Design of horizontally stiffened web plates of plated girders

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Design of Horizontally Stiffened Web Plates of Plated Girders.

Die Bemessung der waagerecht ausgesteiften Stegbleche vollwandiger Träger.

Dimensionnement des âmes renforcées horizontalement dans les poutres à âme pleine.

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I. Introduction.

We shall proceed to investigate the web in one of the middle bays of a solid girder (Fig. 1a) whose load is translated by a series of decking girders, and whose dead weight is replaced in the usual way by a system of concentrated forces acting at the loci of the transverse girders. The transverse force Q is then constant within the width of the bay a, and the bending moment increases, within the width of the bay, in a straight line from the left hand terminal value M_l to the right hand terminal value $M_r = M_l + Q \cdot a$.

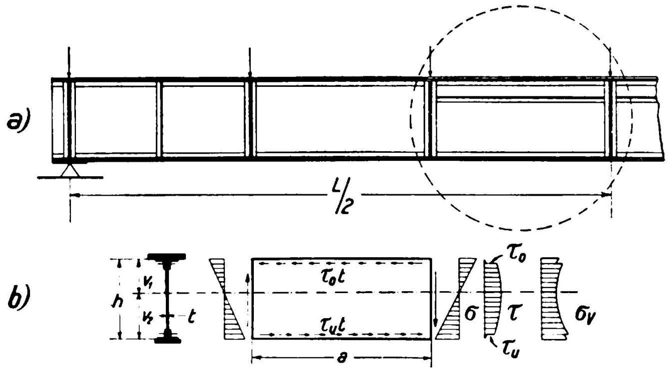

Fig. 1.
The stressing of a web plate forming the central panel of a plate girder.

In the bay investigated, the cross-section of the girder is composed of the two boom cross-sections F_{Go} and F_{Gu}, and the cross-section of the web $F_{St} = t \cdot h$ (Fig. 1 b), and has a moment of inertia J_0 with reference to the main axis of bending. The maximum bending stresses occurring at the ends of the web investigated amount to max $\sigma = \frac{M_r v_2}{J_0}$ and min $\sigma = \frac{-M_r v_1}{J_0}$. The booms, riveted or welded to the web-plates, transmit horizontal shearing forces to the web, and these we wish to distribute uniformly along the line of rivets or the weld inside the bay investigated; the intensity of this distribution is then $\tau_o \cdot t = \frac{Q \cdot S_{Go}}{J_o}$ and $\tau_u \cdot t = \frac{Q \cdot S_{Gu}}{J_o}$, where S_{Go} and S_{Gu} are the static moments of the surfaces of the booms F_{Go} and F_{Gu} with reference to the neutral axis of the bending stresses. The distribution of the shearing stresses τ along the height h of the web is the same in all cross-sections of the bay investigated (Fig. 1 b). The maximum shearing stress is reached in the centre-line of the girder, being $\max \tau = \frac{c \cdot Q}{t \cdot h}$ where c is fixed by the relation $c = \frac{h}{J_o}\left(S_{Go} + \frac{t \cdot v_1^2}{2}\right)$ and falls between 1.00 and 1.50[1].

If we are to avoid local plastification of the web-plate under a given load (ignoring the concentrations of stress at the rivetholes) and base our investigation on the hypothesis of plasticity for constant deformation energy (Plastizitätshypothese der konstanten Gestaltänderungsenergie), what we have to do is to calculate the maximum "comparison stress" $\sigma_V = \sqrt{\sigma^2 + 3\tau^2}$ occurring within the bay investigated, and dimension the web-plate so that σ_V does not reach the yield point σ_F of the structural steel used. If the bending stresses are high compared to the shearing stresses, then the comparison stress, distributed as in the curve Fig. 1 b, becomes highest at the centre line of the rivets or the weld. If, however, the bending stresses are low compared to the shearing stresses, the maximum value $\max \sigma_V$ occurs at the axis of gravity of the girder and, in the bay investigated, amounts to $\max \sigma_V = \max \tau \cdot \sqrt{3} = c\sqrt{3} \cdot \frac{Q}{th}$. In the case of a live load (bridges and crane gantries), the maximum comparison stress in a web-plate element depends on the position of the train load. For ascertaining the 'most unfavourable position of load' we can use the influence line which, for the quantity $\sigma_V^2 = (\sigma^2 + 3\tau^2)$, can be determined with the aid of the known influence lines for M_r and Q.

Besides the problem of avoiding local plasticity effects, and the problem of obviating excessive pressures at the walls of rivet-holes where the rivets connect angle and web-plate (which problem may be excluded in the region of the centre bays of girders, owing to the relative smallness of the shear forces), the question of *ensuring stability* is of fundamental importance in the dimensioning of the web. To prevent repeated over-stressing of the girder and its riveted or welded connections, we must exclude the possibility of the web-plate bulging under the service load. The theoretical determination of the limit of stability depends very largely on extensive idealisation of the web-plate as regards its geometrical and material properties, its position, and its load, so that we are compelled to include in our calculations a coefficient of safety against bulging v_b, in order to cover any unavoidable discrepancy between idealised assumption and actuality. This bulge safety factor refers to the gross stresses (stress components without regard to the weakening of the rivet hole) and, in a carefully designed structure, must undoubtedly be smaller than the average buckling safety factor v_k of members under compression, since the carrying capacity of plates supported at their periphery lies considerably above

[1] *E. Chwalla:* Der Bauingenieur, Vol. 17, 1936, p. 81.

the limit of stability, on account of the high deformation of the middle surface associated with bulging.

The limit of stability is usually raised by strengthening the web by vertical, horizontal or inclined stiffeners, or by a lattice arrangement of stiffeners. Vertical stiffening is usually adopted for the end bays of the girder, where the shearing stresses attain comparatively high values. (As regards the theory of stability of webs in the end bay, see the paper mentioned in Footnote 1.) For the middle bays of girders, on the other hand, horizontal stiffening located on the bending pressure side has been found extremely suitable under certain conditions. The thicknesses of the webs and stiffeners adopted for the wide-span solid girders designed or constructed in recent years have been collated by *Karner*[2], while the theory of the stability of thin plates is dealt with in publications mentioned in Footnotes 1, 4 and 5.

II. Idealizing the Web Plate for the Investigation of Stability.

We shall take a web plate having a constant thickness t and a truly flat middle surface, and which consists of a homogeneous, insotropic material. We shall substitute a constant bending moment for the bending moment (with linear variability) coming on to the web plate within the width of the bay a; while the shearing forces arising from the upper and lower booms (and which, under certain conditions, are variable) will be replaced by the shearing stresses $\tau \cdot t$ which are the same on both sides, and we shall assume that these stresses act directly at the two lengthwise edges of the web plate. The neutral axis line

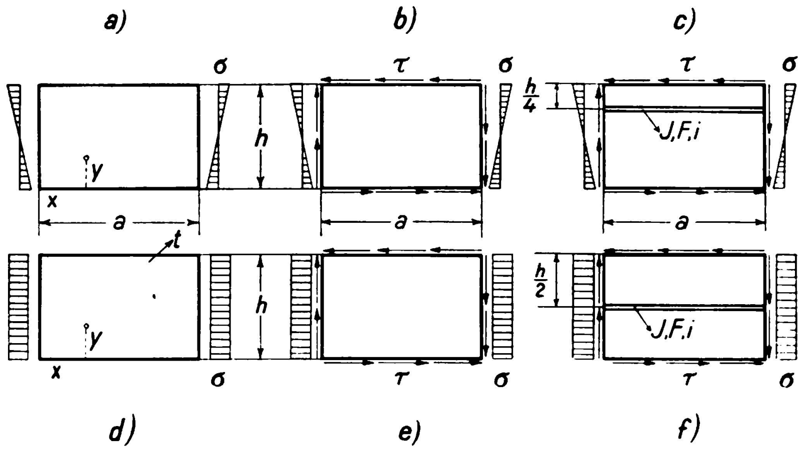

Fig. 2.

Various cases of idealized stressing of plates.

of bending stress then comes at mid-height, and the web plate is under purely bending and shearing stresses (Fig. 2 b), σ and τ being calculated without regard to weakening by the rivet holes. We shall also assume the maximum comparison stress σ_v occurring within the bay investigated to be below the limit of proportionality and the elastic limit of the steel used, under the critical

[2] *L. Karner:* Abhandlungen d. Int. Vereinigung f. Brückenbau u. Hochbau, Vol. I, 1932, p. 297.

load sought; so that the change in stability is consummated within the range of stress for which Hooke's law applies. The stiffness of the web plate is then:

$$D = \frac{E t^3}{12 (1 - \mu^2)} \tag{1}$$

where μ is the ratio of transverse contraction to longitudinal expansion. For $E = 2100$ t/cm² and $\mu = 0.3$, $D = 192.3 \cdot t^3$ in tcm.

