniently choose the two initial members in accordance with which the given force F has been devided, the fransfer of force F can be made as indicated without ever encountering a free edge. If we meet with a free edge on Γ_2 supposed to be arranged for single support, we can still make the division between the second member ending on the vertex considered on Γ_2 and the direction of the reaction of the single support. The operation thus carried out is called a reflection on the single support.

By continuing in the same manner, we finally transmit the force F to a whole zone of double supports. We thus obtain for the system a state of equilibrium which is consistent with the reactions at the supports, and the equilibrium will be stable. Operating in the same manner for each loaded knot of the reticular system, we determine the state of equilibrium for the complementary system of loads only by dividing forces according to the parallelogram of forces. The corresponding diagram can be easily drawn by projecting on an arbitrary plane.

The geometrical determination of the stresses mentioned above allows of considering the equilibrium of a shell of the third group as resulting from a propagation of stresses according to the asymptotical arcs and starting from the free edges so as to arrive at the double supports by reflection on the simple edges. Such behaviour is similar to the propagation by means of waves of the phenomena following the rule of linear equations to the partial derivates of the second order with real characteristics, and is also essentially due to the real nature of the characteristics of the equations governing the equilibrium of stresses in the shell under consideration.

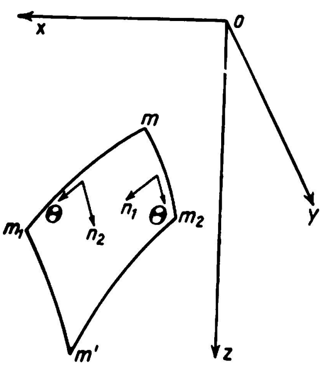

Fig. 1.

10) *Elementary examples of shells of the third group.*

The most simple example of a shell of the third group is the hyperbolic paraboloid. This shell is characterized by its property that shear along to the rectilinear generatrices, within a certain coefficient constant for the whole sur-

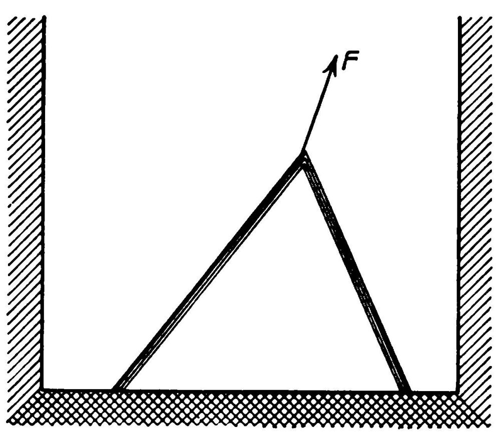

Fig. 2.
Mode of propagation of tangential stresses, in a ruled quadric surface.

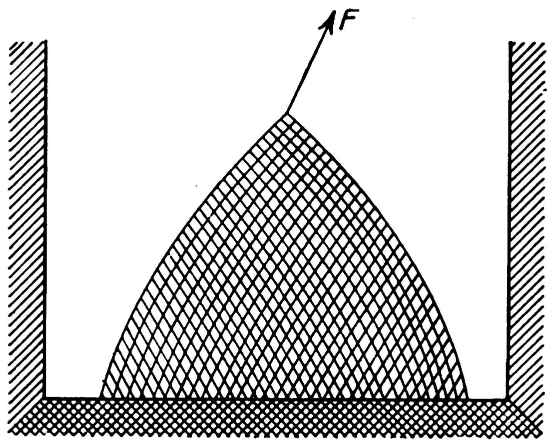

Fig. 3.
Mode of propagation of tangential stresses in any other surface of the third group.

face, is equal to the component along the axis of the paraboloid of the applied load, brought back to the unit of surface projected on an arbitrary plane not parallel to the axis. On the other hand, the stresses due to the complementary system of loads propagate each generatrix without any interference between the generatrices, so that a tengential stress applied to a small element of the shell acts only on the bands produced by the generatrices encountered. The simplest form of shell of the third group after the hyperbolie paraboloid is the hyperboloid. Just like the paraboloid, this shell has the property that stresses due to the complementary system propagate each generatrix without any interference with other generatrices. It only differs from the hyperbolic paraboloid in a more intricate expression of the coefficient of proportionality between shear and density of the applied load.

Then follow the undevelopable straight-line surfaces, and first of all the conoids. For these surfaces, the coefficient of proportionality between shear on the asymptotical lines and density of the applied load is expressed in a much more complicated form than for the preceding surfaces, but the most distinguishing character of such surfaces is that the stresses due to the complementary system propagate by opening on the surface; the unrectilinear asymptotical lines abutting on the rectilinear generatrices of the surface, so that a tangential effort applied to a small element affects a whole zone distributed over the surface, just as for the most general surfaces of the third group.

Figures (2) and (3) show the difference between straight-line quadric surfaces and other surfaces of the third group as regards the views expressed above.

11) *Conclusions.*

With the exception of the developables straight-line surfaces, such as cylinders and cones which form a very particular class of shells, all the shells with double curvature can be divided into two important classes, according to the sign of the total curvature. In these two classes, the asymptotical lines play the essential part in the transmission of tangential stresses and therefore in the determination of the nature of the reactions of supports corresponding to well defined and stable conditions of equilibrium. When the asymptotical lines are imaginary, the shell cannot admit of free edges, but can be limited by edges all arranged as simple supports. A common example of such supports is given by a tympan or flat slab of great stiffness in its own plane and without any appreciable stiffness perpendicularly to this plane. When the asymptotical lines are real, the edges of the shell are to be divided into free edges, edges with simple supports and edges with double supports, according to the well determined principles we have mentioned.

As double supports might involve difficulties as regards design, it is beneficial to reduce their importance as far as possible, and this can be done in different ways when conveniently choosing the outline of the surface.

If we strictly consider the facilities of calculation, the views expressed above show that, among the shells with double curvature, straight-line quadric surfaces are those which lead to the most elementary calculation.

Summary.

The problems offered by the design of thin curved shells of reinforced concrete consist, in the first place, of purely statical problems, independent from the theory of elasticity. We shall deal later on with these problems, as a whole, excluding such other questions as concerning the application in practise of shells under consideration of existing deformations and particularly the problem of comptability of deformations due to stresses calculated by means of ordinary statics.

We shall apply the hypothesis, generally accepted, of a uniform distribution of stresses on any transverse section in such a manner that the shell can be considered as being replaced by mid-surface of the shell.

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