# Bending, torsion and buckling of bars composed of thin walls

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# Bending, Torsion and Buckling of Bars Composed of Thin Walls.

# Biegung, Drillung und Knickung von Stäben aus dünnen Wänden.

# Flexion, torsion et flambage des barres composées de parois minces.

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## 1) Introduction.

The fact that the distribution of longitudinal stresses in the cross-section of certain shapes of girders does not correspond with the distribution as determined by the theory of bending laid down by Navier, was noted for the first time by *Bach*[1] on basis of experiments. At about the same time *R. Sonntag*[2], independently of *Bach*, drew attention to the phenomena of torque — when bending bars of ⌊, [ and ⌉⌊-cross section — and he computed the actual stresses set up. The differential equation of the problem of torque for a symmetrical I-girder was derived by *Timoshenko*[3]. *Weber*[4] generalized this investigation to any two-flanged beams, that is, for [, ⌉⌊ and ⊥-cross sections. The tilting problem of the I-beam was treated by *Timoshenko*[3]. *Ostenfeld*[5] investigated the stability of ⊥, I, ⌉⌊ and [-shaped cross-sections while *Eggenschwyler*[6] discussed the problem of torque of bars of rectangular box-shaped cross-section.

The classical theory of bending and torsion of straight prismatical bars is based on the assumption that the longitudinal bending stresses are distributed over the cross-section of the bar according to a linear law and that in the case of torsion only, no longitudinal stresses occur, provided that the small quantities of second order are disregarded. This assumption, however, does not hold in the case of bars made up of thin sheets, and the investigations referred to above

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[1] *C. Bach:* Versuche über die tatsächliche Widerstandsfähigkeit von Balken mit [-förmigem Querschnitt (Experiments to ascertain the actual strength of girders of [-shaped cross-section), Zeitschrift des Vereins deutscher Ingenieure 1909, p. 1790 and 1910, p. 382.

[2] *R. Sonntag:* Biegung, Schub und Scherung (Bending, torsion and shear), Berlin 1909.

[3] *S. Timoshenko:* Einige Stabilitätsprobleme der Elastizitätstheorie (On some problems of stability in the theory of elasticity), Zeitschrift für Math. und Phys. 1910, p. 361.

[4] *C. Weber:* Übertragung der Drehmomente in Balken mit doppelflanschigem Querschnitt (Transmission of torque in girders of double-flanged cross-section), Zeitschrift für angew. Math. und Mech. 1926, p. 85.

[5] *A. Ostenfeld:* Mitt. Nr. 5 und 6 des Lab. für Baustatik der Techn. Hochschule in Kopenhagen 1931 und 1932.

[6] *A. Eggenschwyler:* Über die Verdrehungsbeanspruchung rechteckiger Kastenquerschnitte (On the torsional loading of rectangular box-shaped cross-sections), Eisenbau 1918, p. 45.

are a proof for this statement. As it is a general practice in structural steel engineering to use girders built up of single plates put together it appears necessary that the actual behaviour of such girders should be studied without restricting such investigations to particular sectional shapes.

The following abstracted representation is the first attempt made to deal with this problem of bending and torsion of thin-walled bars in as general a manner as possible. This investigation enables to elucidate the underlying connections and to furnish fundamental formulae of general validity suitable for the study of the special problems in question i. e. the problems of stable and unstable equilibrium.

The object of the investigation consequently are prismatical bars consisting of several lengthy rectangular discs with cross-sections similar to those shown in Fig. 1. Fig. 1a is an example of an *open* or *singly-connected section.* Fig. 1b

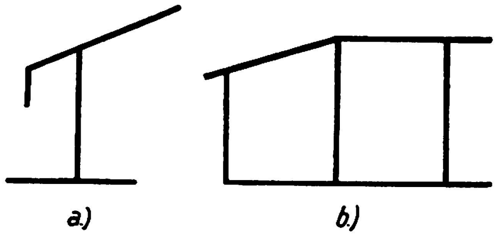


Fig. 1.

shows a *closed* or *multiply-connected section.* The thickness of the individual disc may vary within it; however, it is always assumed that the thickness is small as compared to the height of the disc.

We adopt from the *Navier*'s bending theory the assumption that the geometrical shape of the section will be maintained after deformation due to loading. We shall refer later on again to this fundamental view-point. The requirement that planes originally plane remain so, is not retained. All we do is to assume that the base of *Navier*'s theory of bending holds for each single disc. This assumption, however, cannot be exactly fulfilled at points where two or more discs join, owing to the thickness of the discs being different from zero, yet this thickness shall be assumed as being so small that these deviations can be left out of consideration. We consider only the flexures of the disc in its own plane, whereas flexure which is perpendicular to the plane of the discs will be disregarded on account of the very low resistance to bending in this direction. From the shearing stresses only that part is taken into consideration which is induced by torsion, while bending-shearing stresses are neglected because of the very slight influence they exert on the change in the shape of the bar, if the latter is long compared with the dimensions of its cross-section. The results therefore do not hold for short bars.

To obtain the differential equation of the problem in its most general shape we shall take a principle of variation, well known in mechanics, as our starting point. In this way we secure the advantage to apply the general result also to formulate approximate solutions according to the method by *Ritz,* in cases where an exact solution of the differential equations obtained by variation offers difficulties.

Le $A_a$ be the work done by external forces and $A_i$ that of the internal forces (energy of deformation), then the equation, expressing the extreme-requirement reads as follows:

$$\delta\,(A_i - \lambda\,A_a) = 0 \tag{1}$$

in which $\lambda$ is a multiplicator depending on the special conditions of the problem. In the cases, which we consider here, $\lambda$ can be established a priori.

In the case of deformation of first order in a problem involving *flexure* or *twisting* of the straight bar. $A_i$ and $A_a$ are homogeneous functions of the second, respectively first degree of the quantities of deformation[7]. Due to this fact it follows that the multiplicator has the value 2, as is well known.

When dealing with a problem of *unstable equilibrium* (buckling problem) in which only deformations of first order occur in the equations for equilibrium, then: $\lambda = 1$.

The equation (1) therefore reads as follows:

In the case of stable equilibrium $\quad \delta\,(A_i - 2\,A_a) = 0 \quad$ (2)

In the case of unstable equilibrium $\quad \delta\,(A_i - A_a) = 0 \quad$ (3)

The main part of the following investigation consist in formulating the expression for the energy of deformation $A_i$ and the energy of deformation of the external forces $A_a$. When determining $A_i$ a distinction must be made between bars of singly-connected section and those of multiply-connected section.

## 2) Energy of Deformation $A_i$.

*Bars of singly-connected section.*

If one single disc of a bar is considered, this bar as a whole deforms and the central line will show a flexure $K_i$ in the plane of the disc and an extension $\bar{\varepsilon}_i$. At the same time the whole disc rotates a certain amount denoted by the angle $\varphi$. Let $J_i$ be the moment of inertia of the cross-section of the disc the height being $h_i$ and the thickness $\delta_i$, the latter varying as a rule according to the heigth — with regard to the centre of gravity — axis, the latter being perpendicular to the plane of the disc, $F_i$ the cross-sectional area, $J_{di}$ the torsional resistance of the disc, then the formula for the total energy of deformation $A_i$ for the bar of the length l, consisting of n discs reads as follows:

$$A_i = \frac{1}{2}\int_0^l \left[\sum_{i=1}^{n} (E\,J_i\,K_i^2 + E\,F_i\,\bar{\varepsilon}_i^2 + G\,J_{di}\,\varphi'^2)\right] dz, \tag{4}$$

in which $d_z$ denotes the distance between two cross-sections of the bars infinitely close to each other.

The quantities $K_i$ and $\bar{\varepsilon}_i$, however, are not independent one from the other, but are linked up by certain conditions of transition. The longitudinal stresses and with them the longitudinal strains of the two discs must coincide at edges

[7] This holds true under the limitation, that in the equations for equilibrium the magnitude of the deformations is neglected as compared to the dimensions of the girder.

where they join. If "a" is the distance between the centre of gravity-axis of the disc with respect to the edge, then for each edge of the bar, see Fig. 2, we have:

$$\bar{\varepsilon}_i + a'_i K_i = \bar{\varepsilon}_{i+1} - a_{i+1} K_{i+1} \qquad (i = 1, 2, \ldots n-1) \tag{5}$$

Given n plates, n such equations can be formulated. Furthermore, if S is the longitudinal force acting in the bar, then the average strain $\varepsilon_m$ will be

$$\varepsilon_m = \frac{S}{EF}, \quad \text{where} \quad F = \sum_i F_i.$$

There exists furthermore the requirement of equilibrium

$$E \sum_i F_i \bar{\varepsilon}_i = S = E F \varepsilon_m \tag{6}$$

With $\bar{\varepsilon}_i = \varepsilon_m + \varepsilon_i$, the equations (5) and (6) assume the following relations:

$$\varepsilon_i + a' K_i = \varepsilon_{i+1} - a_{i+1} K_{i+1} \tag{5'}$$

$$\sum_i \varepsilon_i F_i = 0 \tag{6'}$$

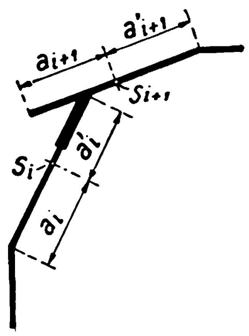



Fig. 2.