We shall assume that, at the edges $x = 0$ and $x = a$, the rectangular web plate is laterally restrained by the vertical stiffeners (Fig. 1 a), but freely pivotable, as the torsional stiffness of the vertical stiffeners is relatively small, and also that no appreciable restraint can be achieved by the adjoining web plate, which, itself, is approaching its own limit of stability at the load considered. Furthermore, as *Schleicher*[3] has shown, the effect of restraining the narrow sides is practically negligible for plates with larger side ratios a/h.

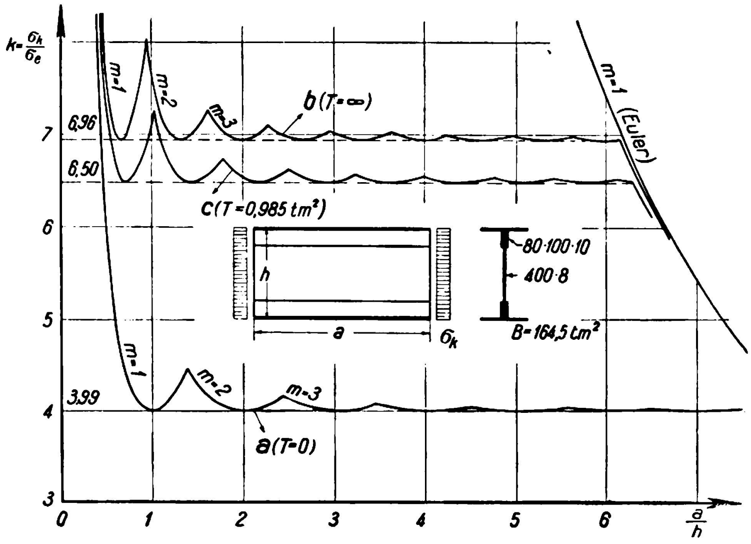

Fig. 3.

Buckling of compressed plates, strengthened by angles at the edges (plated compression members).

At the edges $y = 0$ and $y = h$, the web plate is supported by the booms which are stiff against bending and torsion. To elucidate the influence of this method of supporting upon the extent of the bulging strenght, the writer has investigated the stability of a uniformly compressed rectangular plate with pairs of end angles stiff against torsion and bending[4]. He found that, within *Hooke*'s range, the lateral bending stiffness B of the booms is always sufficient to act in practice in the same way as a lateral restraint of the edges of the plates ($B = \infty$). This may be seen from Figs. 3 and 4 (and Figs. 5 and 6 as well). On the abscissae are plotted the side ratios a/h of the plate, and on the ordinates

[3] *F. Schleicher:* Mitteilungen aus den Forschungsanstalten des Gutehoffnungshütte-Konzerns, Vol. 1, Nr. 8, Nürnberg 1931.

[4] *E. Chwalla:* Ingenieur-Archiv, Vol. V., 1934, p. 54.

the bulging coefficients k, that is to say, the ratio of the minimum critical compressive stress to the reference quantity

$$\sigma_e = \frac{\pi^2 D}{h^2 t} \tag{2}$$

(*Euler*'s buckling stress for a plate strip of unit width and length h amounts, for E = 2100 t/cm² and μ = 0.3 simply to σ_e = 1898 $(t/h)^2$ in t/cm²). Curves 'a', 'b' and 'c' in Fig. 3, referring to a boom having a relatively small amount of lateral stiffness against bending (B = 164.5 tm²), are only slightly lower than the corresponding curves in Fig. 4, which refer to the extreme case of B = ∞. The torsional stiffness T of the booms is usually also sufficient to give a comparatively large amount of restraint over the range for which

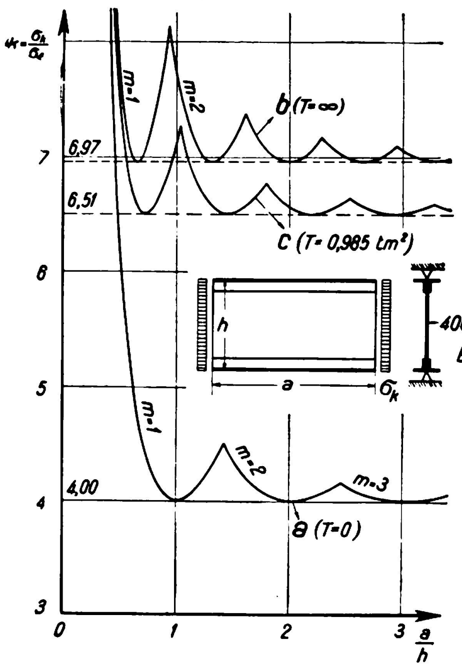

Fig. 4.
Buckling of compressed plates, strengthened by angles at the edges.

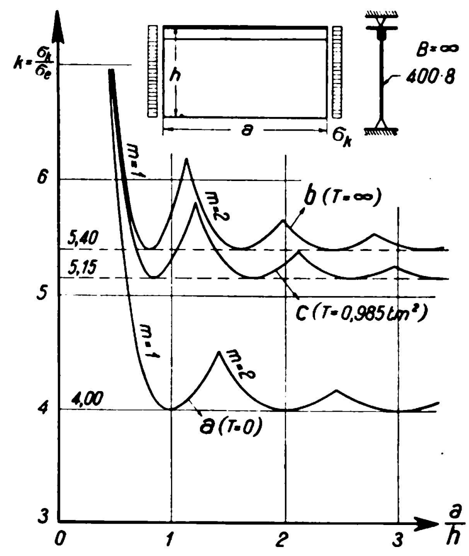

Fig. 5.
Buckling of plates supported on all sides, reinforced on one side.

Hooke's law is applicable. This is apparent from Figs. 3, 4 and 5, in which the curves 'c' refer to a boom having a comparatively small amount of torsional rigidity (T = 0.985 tm²), and yet come very little lower than the curves 'b' referring to the extreme case of infinite torsional rigidity T = ∞. This highly restraining effect of booms having a comparatively small amount of torsional rigidity may be attributed to the relatively short length of the sinusoidal half-waves formed when the web plates bulge (in Figs. 3, 4, 5 and 6, 'm' denotes the number of these half-waves). To these successive bulges in the lengthwise direction of the bulging web are alternately coordinated positive and negative angles of torsion, so that, when the web plate does bulge, the booms must be alternately twisted in either direction within torsional zero positions which follow closely on each other.

The effect of a restraint within *Hooke*'s range upon the limit of stability is, however, much smaller than the effect which a restraint is capable of exerting on *Euler*'s buckling load of compressed bars. Comparing curves 'a' (edges of the plates supported without restraint) and 'b' (lengthwise edges rigidly supported) of Figs. 3, 4 or 6, it will be noted that, under compressive or bending stress, an average increase in the limit of stability of only 1.7 times can be attained *even by a rigid method of supporting;* the same average figure is also obtained in the case of a pure shearing load[5]. In most practical cases, the increase in bulging strength attainable by restraining the web plate is considerably less, since the booms are as a rule already considerably plasticised under the b-fold service load involved in the investigation of stability. It is therefore advisable, when investigating the stability of webs, to regard the

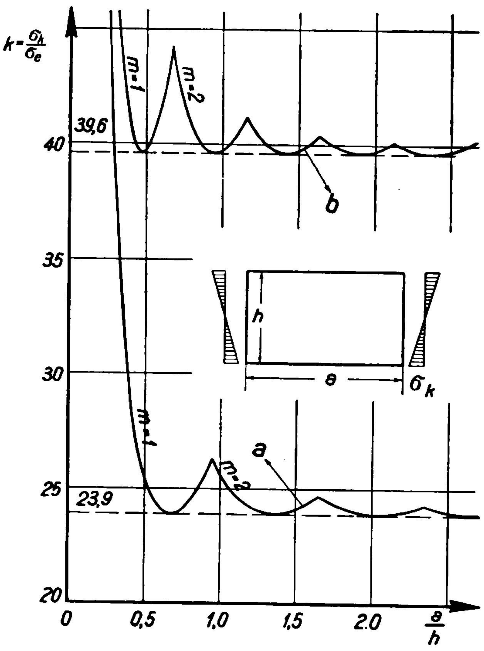

Fig. 6.
Buckling of plate, with bending in the plane of the plate.

lengthwise edges as being restrained laterally, but as *being supported so as to be freely* pivotable. Instead of taking an ideal height (the height fixed by the mutual distance apart of the torsional axes), it is usual to take the actual or effective height and, with riveted girders, frequently the height measured between the centre-lines of the rivets.

If the elastic stability of a rectangular plate, supported so as to be freely pivotable at all four edges, be investigated in terms of a purely bending stress (Fig. 2 a), then, according to *Timoshenko*[6], the solution given by curve 'a' in Fig. 6 will be obtained. As already mentioned, the coefficient of bulging, k, (i. e., the ratio of the minimum critical bending stress σ_k at the edges to the reference quantity σ_e as equation 2) is plotted in terms of the side ratio a/h. The curve shown forms the lower marginal line of an assemblage of closely related curves (by affinity) arranged about the parameter m (i. e., the number

[5] Cf. *C. S. Heck* and *H. Ebner*'s Compilation, Luftfahrtforschung, Vol. 11, 1935, p. 211.
[6] *S. Timoshenko:* Der Eisenbau, Vol. 12, 1921, p. 147.

of sinusoidal half-waves forming in the lengthwise direction). The minimum value of the critical edge stress is min $\sigma_k = 23.9\ \sigma_e$, being attained for plates with the side ratio a/h = 0.667 . m, m = 1, 2, 3 .... For a square plate having a critical edge stress of $\sigma_k = 25.54 \cdot \sigma_e$, the case of m = 2 applies. The curved surface pertaining to this limit of stability has been shown in Fig. 7 in the form of a contour map. The figures appended to the several level lines represent proportions, and should be regarded as being infinitely small at the limit of stability. At the place x = a/2, a 'nodal line' is developed on which are located the points of the middle surface of the plate which do not undergo any lateral deflection (bending) when bulging occurs.

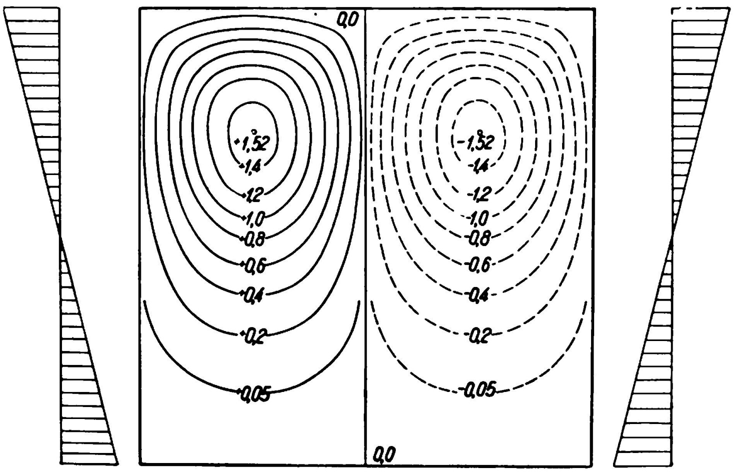

Fig. 7.

Deflection surface of a square plate subjected to bending.

Curve 'b' in Fig. 6 refers to the case of rigidly restrained lengthwise edges investigated by *Nölke*[7]. The bulging coefficient k and the number of half-waves are higher here than they were previously. The minimum value of the critical edge stress is *min* $\sigma_k = 39.6 \cdot \sigma_e$, and is attained with plates having the side ratio a/h = 0.475 . m, m = 1, 2, 3 ...

The curves in Fig. 6 show that vertical stiffeners themselves are not a suitable method of increasing the limit of stability when they are rigid enough against bending to compel a vertical 'nodal line' to form at the place where they act. *Timoshenko*[6] and other experts[8] therefore suggested increasing the bulging strength of webs which are subjected mainly to bending stresses, by means of horizontal stiffeners arranged on the bending-compression side. As these stiffeners are riveted or welded to the web plate, they are subjected to approximately the same compressive stress as acts on the web plate at the place where the stiffeners are located. In calculating the dimensions of the stiffeners, this compressive stress considerably affects the modus operandi of the stiffener, and must be fully allowed for. When ascertaining the *bending strength of the girder* (calculating J_o), on the other hand, it is advisable to consider

[7] *K. Nölke:* Der Bauingenieur, Vol. 17, 1936, p. 111.

[8] Cf. *F. Schleicher:* Der Bauingenieur, Vol. 15, 1934, p. 505; *F. Wansleben:* Der Stahlbau, Vol. 8, 1935, p. 110; „Stahlbau-Kalender 1936", Verlag W. Ernst & Sohn, Berlin, p. 380, etc.

longitudinal stiffness only in cases in which this particular stiffness suffers no interruption at the places where the longitudinal stiffeners are located.

The stiffener can be applied to one or both sides of the wall of the web. When arranged on one side, and when the web bulges, the bending strength of an ideal stiffener applies, rather than the bending rigidity (stiffness against bending) of the stiffener regarded as being detached from the plate. This 'ideal stiffener' is composed of the single stiffener and an adjoining strip of the web plate of definite width. The determination of the width of this cooperating strip of the plate comes within the scope of a wellknown elastic-statical problem[9]. If the stiffener is not loaded axially (e. g., in the case of stiffened plates under a purely shearing stress, or vertically stiffened plates with a compressive load in linear distribution), then there is a limit of stability with a 'branching', even in plates with the stiffeners arranged on one side. If, however, the stiffener *is* axially loaded, the existence of stability limits with 'branches' may only be directly affirmed in cases where the stiffener is arranged on both sides of the wall of the plate and in such a way that its centre-line falls within the medial plane of the plate.

III. Stability of a horizontally stiffened Square Plate subjected to Bending Stresses in its Plane.

We shall now investigate a rectangular plate of length a and height h, freely supported at all four edges (*Navier*'s edge conditions), and subjected in its plane to a pure bending stress. The plate is stiffened by a horizontal stiffener located at a distance of $0.25 \cdot h$ from the edge subject to bending pressure, and is arranged on either side of the wall of the plate so that its centre line comes in the middle plane of the plate (Fig. 2c, with $\tau = 0$). We will call the cross-sectional area of the stiffener F, the bending rigidity manifested when deflections take place at right angles to the plane of the plate, EJ, and the radius of gyration of the cross-section $i = \sqrt{\frac{J}{F}}$. So as to be able to count upon proportional values, we shall refer these constants to corresponding constants of the plate, thus obtaining the auxiliary quantities:

$$\delta = \frac{F}{th}, \quad \gamma = \frac{EJ}{Dh}, \quad \frac{i}{t} = \sqrt{\frac{\gamma}{12(1-\mu^2)\delta}} \tag{3}$$

The writer has investigated the stability of this plate on the basis of the energy criterion[10], and has obtained the solution given in Fig. 8 (thick line), for the case of a/h; 0.8, $F = 0.12$ t h. The curves shows the increase in the fibre bending stress σ_k at the lowest limit of stability when the rigidity of the stiffener against bending increases for a constant cross-sectional area of the stiffener. On the abscissae is plotted the ratio of the radius of inertia of the stiffener i to the thickness of the plate t, and on the ordinates the bulging coefficient k (ratio of the critical edge stress σ_k to the reference quantity σ_e as equation 2).

[9] Cf. *E. Chwalla:* Der Stahlbau, Vol. 9, 1936, p. 73.

[10] *E. Chwalla:* Der Stahlbau, Vol. 9, 1936.

Branch (I) of the curve refers to the critical conditions of equilibrium in which the plate begins to bulge out after a single bulge has occurred (cf. Fig. 9), and the stiffener therefore undergoes a sinusoidal curvature outwards. If, perchance, i/t should be equal to $(i/t)_0 = 0.847$, then we have a special case in which the stiffener, regarded as being detached from the plate, reaches, in its critical state, its *Euler* buckling limit (equivalent to the half-wave frequency m = 1, and is therefore incapable of affecting the bulging of the plate, either favourably or adversely. The critical fibre stress of the plate is in this case $\sigma_k = k \cdot \sigma_e = 24.47 \cdot \sigma_e$, so that, in its critical state of equilibrium, the compressive stress on the plate is $0.5 \cdot \sigma_k = 12.24 \cdot \sigma_e$. This, as will readily be noted, agress with *Euler*'s buckling stress of the isolated stiffener. Since, in view of the small side ratio, the plate investigated would also bulge in the

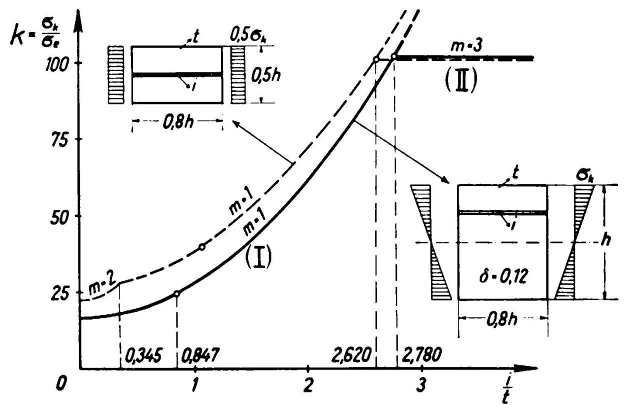

Fig. 8.
The relation between warping (buckling) stress and the rigidity against bending of the horizontal stiffener.

unstiffened state after m = 1 half-waves, the bulging coefficient of k = 24.47 found for our special case is identical with the bulging coefficient (*Timoshenko*'s theory[6]) applicable to the unstiffened plate. The stiffened plate has in this case the same buckling strength as the unstiffened plate.