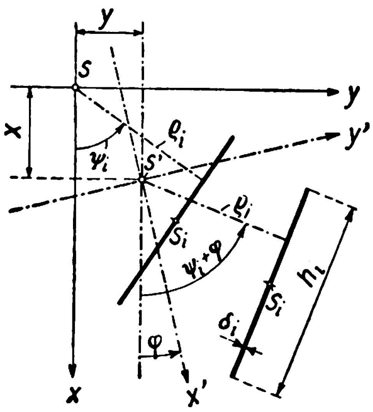



Fig. 3.

If, on the other hand, $\varepsilon_i = \varepsilon_m + \bar{\varepsilon}_i$ is inserted into equation (4), we obtain by taking into consideration the connection (6′) and if we further write

$$J_d = \Sigma J_{di}:$$

$$A_i = \frac{1}{2} \int_0^l \left[ \sum_i (E J_i K_i^2 + E F_i \varepsilon_i^2) + E F \varepsilon_m^2 + G J_d \varphi'^2 \right] dz \tag{7}$$

The n equations (5′) and (6′) enable us to express the n quantities $\varepsilon_i$ by the n quantities $K_i$, with the result that $\varepsilon_i$ appears to be a linear function of $K_i$.

Before further proceeding it is necessary to establish a relationship between the flexures $K_i$ of the various discs and the deformation of the whole bar. We select for our further considerations a left-handed system of coordinates with its z-axis coinciding with the centre of gravity-axis of the bar, and with its x- and y-axes lying in the cross-section considered in such a way that they coincide with its principal axes of inertia. During deformation due to loading the centre of gravity S of the cross-section considered is displaced and comes to the position at S′ see Fig. 3. Let x and y be the components of this displacement. In addition, the cross-section twists, the angle of twisting being denoted here by $\varphi$. The angle $\varphi$ is taken as positive in the direction of the positive x-axis towards the positive y-axis. As we have assumed that the geometrical shape of the cross-section remains unaltered, the three dimensions x, y, $\varphi$ suffice in order to determine the position of any point of the cross-section. Fig. 3 shows the disc "i" with its centre of gravity $S_i$, before and after displacement and twisting. When applying an optional load, the axis of the bar previously straight will become a spatial curve. If the deformations are taken as small, the projections of the curve in the x- or y-planes have the following curvatures: $x'' = \frac{d^2x}{dz^2}$ and $y'' = \frac{d^2y}{dz^2}$. Now, however, the curvature $K_i$ of the disc i is merely the projection of the curvature of the axis of the bar on the distorted plane of the disc. Consequently, in considering also Fig. 3, it follows that:

$$K_i = x'' \sin(\psi_i + \varphi) - y'' \cos(\psi_i + \varphi) - \rho_i \varphi'' \tag{8}$$

This equation offers the possibility of simplification in such cases where $\varphi$ as compared to $\psi$ can be neglected. This is true for the majority of the problems concerning *stable equilibrium.* Equation (8) can then be written in the simplified form:

$$K_i = x'' \sin \psi_i - y'' \cos \psi_i - \rho_i \varphi'' \tag{8'}$$

Inserting the relation (8′) into equation (7) we obtain the following formula for $A_i$:

$$A_i = \frac{E}{2}\int_0^1 \Big[\alpha_{xx} x''^2 + \alpha_{yy} y''^2 + \alpha_{xy} x'' y'' + \alpha_{x\varphi} x'' \varphi'' + \alpha_{y\varphi} y'' \varphi'' + $$
$$+ \alpha_{\varphi\varphi} \varphi''^2 + F \varepsilon_m^2 + \frac{G}{E} J_d \varphi'^2\Big] dz \tag{9}$$

It can now be proved in a general way that:

$$\alpha_{xx} = J_y, \quad \alpha_{yy} = J_x, \quad \alpha_{xy} = J_{xy} = 0,$$

where $J_x$ and $J_y$ represent the moments of inertia of the cross-section of the bar with respect to the principal axes x, y. $J_{xy}$ is the centrifugal moment. Moreover, in cross-sections which show symmetry with regard to the x- or the y-axis respectively, $\alpha_{x\varphi} = 0$ or $\alpha_{y\varphi} = 0$[8] respectively.

*Thus, for problems concerning stable equilibrium, the equation for the internal work done* will read as follows, provided we introduce the final notations:

$$\alpha_{x\varphi} = B_x, \quad \alpha_{y\varphi} = B_y \text{ and } \alpha_{\varphi\varphi} = B_\varphi,$$

$$A_i = \frac{E}{2}\int_0^1 \Big[J_y x''^2 + J_x y''^2 + B_x x'' \varphi'' + B_y y'' \varphi'' + B_\varphi \varphi''^2 + F \varepsilon_m^2 + \frac{G}{E} J_d \varphi'^2\Big] dz \tag{10}$$

Returning to the more precise equation (8) which is to be used in cases of stability problems we may write in this equation, owing to the quantity $\varphi$ being small as supposed:

$$\sin(\psi_i + \varphi) = \sin \psi_i + \varphi \cos \psi_i$$
$$\cos(\psi_i + \varphi) = \cos \psi_i - \varphi \sin \psi_i$$

[8] Space does not permit to prove this statement here. The proof will be published elsewhere.

Thus we obtain for "$K_i$" the expression:

$$K_i = (x'' + y''\,\varphi)\sin\psi_i + (x''\,\varphi - y'')\cos\psi_i - \rho_i\,\varphi'' \qquad (11)$$

The insertion of relation (11) into equation (7) yields a formula for $A_i$, which is similarly constructed as equation (10), with the exception that $x''$ and $y''$ are replaced by $x'' + y''\,\varphi$ and $x''\,\varphi - y''$ respectively. *In cases of stability problems the equation for the internal work done therefore reads as follows:*

$$A_i = \frac{E}{2}\int_0^l\Big[J_y(x''+y''\varphi)^2 + J_x(x''\varphi - y'')^2 + B_x(x''+y''\varphi)\varphi'' + B_y(x''\varphi - y'')\varphi'' + B_\varphi\,\varphi''^2 + F\varepsilon_m^2 + \frac{G}{E}J_d\,\varphi'^2\Big]dz \qquad (12)$$

*Bars with multiply-connected cross-section.*

In the case of a closed cross-section, the shear-stresses set up due to the twisting of the bar — produce bending of the various discs, while with bars of an open cross-section, the bending is related only with the longitudinal stresses. In the case of the open cross-section, the torsional-shearing stresses of the cross-section of the disc flow in opposite directions along the longitudinal edges, whereas in the case of the closed cross-section they flow in the same direction. In the latter case they have the same direction as the bending-shearing stresses. Thus, in the cases here considered, the flexure of a disc in its own plane is made up therefore of two parts, namely, of the curvatures resulting from the longitudinal stresses $\overline{K}_i$ and the curvatures resulting from the torsional-shearing stresses $\overline{\overline{K}}_i$. The shearing force per unit length of the cross-section of the disc, induced by twisting, shall be denoted by $T_i$, Fig. 4.

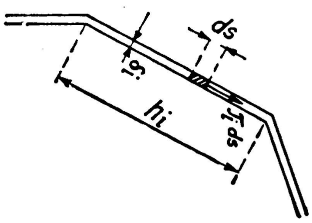



Fig. 4.

The expression for the energy of deformation $A_i$ can now be formulated as follows:

$$A_i = \frac{1}{2}\int_0^l\Big[\sum_i(EJ_i\overline{K}_i^2 + EF_i\varepsilon_i^2) + F\varepsilon_m^2 + \sum_i\frac{1}{G}\frac{h_i}{\delta_i}T_i^2\Big]dz, \qquad (13)$$

In this equation the torsional-resistance of the various discs is neglected due to the fact that it is small as compared to the torsional-resistance of the closed cross-section of the bar, which is represented by the last term in equation (13).

For the purpose of transforming equation (13) further we use once more the transitional relations eq. (5′) and the equations (6′) expressing the equilibrium requirements, which we write once more, this time in the form:

$$\varepsilon_i + a'_i\overline{K}_i = \varepsilon_{i+1} - a_{i+1}\overline{K}_{i+1} \qquad (14)$$

$$\sum\varepsilon_i F_i = 0 \qquad (14')$$

If a cross-section of the bar, consisting of n parts, is connected r-fold (in a simple ring $r = 2$), then there exist $n + r - 2$ equations (14) and one equation (14′), i. e. altogether $n + r - 1$ relations, with n unknown quantities $\varepsilon$. The number of equations is $(r - 1)$ times greater than the number of unknown quantities. If, however, the formulas (14), refering to the corners of a closed sequence of disc are added, the quantities $\varepsilon_i$ will be reduced and we obtain in the case of a r-fold multiply-connected section $r - 1$ relations of the type:

$$\sum_i{}' h_i \overline{K_i} = 0, \tag{15}$$

in which $h_i$ is the height of the disc and in which the dash of the plus-sign shall indicate that the sum is to be taken only with respect to the parts of a closed sequence of discs. These equations are free from the "$\varepsilon_i$" and represent $r - 1$ linear relations of the "$K_i$". The equations (14) are thus seen to be not independent of each other. After elimination of equations (15), which contain no unknown quantities "$\varepsilon_i$" there are remaining altogether n equations, which have a solution only in the case that the requirements expressed by equations (15) are complied with. Therefore, if we are successful to comply with the conditions (15), on which the "$K_i$" depend — and it will finally be shown that this is the case actually — then the equations (14) and (14′) represent a linear system of equations as regards the unknown quantity $\varepsilon_i$, that is free from contradiction. When computing $\varepsilon_i$, one of the equations can be suppressed in each group of equations (14) belonging to a closed discring.

The formula for $A_i$ furthermore contains the n quantities $T_i$. In order to establish the missing relations for the "$T_i$", we will consider a single annular sequence of discs. Assuming this sequence to be interrupted by a cut along an edge in such a way that we now have a bar with singly-connected cross-section, then due to loading of this cut bar, the two edges of the cut will be displaced against one another in the direction of z. However, in the closed sequence of discs the two edges have to fit each other. If the displacement of a point of the disc in the longitudinal direction of the bar (direction of the z-axis) is denoted by $\xi$, then the passage through the multiply-connected cross-section must be represented by the equation:

$$\int_L \frac{\partial \zeta}{\partial s}\, ds = 0 \tag{16}$$

This equations holds for every annular sequences of discs, so that we have at our disposal $r - 1$ conditions (16).

We then write: $\xi = \xi' + \xi''$, where $\xi'$ expresses the displacement resulting from the bending-stresses $\sigma$ and $\zeta''$ the displacement resulting from the shearing-stresses $T_i/\delta_i$. For each disc we may write:

$$\frac{\partial \zeta'}{\partial s} = \int_0^z \overline{K_i}\, dz \qquad \text{and} \qquad \frac{\partial \zeta''}{\partial s} = \frac{T_i}{G\,\delta_i} - \rho_i \varphi',$$

whereby $\rho_i$ denottes the distance of the disc from the centre of gravity-axis of the bar. The insertion into equation (16) yields in the first place:

$$\int\limits_L ds \int\limits_o^z \overline{K_i}\, dz + \int\limits_L \frac{T_i}{G\,\delta_i}\, ds - \int\limits_L \rho_i\, \varphi'\, ds = 0.$$

If in the first term the sequence of integration is interchanged and the integration of the line-integrals is carried out for the discs one after the other, we obtain:

$$\int\limits_o^z \left(\sum_i{}' h_i \overline{K_i}\right) dz + \frac{1}{G}\sum_i{}' \frac{T_i h_i}{\delta_i} - \varphi' \sum{}' \rho_i h_i = 0,$$

The sums in each case refer to an annular sequence of discs. Since in consequence of equation (15) the first term is zero, the following (r — 1) relations remain

$$\frac{1}{G}\sum_i{}' \frac{T_i h_i}{\delta_i} - \varphi' \sum_i{}' \rho_i h_i = 0 \tag{17}$$

To the shearing forces $T_i$, are co-ordinated shearing forces of the same magnitude, acting in the direction of z. Thus for each edge, in which two or more discs join, the equations, representing the equilibrium requirement, read as follows:

$$\sum_i{}'' T_i = 0, \tag{18}$$

In these equations the sum refers only to the shearing forces of the adjacent discs, joining at the edge. We shall have a total of $n - r + 1$ equations (18), so that the total number of equations (17) and (18) just amounts to n. As they are linear, their solution yields the unknown quantities $T_i$ in the simple form

$$T_i = \beta_i\, G\, \varphi' \tag{19}$$

The coefficients $\beta_i$ depend only on the shape of the section.

The actual existence of the formula (15) has still to be proved. For this purpose we determine the curvatures $\overline{\overline{K}}_i$ as set up by the shearing forces, namely

$$\overline{\overline{K}}_i = -\frac{T_i'}{G\,\delta_i}, \quad \text{in which} \quad T_i' = \frac{\partial T_i}{\partial z}.$$

The total curvature is: $K_i = \overline{K}_i + \overline{\overline{K}}_i$, consequently: $\overline{K}_i = K_i - \overline{\overline{K}}_i$. For $K_i$ applies equation (8) and consequently:

$$\overline{K_i} = x'' \sin(\psi_i + \varphi) - y'' \cos(\psi_i + \varphi) - \rho_i\, \varphi'' + \frac{T_i'}{G\,\delta_i} \tag{20}$$

Relation (20) inserted into formula (15) yields:

$$x'' \sum_i{}' h_i \sin(\psi_i + \varphi) - y'' \sum_i{}' h_i \cos(\psi_i + \varphi) - \varphi'' \sum_i{}' \rho_i h_i + \frac{1}{G}\sum_i{}' \frac{T_i' h_i}{\delta_i} = 0.$$

The first two sums are the length of the projection of a closed line on to the x- or y-axis and consequently are zero. The remainder is obtained on basis of equation (17) by differentiation with respect to z. Thus it appears conclusively proved, that equation (15) is identically satisfied.

Inserting finally equation (19) into equation (20), we obtain

$$\overline{K_i} = x'' \sin(\psi_i + \varphi) - y'' \cos(\psi_i + \varphi) + \left(\frac{\beta_i}{\delta_i} - \rho_i\right)\varphi'' \tag{20'}$$

i. e. a formula for closed cross-sections, which corresponds to formula (8).

If $\varphi$ is suppressed as regards $\psi$, equation (20) assumes a simpler form, and then reads as follows:

$$\overline{K_i} = x'' \sin \psi_i - y'' \cos \psi_i + \left(\frac{\beta_i}{\delta_i} - \rho_i\right)\varphi'' \tag{20''}$$

On basis of equations (14) and (14′), the "$\varepsilon_i$" can now be determined as a function of the "$\overline{K}_i$". In this way and on the basis of formula (13), taking furthermore into consideration formula (19) and (20″), the following *equation for the internal work (stable equilibrium)* is obtained:

$$A_i = \frac{E}{2}\int_0^l \left[J_y x''^2 + J_x y''^2 + B_x x'' \varphi'' + B_y y'' \varphi'' + B_\varphi \varphi''^2 + F \varepsilon_m^2 + \frac{G}{E} J_d \varphi'^2\right] dz \tag{21}$$

$$\text{in which} \quad J_d = \sum_i \frac{\beta_i^2 h_i}{\delta_i} \tag{21'}$$

Formula (21) has a structure that is similar to formula (10), with the exception that $J_d$ has a different meaning from that in equation (10). Likewise formula (12), which has to be considered in stability problems, has also to be taken into account in the cases of closed bars, provided $J_d$ is defined by formula (21′).

If within a disc $\delta_i$ is variable, $\delta_i$ must be replaced in all the formulae of the preceding investigation on bars of closed cross-section by the average $\delta_{im} = \frac{F_i}{h_i}$.
A more exact consideration of the variableness of $\delta_i$ presents no difficulties of fundamental nature, but a more precise computation would lead to cumbersome formulae.

## 3. Determination of the cross-sectional functions $B_x$, $B_y$ and $B_\varphi$.

The calculation of the cross-sectional functions $B_x$, $B_y$ and $B_\varphi$ is given by way of example in connection with a symmetrical ⊓-cross-section. The x-axis is selected as the axis of symmetry. The three plates of this cross-section are denoted by 1, 2, 3, see Fig. 5. We will assume that the discs 2 and 3 are of variable thickness. In accordance with this the distance "a" is: $a \neq h/2$.