If, for a given cross-sectional area $F = 0.12\ t\ h$, the sectional shape of the stiffener is selected so that the ratio $i/t > 0.847$, then the plate is capable of supporting itself on the stiffener; and the buckling strength of the plate in this case is increased in accordance with curve (I). If, however, the cross-sectional shape of the stiffener is selected so that the ratio $i/t < 0.847$, despite the large cross-sectional area $F = 0.12\ t\ h$ (dividing the stiffener up into a series of thin plates or laminae resting flat on the wall on either side of the plate), then *the compressed stiffener is supported by the plate.* In the extreme case of $i/t = 0$, the limit of stability of the plate would in this case drop to $\sigma_k = 16.385 \cdot \sigma_e$, and would therefore be 33 per cent. lower than the limit of stability of the unstiffened plate.

If i/t increases considerably and $F = 0.12\ t\ h$ remains constant, then we reach an extreme case where *two different forms of bulge or curvature* exist at the limit of stability, and the plate is able, *at the same critical load*, to bulge in either way. The type of bulge (I) already mentioned assumes, in the longitudinal section, a sinusoidal shape (Fig. 9), so that, when it bulges, the stiffener bends in the form of a sinusoidal half-wave. The potential energy stored up in the bulged plate is composed in this case of the energy stored up in the deflected (bulged) plate and the energy stored up in the bulged stiffener. Type (II) bulge or arch, on the other hand, is split up into a number of smaller bulges, characterised by the development of a horizontal 'nodal line' *at the locus of the stiffener,* so that the stiffener does not undergo any lateral deflection at the limit of stability (Fig. 10). In this particular case, the potential energy accumulated in the bulged plate consists only of the energy stored up in the bulged plate itself, but this energy is comparatively high in amount. The extreme case, where both types of bulge are equivalent as regards energy, so that both forms can develop under the same critical load, is attained

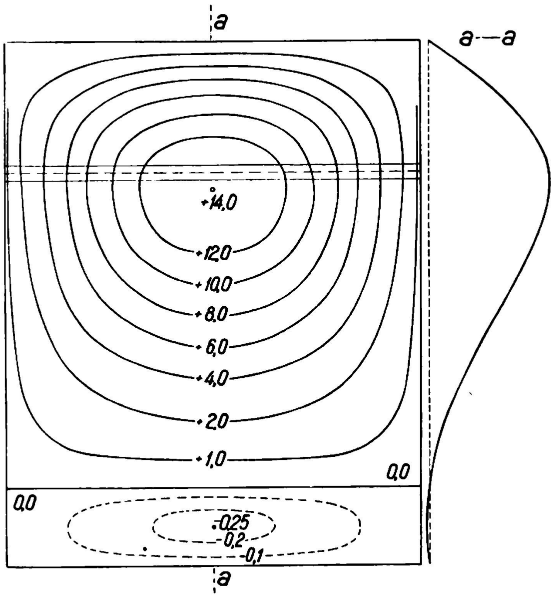

Fig. 9.
Deflection surface of a horizontally stiffened plate, subjected to bending (Deflection form I),

for $i/t = (i/t)_{I, II} = 2.730$, $k = 101.85$ for the plate investigated ($a/h = 0.8$, $F = 0.12\ t\ h$, cf. Fig. 8). The two types of bulging referring to this limit of stability are shown in Figs. 9 and 10 in the form of contour plans. The nodes attributed to the several level lines should again only be taken as proportional values, and the absolute amounts of bulge should be regarded as infinitely small at the limit of stability. Besides the two contour plans, the cross-sections a — a of the curved surfaces are shown.

The special value $(i/t)_{I, II}$ generally depends on the side ratio of the plate and the auxiliary quantity $\delta = F/th$, decreasing both with a/h and δ. As we shall show when we come to investigate a similar loading problem (Fig. 15), it is connected with a maximum value $\max (i/t)_{I, II}$ which depends on the side ratio and is determined only by δ. By making the horizontal strip so rigid against bending that $i/t \geqq \max (i/t)_{I,II}$, then it is able, with plates of any side ratio, to compel the formation of the N° II. type of bulge in the critical state of equilibrium, i. e., the bulging where the stiffener undergoes no lateral deflection.

The bulging coefficient pertaining to the bulge form (II.) is $k = 101.85$, and is 4.16 times as large as the bulging coefficient of the unstiffened plate. The number of sinusoidal half-waves formed in the lengthwise direction is $m = 3$ (cf. Fig. 10), though the bulging strength in the case of $m = 4$ would be very little larger ($k = 103.49$). Since the bulge (II.) exhibits a horizontal 'nodal line' at the locus of the stiffener, the bulging coefficient $k = 101.85$ is independent of the bending rigidity EJ of the stiffener, and, hence, of the value i/t also. Every stiffener for which $i/t \geq (\ /t)_{I,\,II}$ is capable of compelling this horizontal 'nodal line' in the arched surface and therefore acts similarly to a stiffener of 'infinite stiffnes against bending'.

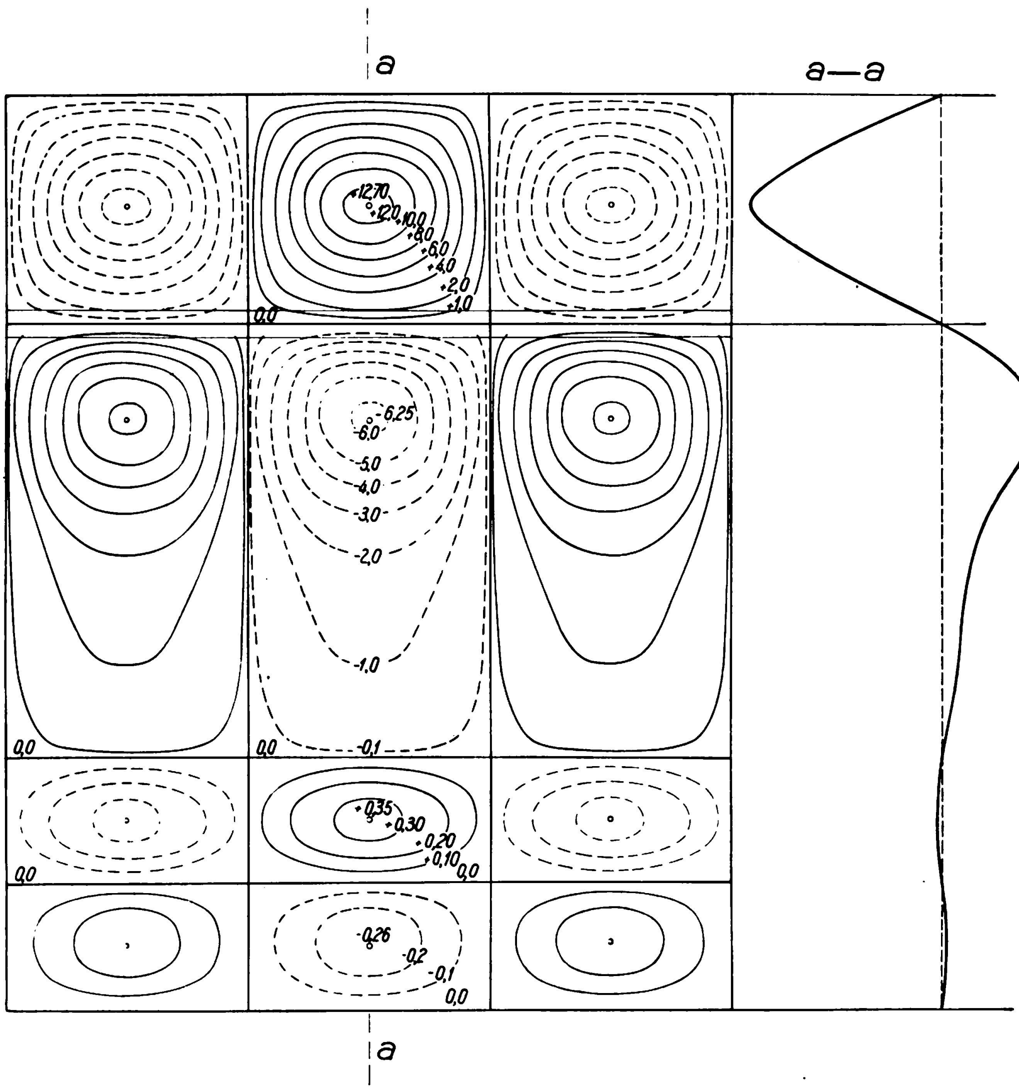

Fig. 10.

Deflection surface of a horizontally stiffened plate, subjected to bending (Deflection form II).

It may be well to note that our investigation refers merely to the *limit* of elastic stability, or simply to the possibility of the formation of infinitely few bulged forms of equilibrium. The problem as to how the bulge form would develop under a *hypercritical* load, i. e., for bending fibre stresses $\sigma > \sigma$, can not be answered, in terms of our linearised stability theory. Bulging with a finite rise (Auswölbungen mit endlich großem Auswölbungspfeil) is related to finite displacements of the edge points in the direction of the original plane of the plate (at right angles to the original edge line), so that, in this case, the solution of the problem must depend upon whether and to what extent these displacements (the 'contraction' of the edge lines) are actually possible in the case investigated. These considerations are of some importance not only when assessing the practical significance of the No. II type of bulge and the limiting state $i/t = (i/t)_{I, II}$, but also when comparing the results derived from the linearised stability theory with results obtained experimentally.

IV. The Effect of Additional Shearing Stresses on the Limit of Stability.

The web of a solid girder is not only subjected to bending stresses, but also to shearing stresses as well, and these latter may have an appreciable effect on the limit of stability and the shape of the curved surface. Here we have an 'unfavourable' combination of the values σ and τ, which refer to the *lowest* limit of stability and can usually only be determined indirectly. In practice, the designer usually restricts himself to the consideration of two load positions only (the one for which the bending moment and the one for which the shear force becomes an extreme in the bay investigated).

The problem of the stability of a square plate subjected in its plane to a pure bending and a pure shearing stress, has been investigated and solved by *Timoshenko*[11] and *Stein*[11] for the case where the edges of the plate are supported free from restraint. This solution may be represented in convenient form by calculating the pairs of values σ_k, τ_k coordinated with the limit of stability, and, in addition, the bulging stresses σ_{ko}, τ_{ko} which would apply to the plate investigated if the latter were subjected *exclusively* to a purely bending stress or *exclusively* to a purely shearing stress respectively. The proportions σ_k/σ_{ko} and τ_k/τ_{ko} then determine the coordinates of a point on the curve $\sigma_k/\sigma_{ko} = \Phi(\tau_k/\tau_{ko})$ which belongs to the parameter a/h and determines the solution of the stability problem for all combinations of the quantities σ and τ. Changing the sign of the bending moment or the shearing force obviously does not affect the limit of stability, so that this curve is symmetrical in shape with respect to the two coordinated axes, and intersects them at right angles at the points $\sigma_k/\sigma_{ko} = 1.00$ or $\tau_k/\tau_{ko} = 1.00$.

The solution found by *Timoshenko* for the case $a/h = 0.5$ and 1.00 has been represented in this way in Fig. 11. These curves, like the ones plotted for other side ratios, line within the region bounded by dotted lines, so that, generally speaking, we shall be pretty safe in approximating the curves for all a/h by an are of 'unity' radius. When calculating the web, then, all we have to do is to work out the gross stresses ν_b, σ, $\nu_b \cdot \tau$ occurring at ν_b-times the service load, and the bulging stresses σ_{ko} and τ_{ko}. If the point determined by the pairs of coördinates $\nu_b \sigma/\sigma_{ko} \cdot \nu_b \tau/\tau_{ko}$ comes upon or below the arc in Fig. 11; that is to say, is $\nu_b \sigma/\sigma_{ko} \leqq \sqrt{1 - (\nu_b \tau/\tau_{ko})^2}$, then the desired bulging safety factor is guaranteed with any values of a/h. In place of the arc, we can also use a three-sided polygon for approximating the solution curves. In this case, $\nu_b \cdot \sigma$ and $\nu_b \cdot \tau$ must be selected so that three

[11] *S. Timoshenko:* Miscell. Papers pres. Amer. Soc. Mech. Engr. Meetings, 1933, Paper N° 3, and Engineering 138, 1934, p. 207; *O. Stein:* Der Stahlbau, Vol. 7, 1934, p. 57.

definite inequalities related to the three sides of the polygon must be fulfilled.

If the web plate is strengthened in the region of the bending pressure stresses by a horizontal stiffener (Fig. 2c), then the reference value σ_{ko} (which we can ascertain with the help of the theory developed in Part III.) as also the reference value τ_{ko} (which can be determined with the held of the solution developed by *Timoshenko*[6]) will be considerably higher than

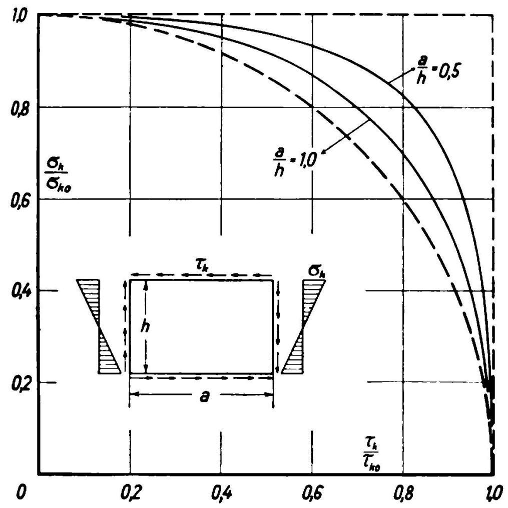

Fig. 11.
Mutual dependence of warping stresses due to bending and shear efforts.

before. The curve $\sigma_k/\sigma_{ko} = \Phi(\tau_k/\tau_{ko})$ now obtained meets the coördinate axes at the points $\sigma_k/\sigma_{ko} = 1.00$ or $\tau_k/\tau_{ko} = 1.00$ as before, but no longer intersects the axis of the abscissae at right angles, because the limit of stability now depends on the sign of the bending moment owing to the fact that the stiffener is arranged on the *bending pressure* side only.

V. Stability of a Rectangular Plate subjected to uniform Compressive Stress and, additionally, to Shearing Stress.

We have investigated a rectangular plate of side ratio a/h, supported free from restraint at all four edges, and subjected to normal stresses σ distributed evenly over the height h, and by uniformly distributed shearing stresses τ (Fig. 20). The problem of the stability of this plate has been solved by proximation by *Wagner*[12] and *Wansleben*[8] in terms of a plate strip of infinite length, and by the present writer[10] for plates of any given side ratio. As in the case of Section IV., this solution can be conveniently presented by calculating the pairs of values σ_k, τ_k pertaining to the limit of stability found, as also the bulging stresses σ_{ko}, τ_{ko} which would apply if the plate investigated were subjected *exclusively* to purely compressive stresses, or *exclusively* to purely shearing stresses respectively. The proportions σ_k/σ_{ko}, τ_k/τ_{ko} then determine the coordinates of a point on the curve $\sigma_k/\sigma_{ko} = \Psi(\tau_k/\tau_{ko})$ applying to the parameter a/h and solving the stability problem for all combinations of σ and τ. Since the limit of

[12] *H. Wagner:* Jahrbuch d. wiss. Ges. f. Luftfahrt, 1928, p. 113.

stability is not affected by a change of sign in the shearing stress, this curve assumes a shape which is symmetrical to the axis of the ordinates, and intersects the latter at right angles at the point $\sigma_k/\sigma_{ko} = 1.00$. The axis of the abscissae is intersected at an acute angle at the point $\tau_k/\tau_{ko} = 1.00$, as the limit of stability is altered by a change in the sign of the normal stress (transition from the case of 'compression and shear' to the case of 'tension and shear').

In the case of plates having a large side ratio, the bulged surfaces formed at the limit of stability are split up into a series of bulges separated by nodal lines. In contrast to the results shown in Figs. 3 to 10 (referring to plates with normal stresses distributed linearly over the height h), plates subjected additionally to shearing stresses develop bulged surfaces in which the longitudinal section no longer assumes the form of a simple sine line and whose nodal lines are no longer straight. To stress this difference, we shall call the number of half-waves m' instead of m. The general equation governing the limits of elastic stability falls into two conditions of bulging independent of each other, just as it does in the case of a purely shearing stress. One of these conditions is related to the bulged surfaces with $m' = 1, 3, 5 \ldots.$, and the other to the bulged surfaces with $m' = 2, 4, 6 \ldots.$. Depending upon the side ratio of the plate and the extent of the additional shearing stress, either one or the other leads to the minimum limit of stability.