Starting with equations (5′) and (6′), we shall find that in the case under consideration, beginning with disc 2 and proceeding in the positive counting-direction from $x \longrightarrow y$, these equations will read as follows:

$$\left.\begin{aligned} \varepsilon_2 + aK_2 &= \varepsilon_1 - bK_1 \\ \varepsilon_1 + bK_1 &= \varepsilon_3 - aK_3 \\ F_1\varepsilon_1 + F_2(\varepsilon_2 + \varepsilon_3) &= 0 \end{aligned}\right\} \tag{a}$$

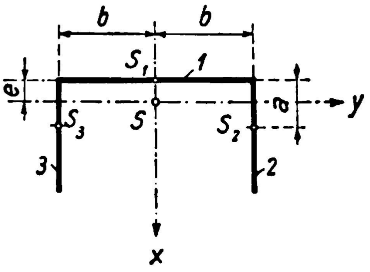



Fig. 5.

If $\varepsilon_2$ and $\varepsilon_3$ are expressed by $\varepsilon_1$, the following relations are obtained:

$$\varepsilon_2 = \varepsilon_1 - bK_1 - aK_2\,, \quad \varepsilon_3 = \varepsilon_1 + bK_1 + aK_3 \tag{b}$$

Inserting this into the third equation of (a) we obtain with $F = F_1 + F_2 + F_3$

$$\varepsilon_1 = \frac{F_2 a}{F}(K_2 - K_3), \tag{c}$$

Due to this, the "$\varepsilon_2$" and "$\varepsilon_3$" are also determined as functions of the "$K_i$".

Now we proceed in computing the sum $\Sigma F_i \varepsilon_i^2$ of equation (7):

$$\sum_1^3 F_i \varepsilon_i^2 = \frac{3}{F}[F_2 a(K_2 - K_3)]^2 + F_2[2b^2K_1^2 + 2abK_1(K_2+K_3) + a^2(K_2^2+K_3^2)] \tag{d}$$

The curvatures $K_i$ have to be determined on basis of equation (8′), whereby the following numerical values are to be introduced for the angle $\psi_i$:

$$\psi_1 = \pi, \qquad \psi_2 = \frac{\pi}{2}, \qquad \psi_3 = \frac{3\pi}{2}.$$

With these $\psi_i$-values we obtain with due consideration of Fig. 5:

$$K_1 = y'' - e\varphi'' \qquad K_2 = x'' - b\varphi'' \qquad K_3 = -x'' - b\varphi''.$$

In this manner the sum:

$$\sum_1^3 J_i K_i^2 = J_1(y'' - 2ey''\varphi'' + e^2\varphi''^2) + 2J_2(x''^2 + b^2\varphi''^2) \tag{e}$$

and by using the formula (d) derived above, the sum:

$$\sum_1^3 F_i \varepsilon_i^2 = -\frac{4}{F}F_2^2 a^2 x''^2 + F_2[2b^2(y''^2 - 2ey''\varphi'' + e^2\varphi''^2) - 4ab^2(y''\varphi'' - e\varphi''^2)$$
$$+ a^2(2x''^2 + 2b^2\varphi''^2)]. \tag{f}$$

is obtained.

If the coefficients of the products $x''\varphi''$ and $y''\varphi''$ respectively, which were expressed above by $B_x$ and $B_y$, are extracted from the sums (e) and (f), we find:

$$\left.\begin{aligned} B_x &= 0 \\ B_y &= -2J_1 e - 4F_2 b^2(a+e) = -2e(J_x + Fb^2) \end{aligned}\right\} \tag{22}$$

$J_x$ and $J_y$ denote the principal moments of inertia of the ⊓-cross-section with respect to the x-and y-axis respectively.

If all coefficients of $\varphi''^2$ are collectively considered, we obtain in a similar manner:

$$B_\varphi = J_1\, e^2 + 2\, J_2\, b^2 + 2\, b^2\, F_2\, (a + e)^2 = J_x\, e^2 + J_y\, b^2 + 3\, F\, b^2\, e^2. \tag{23}$$

Below are placed together the cross-sectional functions $B_x$, $B_y$ and $B_\varphi$ for a few additional cross-sectional shapes with one axis of symmetry, frequently encountered in steel structures, which have been determined as outlined above.

*Symmetrical cross-section (Fig. 6).*

$$\left.\begin{aligned} B_x &= 0 \\ B_y &= 2\,\{-J_1 e + 2\,J_4 (h - e) - 2\,F_2\, b^2 (a + e) \\ &\qquad - 2\,F_4 (b - c)\,[c\,(h - e) + b\,(h + e)]\} \\ B_\varphi &= J_1 e^2 + 2\,J_2\, b^2 + 2\,J_4 (h - e)^2 + 2\, b^2\, F_2 (a + e)^2 \\ &\qquad + 2\,F_4\,[c\,(h - e) + b\,(h + e)]^2 \end{aligned}\right\} \tag{24}$$

$J_1$, $J_2$ and $J_4$ are the moments of inertia of the discs 1, 2, and 4, with respect to the centre of gravity-axis, the latter being at right angle to their plane — and $F_1$, $F_2$, $F_4$ are the cross-sectional areas of the discs under consideration[1].

*Unsymmetrical I-cross-section (Fig. 7).*

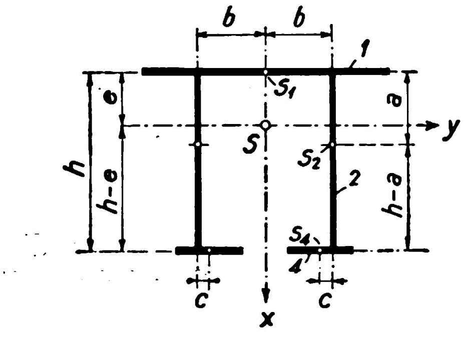



Fig. 6.

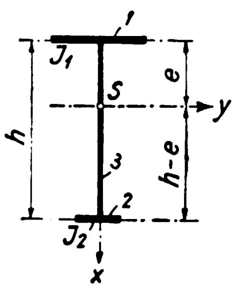



Fig. 7.

$$\left.\begin{aligned} B_x &= 0 \\ B_\varphi &= J_1 e^2 + J_2 (h - e)^2. \\ B_y &= -2\;[J_1 e - J_2 (h - e)] \end{aligned}\right\} \tag{25}$$

$J_1$ and $J_2$ are the moments of inertia of the flanges 1 and 2 respectively with respect to the x-axis.

*T-cross-section (Fig. 8).*

$$\left.\begin{aligned} B_x &= 0 \\ B_y &= -2 J_1 e \\ B_\varphi &= J_1 e^2. \end{aligned}\right\} \tag{26}$$

$J_1$ is the moment of inertia of disc 1, with respect to the x-axis.

---

9 If the centre of gravity of the two lower flanges is outside of the webs, then c must be introduced with the negative sign.

*Isosceles angle (Fig. 9).*

$$\left.\begin{aligned} B_x &= 0 \\ B_y &= 2\sqrt{2}\, e\, J_x \\ B_\varphi &= 2\, e^2 \left(J_x - \frac{F\, b^2}{16}\right) \end{aligned}\right\} \tag{27}$$

In this case $J_x$ is the moment of inertia with respect to the x-axis: F is the cross-sectional area of the angle.

The general developments, on the basis of which the above computations were based, were deduced on the strict assumption of the plate-thickness being

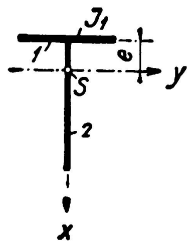


Fig. 8.

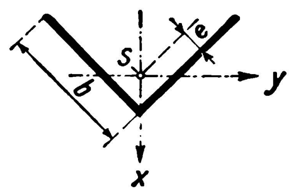


Fig. 9.

variable. The equations (24) to (27) given above are therefore also applicable to riveted girder cross-sections of the usual type, in which the various walls are reinforced in places by addition of corner-irons or other iron straps. Consequently, the distances from the centre of gravity $a_i$, the moments of inertia $J_i$ and the surface $F_i$ must be calculated on basis of the reinforced cross-sections of the discs. The distances a, b, c etc. always refer to the central plane of the discs. The principal moments of inertia $J_x$ and $J_y$ must be determined in the usual manner.

*Multiply-connected cross-section according to Fig. 10.*

In this figure, the projecting parts must be considered as separate discs, because, from the view-point of taking up shearing stresses, they behave dif-

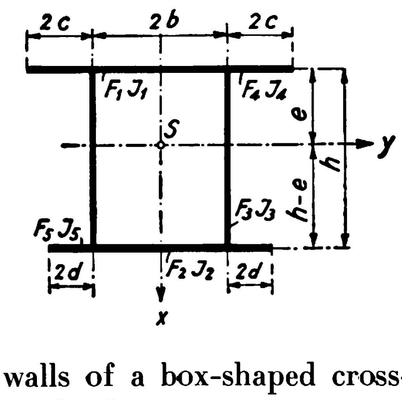


Fig. 10.

ferently from the four walls of a box-shaped cross-section of a closed sequence of discs. In the further developments we assume symmetry with respect to the x-axis.

In order to ascertain "$J_d$" in the expression for the work $A_i$ of equation (21), the shearing forces $J_i$ must be determined first. These shearing forces act only within the closed rectangular sequence of discs, made up of the four walls 1, 2, 3, 4. In order to maintain equilibrium with regard to the longitudinal shearing — forces along the four edges of the ring, the shearing forces in the four discs have to be equivalent within themselves. Thus only one unknown quantity has to be determined, which is given — on the basis of equation (17) — as a function of the angle of twist $\varphi$. Hence:

$$\frac{T}{G}\left[\frac{b}{\delta_1}+\frac{b}{\delta_2}+\frac{2h}{\delta_3}\right]-2\,hb\varphi'=0$$

whence

$$T=\beta G\varphi' \qquad \text{with} \qquad \beta=\frac{2\,hb}{\dfrac{b}{\delta_1}+\dfrac{b}{\delta_2}+\dfrac{2h}{\delta_3}} \tag{28}$$

According to formula (21) we thus obtain:

$$J_d=\frac{4\,h^2\,b^2}{\dfrac{b}{\delta_1}+\dfrac{b}{\delta_2}+\dfrac{2h}{\delta_3}}\ ^{10} \tag{29}$$

The computation of the values $B_x$, $B_y$ and $B_\varphi$ is carried out in a similar manner as in the case of the open cross-sections. As the expression for the curvature $\overline{K}_i$ also contains the shearing-forces $T_i$ as a result of bending, terms relating to the shearing-forces T are found in the cross-sectional functions B. With the notations of Fig. 10, the following expressions are obtained:

$$\left.\begin{aligned}
B_x &= 0\\
B_y &= 2\Big[-(J_1+2\,J_4)\,e+(J_2+2\,J_5)\,(h-e)+\frac{F_3}{4}\,b^2\,(h-2e)\\
&\qquad -\frac{F_4}{2}\,(b+c)^2 e+\frac{F_5}{2}\,(b+d)^2\,(h-e)\Big]\\
B_\varphi &= (J_1+2\,J_4)\,e^2+(J_2+2\,J_5)\,(h-e)^2+2\,J_3\left(\frac{\beta}{\delta_3}-\frac{b}{2}\right)^2\\
&\qquad +\frac{F_3}{8}\,b^2\,(h-2e)^2+\frac{F_4}{2}\,(b+c)^2\,e^2+\frac{F_5}{2}\,(b+d)^2\,(h-e)^2
\end{aligned}\right\} \tag{30}$$

Attention is due to the fact, that $F_1$, $J_1$ and $F_2$, $J_2$ are the surfaces and moments of inertia of the discs 1 and 2 respectively, having the height 2b; $F_4$, $J_4$ and $F_5$, $J_5$ are the surfaces and moments of inertia of the projecting discs 4 and 5 respectively which latter have a height of 2c and 2d respectively. In the case of riveted girders the variability of the cross-section within the different discs must be taken into account in such a manner that an average value of $\delta_{im}=\dfrac{F_i}{h_i}$ is inserted for "$\delta_i$" in the expressions for $\beta_1$, $J_d$ and $B_\varphi$. Furthermore, all that has been said above about bars with open cross-sections with regard to the values $F_i$, $J_i$ and $a_i$ also holds here.

[10] Formula (28) agrees with the know formula of the theory of elasticity for the rectangular ring. See *A.* und *L. Föppl:* Drang und Zwang, Vol. II, 2nd Edition, Munich and Berlin. 1925.

4) Differential equations for bending and torsion.

Let $p_x$ and $p_y$ be the continuous or discontinuous external loading in the direction of the principal axes x and y respectively and $M_d$ the torque moment of the outer forces whereby $p_x$, $p_y$ and $M_d$ shall be considered as functions of the coordinate z, then the expression for the outer work $A_a$ reads as follows:

$$A_a = \frac{1}{2} \int_0^l [p_x x + p_y y - M_d \varphi'] \, dz \tag{31}$$

$A_a$ is a linear function of the displacements x, y, $\varphi$.

The equation representing the equilibrium requirement: $\delta\,(A_i - 2A_a)$, eq. (3), reads as follows, if we introduce $A_i$ from eq. (10):

$$\left.\begin{aligned} \delta \int_0^l \Big[ & \frac{E}{2} (J_y x''^2 + J_x y''^2 + B_x x'' \varphi'' + B_y y'' \varphi'' + B_\varphi \varphi''^2) + \frac{G}{2} J_d \varphi'^2 \\ & - p_x x - p_y y + M_d \varphi' \Big] dz, \end{aligned}\right\} \tag{32}$$

whereby we assume that the longitudinal force $S = 0$, and in consequence $\varepsilon_m = 0$. The execution of the variation results in the following three simultaneous differential equations

$$\left.\begin{aligned} & E J_y \frac{d^4 x}{dz^4} + \frac{E B_x}{2} \frac{d^4 \varphi}{dz^4} - p_x = 0 \\ & E J_x \frac{d^4 y}{dz^4} + \frac{E B_y}{2} \frac{d^4 \varphi}{dz^4} - p_y = 0 \\ & \frac{E B_x}{2} \frac{d^4 x}{dz^4} + \frac{E B_y}{2} \frac{d^4 y}{dz^4} + E B_\varphi \frac{d^4 \varphi}{dz^4} - G J_d \frac{d^2 \varphi}{dz^2} - \frac{d M_d}{dz} = 0 \end{aligned}\right\} \tag{33}$$

The eq. (33) can be considerably simplified by a transformation of the coordinates. If the transformation:

$$x = \bar{x} - \frac{B_x}{2 J_y} \varphi, \qquad y = \bar{y} - \frac{B_y}{2 J_x} \varphi$$

is carried out, that is, if the z-axis is removed parallel to the centre of gravity-axis by the amounts:

$$s_x = \frac{B_y}{2 J_x}, \qquad s_y = -\frac{B_x}{2 J_y} \tag{34}$$

and if the torsional moment of the outer forces with respect to the new axis is denoted by $\overline{M}_d$, then the equations (33) will read as follows:

$$\left.\begin{aligned} & E J_y \frac{d^4 \bar{x}}{dz^4} - p_x = 0 \\ & E J_x \frac{d^4 \bar{y}}{dz^4} - p_y = 0 \\ & E \left[ B_\varphi - \frac{B_x^2}{4 J_y} - \frac{B_y^2}{4 J_x} \right] \frac{d^4 \varphi}{dz^4} - G J_d \frac{d^2 \varphi}{dz^2} - \frac{d \overline{M}_d}{dz} = 0 \end{aligned}\right\} \tag{35}$$

The two first equations are independent of $\varphi$ and in a formal way they are identical with the differential equations of *Navier*'s theory. The third equation is independent of x and y. The principal axis-transformation as has been carried out thus offers in all cases the possibility of separate treatment of bending and twisting and it makes possible to obtain a clear conception of how deformation proceeds and of the distribution of stress.

From the last of the equations (35) it becomes obvious that the twist $\varphi$ becomes zero for $\overline{M}_d = 0$. If on the other hand $p_x = p_y = 0$, then $\bar{x}$ and $\bar{y}$ disappear, the axis of the bar remains straight, the whole bar will twist itself round the angle $\varphi$. Hereby is, according to formula (34), the axis with the displacement coordinates $e_x$ and $e_y$ just the axis of twist of the bar. We shall call it the *axis of torque*. Its point of intersection with the cross-section is called *shearing-centre or lateral force-centre*. It was determined for the first time for the [-iron. Its fundamental significance for the determination of the shapes of bars under consideration, however, did not become obvious, until the transformation described above was effected.

The equations (35) enable us to perceive the following general behaviour of slender bars made of thin sheets: If the outside forces pass through the axis of torque which, as a rule, does not coincide with the centre of gravity-axis, bending stress alone, and no torsional stress will be obtained. The determination of the deformations and stresses is effected on the basis of the two first equations (35), that is in such a manner, as is done according to *Navier*'s theory for loads, which pass through the centre of gravity-axis. In all other cases of loading, both kind of stresses — bending and torsional — are set up, even in cases where $p_x$ and $p_y$ disappear. If the cross-section has an axis of symmetry, then the shearing-centre will be located on this axis. With two symmetrical axes, the axis of torque coincides with the centre of gravity-axis.