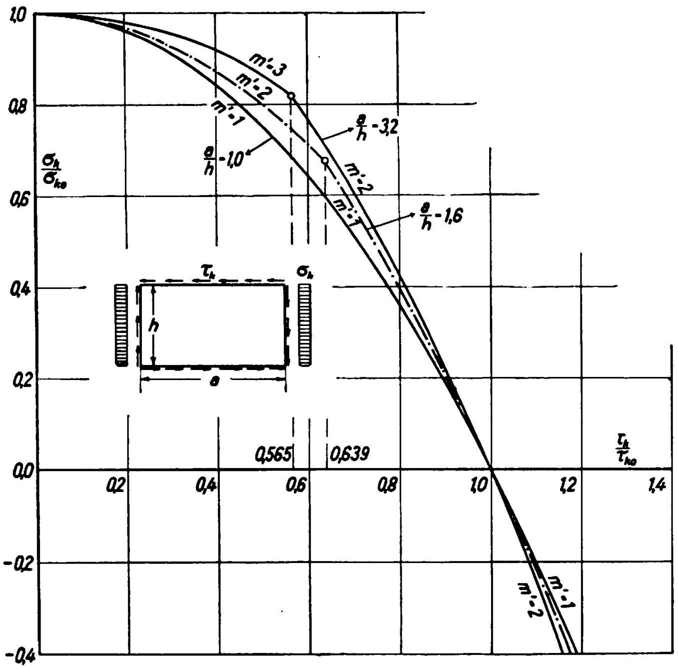

Fig. 12.
Mutual dependence of warping stresses for pure compressive and shear efforts.

Fig. 12 shows the curve applying to the solution $\sigma_k/\sigma_{ko} = \Psi\,(\tau_k/\tau_{ko})$ for rectangular plates of side ratio $a/h = 1.00$, 1.60 and 3.20. When subjected to shearing stress only ($\sigma_k/\sigma_{ko} = 0$) and also in the case of 'compression and shear' and 'tension and shear', square plates bulge in a form which has only one half-wave, so that, in this case, the curve $\sigma_k/\sigma_{ko} = \Psi\,(\tau_k/\tau_{ko})$ consists of of a single branch only ($m' = 1$). In plates of side ratio $a/h = 1.60$

subjected to a purely compressive stress ($\tau_k/\tau_{ko} = 0$) and small additional shearing stresses ($\tau < 0.639 \cdot \tau_{ko}$, $\sigma > 0.672 \cdot \sigma_{ko}$), the bulged surface has *two* half-waves, so that to curve $\sigma_k/\sigma_{ko} = \Psi(\tau_k/\tau_{ko})$ is composed of two branches ($m' = 1$ and $m' = 2$). Plates of side ratio $a/h = 3.20$ subjected to a purely compressive stress ($\tau_k/\tau_{ko} = 0$) or to small additional shearing stresses ($\tau < 0.565 \cdot \tau_{ko}$, $\sigma > 0.820\,\sigma_{ko}$) form a bulged surface with *three* half-waves; and, in the case of large additional shearing stresses, or in the case of "tension and shear", they form a bulged surface with *two* half-waves in the longitudinal direction, so that in this particular case the curve consists of two branches ($m' = 2$ and $m' = 3$).

If we confine ourselves to the stress region $-0.40 = (\sigma_k/\sigma_{ko}) \leqq +1.00$, and ignore the splitting up of the curves into single branches, the curves $\sigma_k/\sigma_{ko} = \Psi(\tau_k/\tau_{ko})$ in Fig. 12 may be approximated by parabolas from the law $\frac{\sigma_k}{\sigma_{ko}} = 1 - \left(\frac{\tau_k}{\tau_{ko}}\right)^{\varkappa}$. A similar law of approximation has already been suggested in connection with a similar investigation into stability (viz., the bulging of a thin-walled pipe under axial load and additional torsional stresses)[13]. For $\varkappa = 2$, we get a square parabola which coincides almost perfectly with the curve $\sigma_k/\sigma_{ko} = \Psi(\tau_k/\tau_{ko})$ referring to the parameter $a/h = 1.00$. Instead of the parabola, we can also use a three-sided polygon for the approximation of the solution curves. The basis of calculation is then formed by *three* inequalities related to the three sides of the polygon.

If $a/h = 1.6$, and if the compressive stress is $\sigma = 2.82 \cdot \sigma_e$ then bulging occurs when the additional shearing stress present reaches the value $\tau = 4.47 \cdot \sigma_e$; since in this case the reference values amount to $\sigma_{ko} = 4.20 \cdot \sigma_e$ and $\tau_{ko} = 7.00 \cdot \sigma_e$, wo then get $\sigma_k/\sigma_{ko} = 0.672$ and $\tau_k/\tau_{ko} = 0.639$ so that we arrive in Fig. 12 at the special case where bulging is possible in two different forms ($m' = 1$ and $m' = 2$) under the same critical load. These two forms of bulging, which are equivalent as regards energy and can develop with the same amount of probability, are shown in Figs. 13 a and 13 b in the form of contour plans. The nodes ascribed to the several level lines should once more only be regarded as proportional values, and the absolute values of the bulging should be regarded as being infinitely small at the limit of stability.

VI. Stability of a horizontally stiffened rectangular Plate under Uniform Compressive Stress.

We have investigated the stability of a rectangular plate of side ratio a/h subjected to uniformly distributed compressive stresses and strengthened by a horizontal stiffener (Fig. 2 f, with $\tau = 0$). The stiffener is assumed to be arranged on either side of the plane of the plate in such a way that the centre line of the stiffener lies in the middle plane of the plate. Let F be its cross-sectional area, EJ its stiffness against bending (referred to deflections at right angies to the plane of the plate), and $i = \sqrt{J/F}$ its cross-sectional radius of gyration.

[13] *P. J. Bridget, C. C. Jerome* and *A. B. Vosseller:* Trans. Amer. Soc. Mech. Engr., 56, 1934, p. 569. APM—56—6.

Provided we are content with a (practically approximately accurate) proximate solution, and we introduce the auxiliary quantities

$$\delta = \frac{F}{th}, \quad \gamma = \frac{EJ}{Dh}, \quad \frac{i}{t} = \sqrt{\frac{\gamma}{12(1-\mu^2)\delta}}, \quad k = \frac{\sigma_k}{\sigma_e},$$
$$\sigma_e = \frac{\pi^2 D}{h^2 t}, \quad \beta_1 = \frac{a}{m \cdot h}. \quad m = 1, 2, 3, \ldots \tag{4}$$

then an examination of the stability of this plate, which we owe to *Timoshenko* [6], leads to the comparatively simple bulge condition:

$$\begin{aligned}(1+4\delta)\,(k\,\beta_1^2)^2 - (k\,\beta_1^2)\,\{4\gamma + (1+2\delta)\cdot[(1+\beta_1^2)^2 + (1+9\,\beta_1^2)^2]\} \\ + 2\gamma\,[(1+\beta_1^2)^2 + (1+9\,\beta_1^2)^2] + (1+\beta_1^2)^2\cdot(1+9\,\beta_1^2)^2 = 0\end{aligned} \tag{5}$$

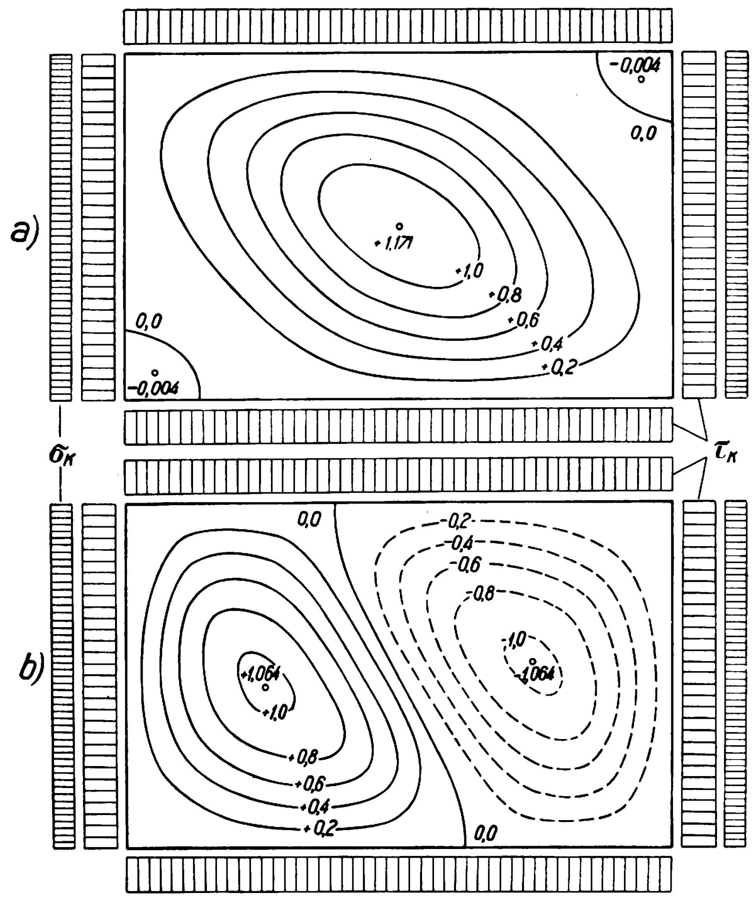

Fig. 13.

Deflection surface of plate subjected to compression and shear.

The solutions derived for the case of $\delta = 0.20$ with the help of Equation (5) are given in Fig. 14 in the form of an assemblage of curves ('bordering lines') coordinated to the parameter. The bulge surfaces pertaining to these solutions will hereafter be referred to as 'Bulge Form I'. In plates of large side ratio, they are split up by vertical 'nodal lines' into a series of single bulges having a sinusoidal longitudinal section. The number of sinusoidal half-waves m governing the lowest limit of stability has been appended to the several branches of the curves in Fig. 14.

The lowermost curve in Fig. 14 refers to the limiting case of $\gamma = 0$, in which the cross-section of the stiffener certainly possesses the surface area $F = 0.20 \cdot th$, but the form of cross-section was selected so unfavourably that the bending rigidity (stiffness against bending) EJ is practically nil (groups of laminae resting flat against the plate). The 'stiffener' being under compressive load must in this case be supported by the plate, so that the bulging coefficient k is smaller than that of the unstiffened plate (curve 'a' in Fig. 4). If, however, the section $F = 0.20 \cdot th$ is shaped so that EJ (and, hence, γ as well) is comparatively large, then the stiffener is able to perform its function and support

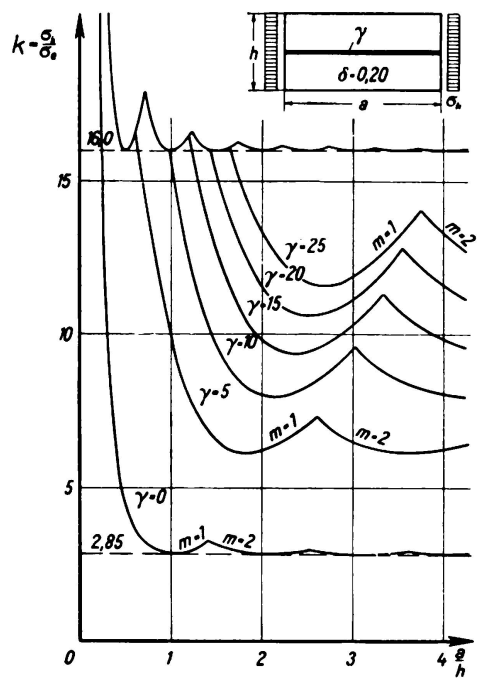

Fig. 14.
Buckling of compressed, horizontally stiffened plates.

the plate at the limit of stability. As in Fig. 8, however, the bulging strength cannot be raised beyond limits in this case either, since for all $\gamma \geqq \gamma_{I,II}$ a 'Bulge Form II' is developed at the lowest limit of stability, and is characterised by a horizontal 'nodal line' occurring at the locus of the stiffener. The stiffener does not undergo any lateral outward bend (deflection), so that the bulging coefficients pertaining to the No. II. form of bulge depend on the quantity γ,, and cannot be increased by fitting a stiffener possessing 'infinite bending rigidity'. The curve corresponding to this extreme case (the uppermost curve in Fig. 14) may be directly derived from curve 'a' in Fig. 4 by remembering that the plate in question is divided by the horizontal 'nodal line' into two strips of height h/2 subjected to the same type of stress, and whose minimum critical compressive stress is min $\sigma_k = 4.00 \cdot \frac{4\pi^2 D}{h^2 t} = 16.00 \cdot \sigma_e$.

The extreme value $\gamma_{I,II}$ depends on the side ratio a/h and on the auxiliary quantity $\delta = F/t\,h$, and is directly proportional to the values a/h and δ. It is combined with a maximum value *max* $\gamma_{I,\,II}$, which depends only on δ, and applies to plates of any side ratio. The limiting value $\gamma_{I,\,II}$ is plotted in terms of the side ratio of the plate, for stiffeners with $\delta = 0.20$ and 0.05, in Fig. 15. Both curves are formed of single branches correlated to the parameters m (number of sinusoidal half-waves developed in the lengthwise direction), and attain their maximum values of $\gamma_{I,\,II} = 51.60$ and 30.15 at the points a/h = 3.17 and 2.80 respectively. Stiffeners having a bending rigidity of $EJ \geqq$ (max $\gamma_{I,\,II}$). D h are sufficiently stiff against bending to compel the formation, at the lowest limit of stability, of a bulged surface having a 'nodal line' at the locus of the stiffener, even in plates of any given side ratio.

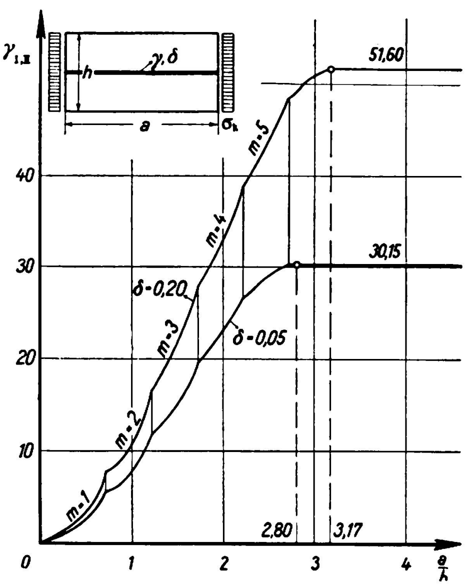

Fig. 15.
Relation between ultimate value of rigidity ratio $\gamma_{I,\,II}$ and the side ratio of the plate.

By taking the curves in Fig. 15 as a basis, there is no difficulty in developing simple proximate formulae for the extreme value $\gamma_{I,\,II}$ and for the maximum value $\gamma_{I,\,II}$ which only depends on δ.

Fig. 16 shows how the bulging coefficient of a plate of given side ratio (a/h = 1.00, 1.60 and 3.00) increases when the stiffener has the cross-sectional area F = 0.20 . t h, but different shapes of section are adopted, so that the ratio i/t assumes different values. Where a/h = 1.00, the 'Bulge Form I' applying to the lowest limit of stability forms only a single half-wave in the longitudinal direction, so that the curve applying to this case consists merely of a single branch in the region $0 \leq i/t \leq 2.16$, and the ordinate of this branch increases from k = 2.85 to k = 16.00. The curve for a/h = 1.60 is composed, in the region $0 \leq i/t \leq 3.26$, of *two* branches, since a bulge form with m = 2 half-waves applies for small values of i/t. In the region mentioned, the ordinates of this curve increase from k = 3.30 to k = 16.07. The curve for a/h = 3.00 is composed, within the region $0 \leq i/t \leq 4.83$, of three branches, because bulged surfaces with m = 3, m = 2, or m = 1 half-waves (depending on the size of i/t), form at the lowest limit of stability; within the region stated, the ordinates of the curve increase from k = 2.85 to k = 16.00.

VII. Stability of a horizontally stiffened rectangular Plate under Uniform Compressive Stress and, additionally, Shearing Stress.

If the plate investigated in Section VI. be subjected to uniformly distributed shearing stresses in addition to the uniformly distributed normal stresses (Fig. 2 f), the lengthwise section of the bulged surfaces pertaining to the limits of stability are no longer formed of single sinusoidal half-waves, and the "nodal lines" cease to be straight and at right angles to the edge of the plate. Similarly to the case of a pure shearing load, and the case of a combined compressive and shearing load in *unstiffened* plates (Section V.), the general equation governing the attainment of limits of stability falls into two conditions of bulging independent of each other, giving bulge surfaces having an odd and an even number of half-waves m′ respectively. Depending on the extent of a/h, δ and γ, either of these two bulge conditions governs the development of the *lowest* limit of stability. The solution can be represented in the form of curves $\sigma_k / \sigma_{ko} = \Psi\,(\tau_k / \tau_{ko})$ but, in this particular case, the shape of the curves depends not only on a/h, but on δ and γ as well.

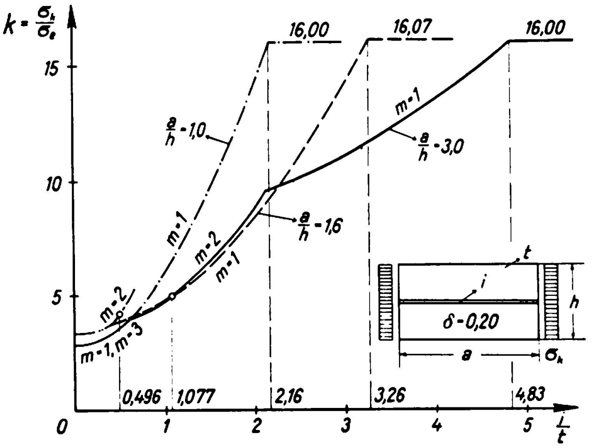

Fig. 16.
Relation between buckling coefficient k and regidity ratio $\frac{i}{t}$.