The equations (34) and (35) apply both to bars with open and closed cross-section. Only when determining the cross-sectional functions $J_d$, $B_x$, $B_y$, $B_\varphi$ attention must be paid to whether the section is singly- or multiply-connected.

For ordinary applications the equations (35) can be converted by integration to

$$\left.\begin{aligned} &E J_y \frac{d^2 x}{dz^2} + M_y = 0 \\ &E J_x \frac{d^2 y}{dz^2} + M_x = 0 \\ &E\left[B_\varphi - \frac{B_x^2}{4 J_y} - \frac{B_y^2}{4 J_x}\right]\frac{d^3\varphi}{dz^3} - G J_d \frac{d\varphi}{dz} - \overline{M}_d = 0 \end{aligned}\right\} \tag{36}$$

$M_x$ and $M_y$ are the moments of the outer forces in the planes zy and zx respectively.

Hence, $\varphi$ is defined by the differential equation of third order:

$$\frac{d^3\varphi}{dz^3} - \alpha^2 \frac{d\varphi}{dz} - \frac{M_d}{\gamma} = 0 \tag{37}$$

in which $$\alpha^2 = \frac{G J_d}{\gamma} \quad \text{and} \quad \gamma = E\left[B_\varphi - \frac{B_x^2}{4 J_y} - \frac{B_y^2}{4 J_x}\right] \tag{37'}$$

The solution of the equation for torsion (37), which is dependent on the state of loading and on the conditions of the supports, depends only on the shape of the bar through the intermediary of the parameters $\alpha$ and $\beta$, which enter into the solution as fixed quantities. The solutions of equation (37) can thus be established independently of the shape of the bar, just as is the case for the solutions of the *Navier* bending formulas. Formula (37) represents a generalisation of the differential equation of torque of symmetrical I-girders as derived by *Timoshenko.*

In connection with the above general discussions, a few remarks may be added regarding the essential assumption made in the early part of this paper concerning the unalterableness of the shape of the cross-section. If three discs are joined to form one bar, it is obvious that each of these discs may be subjected to slight deformations in its plane, without destroying the sequence between the discs and without changing the angles formed each time by two consecutive discs. In the case of more than three, or generally spoken n-discs, n-3 angles change, provided that to the various discs deformations in their own plane are prescribed. On basis of this consideration it follows that in the case of $\perp$, $\llcorner$, [, and ⌐$\llcorner$ shaped bars, the condition of conservation of the cross-sectional shape is automatically complied with, as the deformations to which the individual discs are forced due to loading, do not show any constraint, which would result in a change of the shape of the cross-section. However, in the case of bars made up of four or more discs — regardless whether the section is singly-or multiply-connected — care must be taken to ensure by favourable means that the shape of the section remains unchanged. As a rule this can be attained by inserting transverse straps at fairly large intervals. Judged from a practical view-point they are evidently necessary only at those places where the loading, accompanied by considerable twisting-effects, induces very marked changes of the angle of twist between the different discs.

If two walls of a bar with open cross-section are connected together by means of a strut, as shown by way of example in Fig. 11, such a bar can be considered

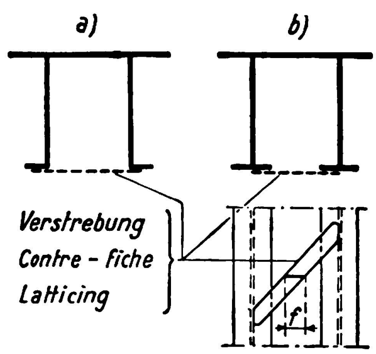



Fig. 11.

as a closed bar. When determining the B-terms according to the equation (30), the values $J_2 = 0$, $F_2 = 0$ must be inserted for a lattice-girder disc such as shown in Fig. 11a. In the case Fig. 11b, the substitute moment of inertia of the lattice girder disc, situated between the two webs, must be introduced for $J_2$, while for $F_2$ the area of the strip-plates, which are located between the two webs has to be inserted. In computing the shearing-force T on basis of formula (28) and $J_d$ on basis of formula (29) $\delta_2$ must be taken equal to the ideal plate thickness, which is obtained by dividing the cross-sectional area f of the strut (Fig. 11b) by the whole height 2b of the disc.

5) Stress-determination in the case of bending and twisting.

The edge-stresses $\sigma$ set up by the moments $M_x$ and $M_y$ are computed in the usual manner as though the loads were passing through the centre of gravity-axis of the bar. If $\overline{M}_d$ differs from zero, which is the case, when the loads do not pass through the axis of torque, then the third of the eq. (35) or eq. (37) respectively must be integradet after insertion of $M_d$ as a function of z, taking into consideration the edge-conditions of the problem. If $\varphi$ is known as a function of z, then the additional edge-stresses and shearing-loads induced by twist can be calculated. For this purpose the curvature of the various discs of the bar has to be computed on basis of formula (8′) or (20″) respectively, x and y being the displacements to which the centre of gravity-axis is subjected when the bar is twisted round the axis of torsion by the angle $\varphi$. Thus the displacements $x = S_y \varphi$ and $y = s_x \varphi$ must be inserted into the formula (8′) or (20″) respectively with the result that we obtain equation:

$$K_i = \left[s_y \sin \psi_i + s_x \cos \psi_i + \left(\frac{\beta_i}{\delta_i} - \rho_i\right)\right] \varphi'' \tag{38}$$

of general validity. For bars with an open cross-section $\beta_i = 0$.

The edge-stresses $\sigma_i$ in the disc i are determined by the relations:

$$\sigma_i = E\,(\varepsilon_i + \mu K_i) \tag{39}$$

in which $\eta$ denotes the distance of the edge-points from the centre of gravity of the disc. Hereby $\eta$ is taken as positive in the positive counting-direction from $x \longrightarrow y$.

If the lengths e, $s_x$, b etc. are numerically inserted into formula (38), expressions of the form $K_i = \mu_i \varphi''$ will be obtained for the "$K_i$". This serves for the calculation of the $\varepsilon_i$-values on the basis of eq. (5′) which also appear in the form

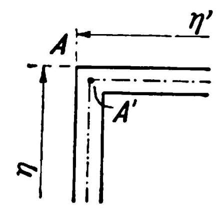



Fig. 12.

$\varepsilon_i = v_i \varphi''$. Consequently, all the numerical values are then known, so that on basis of eq. (39) the longitudinal stresses $\sigma_i$ can be computed for the edge-points under consideration. In this connection it should be noted that while all the dimensions used so far for the calculations, such as the quantities a, b, e, $s_x$ etc. refer to the central planes of the discs, the "$\eta$" refer to the end or outer surfaces of the discs. For a corner, such as shown for example in Fig. 12, the determination of the stress has to be effected by considering the distances $\eta$ or $\eta'$, according to whether the stress in the corner point A is computed as belonging to the vertical or to the horizontal disc respectively. The two stresses determined in this way, naturally enough, do not coincide entirely, as the above solution of the problem before us was obtained on basis of the assumption of infinitely thin discs. The computation would result in a complete coincidation only for the point of intersection A'.

The longitudinal stresses $\sigma$ due to twisting are finally to be superimposed upon the stresses $\sigma$ set up by the moments $M_x$ and $M_y$, which are to be determined in the usual manner.

The torsional shearing-stresses in bars of *open-cross-section* have to be computed for each disc on basis of the formula:

$$\tau_i = G\,\varphi'\max\delta_i \tag{40}$$

in which max $\delta_i$ is the *largest thickness* of the disc considered. For bars of *closed cross-section* the decisive shearing-stresses resulting from $T_i$ are obtained on basis of the formula:

$$\tau_i = \frac{T_i}{\min\delta_i} \tag{41}$$

In this case min $\delta_i$ is the smallest thickness of the disc considered.

In order to prove the correctness of the theory as developed above, the stress determinations which *Bach* has carried out with a [ NP 30, were compared with the stresses as obtained on basis of the equations, derived in this paper. The girder tested had a span of 3 m and was vertically loaded with 1,500 kg at

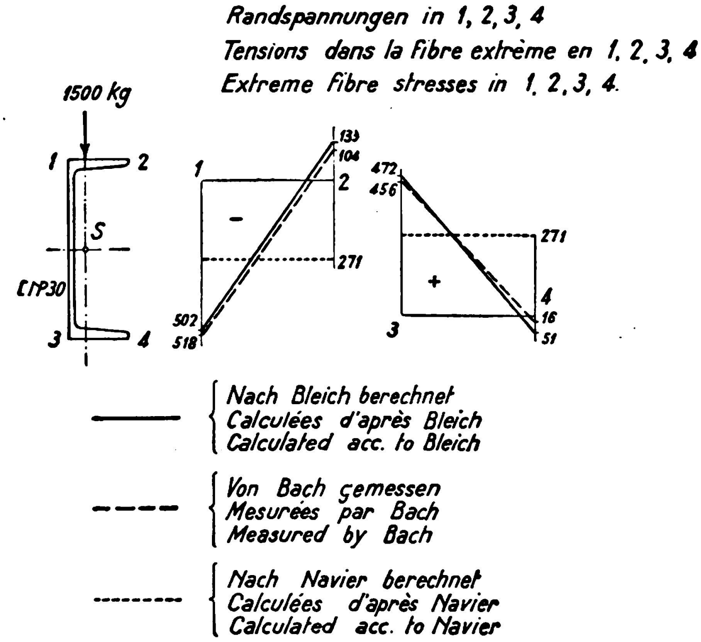



Fig. 13.

three equidistant points as shown by the transverse-section Fig. 13. The magnitude and the distribution of the longitudinal stresses of the upper and lower flange of the [-steel are shown in the same figure. The agreement between the measured and the computed stresses is satisfactory. For the purpose of comparison the uniform bending stress of 271 kg/cm², calculated in the usual manner, is also shown in the stress-diagram. Attention is drawn to the fact that points 1 and 4, in which measurements were made, had a distance of 5 mm from the outer surfaces of the two flanges. In determining the stresses, this position of the points was considered[11].

6. The differential equations for the buckling-problem.

Let the straight bar be loaded with a longitudinal force S acting along the centre of gravity-axis. Taking into consideration, that due to "S" the bar is first compressed to the extent of "$\varepsilon_m$" and that it then, after reaching the limit of stability, starts to buckle, the formula for the external work $A_a$ assumes the following familiar expression:

$$A_a = \int_0^l \left[\frac{1}{2} S \varepsilon_m + \frac{1}{2} S (x'^2 + y'^2)\right] dz$$

In the cases of certain simple buckling problems, which we will consider here in this part of the paper, it will suffice, to refer to eq. (10) for the expression of the internal work, in which the small quantities of deformation of higher order are suppressed.

If due consideration is given to the fact that the term $\frac{1}{2} S \varepsilon_m$ in $A_a$ can be replaced by $\frac{E}{2} F \varepsilon_m^2$ on account of $S = E F \varepsilon_m$ the variation-requirement (3) reads as follows:

$$\delta \int_0^l \left[\frac{E}{2}\left(J_y x''^2 + J_x y''^2 + B_x x'' \varphi'' + B_y y'' \varphi'' + B_\varphi \varphi''^2 + \frac{G}{E} J_d \varphi'^2\right) - \frac{1}{2} S (x'^2 + y'^2)\right] dz = 0.$$

The carrying out of the variation yields the following three differential equations:

$$\left.\begin{aligned} E J_y \frac{d^4 x}{dz^4} + \frac{E B_x}{2} \frac{d^4 \varphi}{dz^4} + S \frac{d^2 x}{dz^2} &= 0, \\ E J_x \frac{d^4 y}{dz^4} + \frac{E B_y}{2} \frac{d^4 \varphi}{dz^4} + S \frac{d^2 y}{dz^2} &= 0, \\ \frac{E B_x}{2} \frac{d^4 x}{dz^4} + \frac{E B_y}{2} \frac{d^4 y}{dz^4} + E B_\varphi \frac{d^2 \varphi}{dz^4} - G J_d \frac{d^2 \varphi}{dz^2} &= 0, \end{aligned}\right\} \quad (42)$$

which, after a double integration, assume the form:

$$\left.\begin{aligned} E J_y \frac{d^2 x}{dz^2} + \frac{E B_x}{2} \frac{d^2 \varphi}{dz^2} + S x &= 0, \\ E J_x \frac{d^2 y}{dz^2} + \frac{E B_y}{2} \frac{d^2 \varphi}{dz^2} + S y &= 0, \\ \frac{E B_x}{2} \frac{d^2 x}{dz^2} + \frac{E B_y}{2} \frac{d^2 y}{dz^2} + E B_\varphi \frac{d^2 \varphi}{dz^2} - G J_d \varphi &= 0 \end{aligned}\right\} \quad (43)$$

[11] Bach's experiment has already been verified by *C. Weber* on basis of the formulae which he derived for two-flanged girders.

Having these three simultaneous differential equations on hand, the buckling problem of thin-walled bars can be considered as solved in a general way. The eq. (43) enable us to perceive the following facts: In the cases where the cross-section of the bar has no axes of symmetry, i. e. in the case both $B_x$ and $B_y$ being different from zero, the bar will be subjected to twist, because $\varphi$ does not disappear. If the cross-section has one axis of symmetry, then the bar — according to the conditions of stiffness-will buckle either in the direction of the axis of symmetry or vertically to it. It is in the latter case that the bar also twists.

In order to obtain further information regarding the behaviour of such bars, we will examine a bar having the length 1, whose cross-section shows symmetry with respect to the x-axis, and which is so fastened at the ends by pins that buckling will take place in the direction of the y-axis and which is furthermore so consolidated also at the ends that there no twisting round the z-axis can take place.

With the trial-solution:

$$y = C_1 \sin \frac{n\pi z}{l}, \quad \varphi = C_2 \sin \frac{n\pi z}{l} \qquad (n = 1, 2, 3 \ldots)$$

we obtain, by way of insertion into the second and third eq. (43), the requirement for buckling:

$$\begin{vmatrix} \left(-\frac{n^2\pi^2}{l^2} EJ_x + S\right) & -\frac{n^2\pi^2}{l^2}\frac{EB_y}{2} \\ -\frac{n^2\pi^2}{l^2}\frac{EB_y}{2} & \left(-\frac{n^2\pi^2}{l^2} EB_\varphi + GJ_d\right) \end{vmatrix} = 0.$$

After computation of the determinant and insertion of the abbreviation $S_E = \frac{n^2\pi^2 EJ_x}{l^2}$, the following formula is obtained for the buckling load $S_k$:

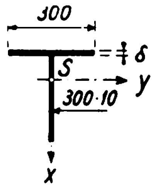
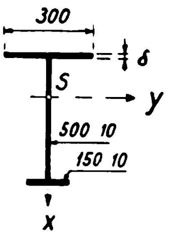
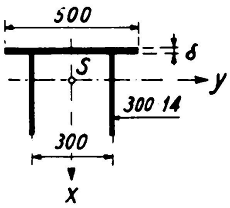
$$S_K = S_E\left(1 - \frac{B_y^2}{4 J_x\left(B_\varphi + \frac{J_x G J_d}{S_E}\right)}\right) \tag{44}$$

The smallest value is obtained by taking $n = 1$, corresponding to $S_E = \frac{\pi^2 EJ_x}{l^2}$. *The actual carrying capacity as regards buckling at right angle to the axis of symmetry is therefore less than the carrying capacity value as computed on basis of the Euler formula.* In determining the magnitude of the second term within the parentheses it will be observed that for certain cross-sectional shapes, such as T-and unsymmetrical I-sections, considerable diminutions of the carrying capacity — as compared with the values determined in the usual manner exist[12].

In the following table $S_k$ as computed according to formula (44) is given for several shapes. The figures were calculated on the assumption that the

[12] This diminution of carrying capacity has already been observed by Ostenfeld. See reference 5) for title of paper, p. 17).

buckling-stress $\sigma_k$, determined on basis of the Euler formula just amounts to 2000 kg/cm². With increasing slenderness of the bar the "$S_k$" approaches *Euler*'s carrying capacity value. The influence of torsional resistance of the cross-section of the bar is very considerable. As this increases, so the carrying capacity $S_k$ increases. Its influence increases with the increase of the degree of slenderness. In the case of closed cross-sections the difference between $S_k$ and $S_E$ is insignificant, owing to the great torsional resistance.

| Shape of cross-section | Ratio of carrying capacity $S_K$ in the case buckling in direction y to Euler's carrying capacity value $S_E$ | | | |
|---|---|---|---|---|
|  | $\delta^{mm} =$ | 10 | 20 | 30 |
| | $\frac{S_K}{S_E} =$ | 0.699 | 0.930 | 0.974 |
|  | $\delta =$ | 10 | 20 | 30 |
| | $\frac{S_K}{S_E} =$ | 0.691 | 0.724 | 0.883 |
|  | $\delta =$ | 10 | 20 | 40 |
| | $\frac{S_K}{S_E} =$ | 0.534 | 0.703 | 0.942 |

If the load is drawn off the centre of gravity of the bar and displaced towards the centre of the shearing-stress, which for instance in the case of a **T**-section is situated at the point of intersection of the two discs, then $S_k$ increases steadily — as we shall prove in the next paragraph — and it finally reaches the maximum value $S_E$, whereby the load just acts in the axis of torque. As regards lateral buckling, the centre of gravity-axis and the axis of torque have interchanged their parts, at least as far as the here considered shapes of bars are concerned. According to these considerations, tall-webbed **⊥**-steels, joined on one side to a flange, therefore probably will possess a greater carrying capacity as computed on the basis of the usual theory for the case of excentric point of load application.

It shall be emphasised, that the preceding considerations and conclusions apply only to elastic buckling.

## 7) Differential equations for the Tilting problem.

In the case of tipping or tilting we conceive of the beginning of a state of unstable equilibrium of a bar subjected to bending, which is characterised by the fact that, when this unstable equilibrium becomes apparent, the bar not only yields laterally, i. e. perpendicular to the plane of flexure, but at the same time also twists. In order to simplify somewhat our presentation we will consider only bars of simple symmetrical section which are loaded in the xz-plane. The bar will yield in the direction of the y-axis in the case that the magnitude of $J_x$ is substantally less than $J_y$, as, is the case, for instance, when an I-girder is loaded in the plane of its web.

Let $p_x$ be the load acting in the x z-plane, $M_y$ the bending moment induced by this load, S an external longitudinal force acting along the axis of the bar, a the height of the point of application of the load p above the cross-sectional centre of gravity, see Fig. 14, whereby "a" also may be negative. The total work done by the external forces, i. e. the work done in both cases of stable and unstable state of equilibrium, is expressed by the following equation:

$$A_a = \frac{1}{2}\int\left[\frac{M_y^2}{EJ_y} + S\varepsilon_m + Sy'^2 + ap_x\varphi^2\right]dz$$

With regard to the internal work $A_i$ it is necessary to go farther back to the formula (12), as the influence of the deformation-terms of higher order can

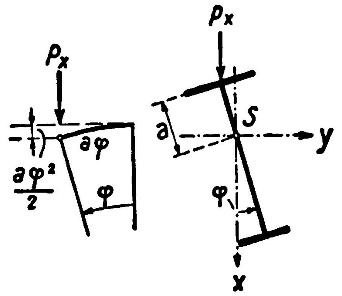



Fig. 14.

no longer be neglected. In accordance herewith and if due attention is given to the fact that on account of the assumed symmetry $B_x = 0$, the expression for the variation-requirement (3) then reads as follows:

$$\delta\frac{1}{2}\int_0^l\Big[E\Big(J_y(x''+y''\varphi)^2 + J_x(x''\varphi - y'')^2 + B_y(x''\varphi - y'')\varphi'' + B_\varphi\varphi''^2 + F\varepsilon_m^2$$
$$+\frac{G}{E}J_d\varphi'^2\Big) - \Big(\frac{M_y^2}{EJ_y} + S\varepsilon_m + Sy'^2 + ap_x\varphi^2\Big)\Big]dz \qquad (45)$$

When tilting takes place, the changes in the quantities x and $\varepsilon_m$ are small as compared with the displacements y and $\varphi$ which are induced only when tilting starts. For this reason we further consider the quantities x and $\varepsilon_m$ as being invariable for the variation; they retain their values of $x'' = -\frac{M_y}{EJ_y}$ and $\varepsilon_m = \frac{S}{EF}$ — which they had attained just shortly before the unstable state was reached — while the tilting state lasts. If these values for x and $\varepsilon_m$ are inserted into formula (44), the following conditional equation will be obtained, in which, however, all terms — small of a higher order than the second — were suppressed:

$$\delta \frac{1}{2}\int_0^l \Big[EJ_x y''^2 - EB_y y''\varphi'' + EB_\varphi \varphi''^2 + GJ_d \varphi'^2 - 2\left(1 - \frac{J_x}{J_y}\right) M_y y''\varphi - a p \varphi^2 - S y'^2\Big] dz.$$

If the variation with respect to y and $\varphi$ is carried out the two following simultaneous differential equations will be obtained:

$$\left.\begin{aligned} & EJ_x \frac{d^4 y}{dz^4} - \frac{EB_y}{2}\frac{d^4\varphi}{dz^4} - \left(1 - \frac{J_x}{J_y}\right)\frac{d^2}{dz^2}(M_y\varphi) + S\frac{d^2 y}{dz^2} = 0, \\ & -\frac{EB_y}{2}\frac{d^4 y}{dz^4} + EB_\varphi \frac{d^4\varphi}{dz^4} - \left(1 - \frac{J_x}{J_y}\right) M_y \frac{d^2 y}{dz^2} - GJ_d \frac{d^2\varphi}{dz^2} - a p_x \varphi = 0, \end{aligned}\right\} \quad (46)$$

on basis of which the general tilting problem for bars having at least one axis of symmetry may be considered as solved[13] in principle.

When dealing with the simplest special case, i. e. with S and $M_y$ constant, $p_x = 0$, it will be found that for a bar of the length l, which is rigidly fixed at the ends in such a manner as to prevent twisting round the z-axis but which can turn freely round the x-axis, one obtains with the trial-solution:

$$y = C_1 \sin\frac{\pi z}{l}, \qquad \varphi = C_2 \sin\frac{\pi z}{l}$$

an equation representing the requirement for stability:

$$\left(EJ_x \frac{\pi^2}{l^2} - S\right)\left(EB_\varphi \frac{\pi^2}{l^2} + GJ_d\right) - \left[\frac{EB_y}{2}\frac{\pi^2}{l^2} - \left(1 - \frac{J_x}{J_y}\right) M_y\right]^2 = 0 \quad (47)$$

On basis of this formula, S or $M_y$ can be determined. Considering the case in which the moments $M_y$ are set up due to the force S acting not in the centre of gravity but at a distance $e = \frac{M_y}{S}$ from it, then we have before us the case of buckling due to excentric compression. With $S_E = \frac{\pi^2 EJ_y}{l^2}$ and $\left(1 - \frac{J_x}{J_y}\right) \approx 1$ the formula (46) reads somewhat simpler as follows:

$$(S_E - S)\left(EB_\varphi \frac{\pi^2}{l^2} + GJ_d\right) - \left(\frac{B_y}{2 J_x} S_E - Se\right)^2 = 0 \quad (48)$$

Now $\frac{B_y}{2 J_x} = S_x$ is the distance of the lateral force-centre from the centre of gravity. Considering the case $e = s_x$, which corresponds to the longitudinal force S acting at the lateral force-centre, then it follows that $S = S_E$ becomes a maximum and equal to *Euler*'s buckling-value, wherewith the statement made in the preceding paragraph is proved.

[13] If $B_y$ is made equal to zero, i. e. if it be assumed that the crossection possesses two axes of symmetry, the functiony can be eliminated in the special case S = o, and the differential equation of the third order derived by *Timoshenko* for the tieting problem of I-girders will result.

## Summary.

The standard theory of bending and torsion of straight bars is based on two assumptions: 1. The longitudinal stresses, induced by bending, are linear functions of the cross-sectional coordinates x and y; 2. In the case of twisting alone, the longitudinal stresses will be zero, if small quantities of the second order are left neglected. These assumptions are not always realised in the case of bars made up of thin sheets. Proceeding from the assumption, that in each of the sheets of which the bar is composed, the basic principles of the *Navier*'s bending theory apply the differential equations for bending and torsion of prismatical bars are derived on the basis of a principle of variation. The equations apply generally, for bars with open, as well as closed cross-sections, singly and multiply-connected. By means of an appropriate transformation of coordinates it becomes possible, to separate the treatment of bending and torsion. This transformation also points out the significance of the lateral force-centre.

After the general representation of the stress-determination for bending and torsion, the differential equations of the buckling problem are derived in a general manner for the shapes of bars in question and the simple case of buckling of a bar having a simple symmetrical cross-section is discussed. It is pointed out that under certain conditions a centrically loaded bar possesses a load carrying capacity as regards buckling, which is often considerably below the Euler load. Finally the differential equations are derived for tilting of bars with cross-sections of at least one axis of symmetry. In this connection it is shown that in the case of the bar being loaded by a longitudinal compressive force S acting along the axis of symmetry, the critical load $S_k$ reaches a maximum value, if S passes through the lateral-force centre.