The reference quantities σ_{ko} and τ_{ko} are again the bulging strenghts applying to the horizontally stiffened plate investigated, in the case of an *exclusively* compressive load (cf. the solution to this in Section VI.), or in the case of an *exclusively* shearing stress (cf. the solution developed by *Timoshenko*[6] for this case) respectively.

Fig. 17 is the curve $\sigma_k / \sigma_{ko} = \Psi\,(\tau_k / \tau_{ko})$ for a plate having a side ratio a/h = 1.60 and a horizontal stiffener with F = 0.24 . t h and i/t = 2.00. The bulging stresses σ_{ko} and τ_{ko} have in this case the values $\sigma_{ko} = 8.83 \cdot \sigma_e$ and $\tau_{ko} = 12.72 \cdot \sigma_e$ respectively, and the number of half-waves developed by the surface of the bulge at the lowest limit of stability is in all cases m′ = 1. For purposes of comparison, the curve has also be plotted for the case of the unstiffened plate (curve 'a/h = 1.60' in Fig. 12), together with the one for the extreme case in which the stiffness against bending is sufficient to compel the formation of the 'Bulge Form II'. Since in this extreme case the plate is split up into two strips of height h/2 stressed in the same way, this curve agrees with the 'a/h = 3.20' curve in Fig. 12. We appreciate that the plate investigated

($\delta = 0.24$, $i/t = 2.00$) is 'more sensitive' to additional shearing stresses than an unstiffened plate ($\delta = 0$, $i/t = 0$) or a much stiffened plate ($\delta = 0.24$, $i/t \longrightarrow \infty$). This is probably due to the fact, in contrast to the two other cases compared, the fitting of the stiffener selected compels the formation of bulged surfaces having the same number of half-waves and a similar shape, both in the assumed case of pure compression and in the case of a purely shearing stress. In comparing the curves in Fig. 17 and assessing the 'sensitivity' just referred to, it must of course be remembered that the coordinates plotted in Fig. 17 are not the absolute values σ_k and τ_k, but the proportions σ_k/σ_{ko} and

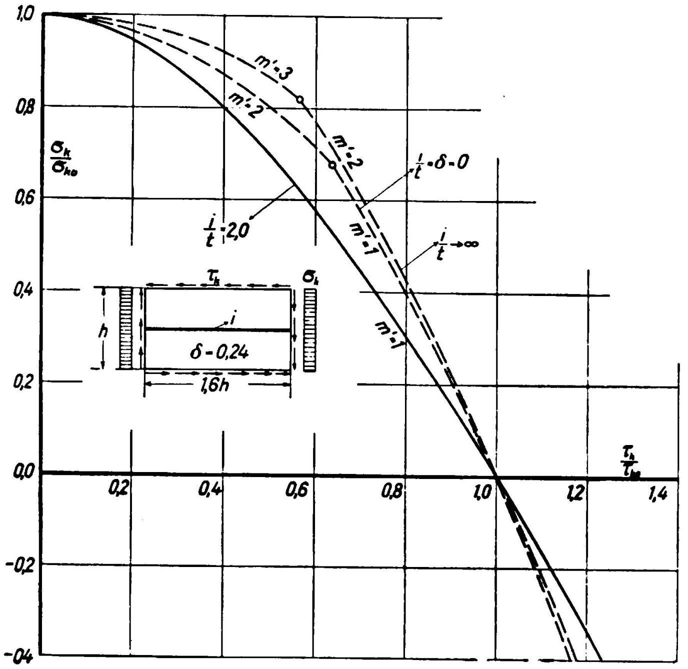

Fig. 17.
Mutual relation between buckling stresses of a stiffened plate produced by pure compression and shear efforts.

τ_k/τ_{ko}, the reference values σ_{ko}, τ_{ko} being different in the three curves, and also being much higher for the stiffened plate than they are for the unstiffed plate.

VIII. The Proximate Computation of horizontally stiffened Webs.

If we investigate a rectangular plate supported without restraint at all four edges, and stressed in a lengthwise direction by compressive stresses of maximum value σ distributed trapezoidally or triangularly, we can compute the critical stress $\sigma = \sigma_k$ applying to the lowest limit of stability, either accurately (with the aid of Timoshenko's method[6]) or approximately by replacing the irregular distribution of the compressive stress by a *uniform* distribution of the compressive stress $\sigma_m = \psi \cdot \sigma$, $\psi < 1$, and ascertaining the lowest limit of stability $(\sigma_m)_k$ for this simple case of loading. As noted by *Shizuo Ban*[14] as well the calculated value $\bar{\sigma}_k = \frac{1}{\psi} \cdot (\sigma_m)_k$ then differs very little from the true value σ_k. This result suggests the idea of computing the

[14] *Shizuo Ban:* Publications of the Int. Ass. for Bridge and Struct. Engrg., Vol. III., 1935, p. 1.

critical edge stress of a rectangular plate (Fig. 2a) subjected in its plane to a pure bending stress, approximately, in terms of a "substitute plate" which we may imagine as being cut out on the bending pressure side of the original plate, and whose height would therefore be $h' = h/2$. The triangularly distributed compressive stress acting on this substitute plate is replaced by the uniformly distributed compressive stress $\sigma' = 0{,}5 \cdot \sigma$, and all quantities relating to the substitute plate will have a tick (′) appended to distinguish them from the quantities referring to the original plate.

To simplify the proximate investigation as far as possible, we shall assume the substitute plate to be supported at all four edges like the original plate, in which case the computed quantities $\bar{\sigma}_k = \sigma'_k / 0.5 = 2 \cdot \sigma'_k$ must be reduced by approximately 30 % so as to get useful proximate values for the critical edge bending stress σ_k sought. This reduction is necessary, because, in the approximation method, the *lower* edge line of the substitute plate has to be regarded as being laterally restrained as well, whereas, when the original plate bulges (cf. Fig. 7), the elements of the plate located on this line undergo a relatively large amount of lateral deflection (bulging). If the lengthwise edges of the original plate are supported free from restraint, then the value we obtain, with the help of the substitute plate, for the minimum critical edge stress is min $\sigma'_k = 4.00 \cdot \sigma'_e = 16.00 \cdot \sigma_e$. (Curve 'a' in Fig. 4), $\bar{\sigma}'_k = 32.00 \cdot \sigma_e$, and, therefore, after making the reduction mentioned above, $0.7 \cdot \bar{\sigma}_k = 22.4 \cdot \sigma_e$ whereas the strict solution (Curve 'a' in Fig. 6) is min $\sigma_k = 23.9 \cdot \sigma_e$. If the lengthwise edges of the original plate are rigidly restrained, we get, in terms of the substitute plate, min $\sigma'_k = 6.97 \cdot \sigma_e = 27.88 \cdot \sigma_e$. (Curve 'b' in Fig. 4), $\bar{\sigma}_k = 55.76 \cdot \sigma_e$, and hence $0.7 \cdot \bar{\sigma}_k = 39.0 \cdot \sigma_e$, whereas the strict value (Curve 'b' in Fig. 6) is min $\sigma_k = 39.6 \cdot \sigma_e$.

If the plate subjected to pure bending stress which we are investigating is supported at the edges free from restraint, and strengthened at the point $y = 0.75 \cdot h$ by a *horizontal stiffener* whose axis is located in the medial plane of the plate (Fig. 2c with $\tau = 0$), our approximation method gives us a substitute plate of height $h' = 0.5\ h$, stiffened in the middle by a horizontal, centrally arranged stiffener and loaded by the uniformly distributed compressive stresses $\sigma' = 0.5 \cdot \sigma$ (Fig. 2f with $\tau = 0$). By calculating the minimum critical compressive stresses σ'_k of this substitute plate with the aid of the theory outlined in Section VI. (i.e., with the help of the simply constructed bulging condition of Equation 5), and reducing by approx. 10—20 % the quantities $\bar{\sigma}_k = \frac{\sigma'_k}{0.5}$ derived in this way, we shall also in this case arrive at useful approximate values for the critical fibre bending stress σ_k of the original plate. If, for instance, $a/h = 0.8$, $F = 0.12 . t\ h$, and $i/t = 2.00$, the figures for the substitute plate work out at $a'/h' = 1.60$, $\delta' = 0.24$, $\gamma' = 12.$ $(1 - \mu^2) \cdot \delta' \cdot (i/t)^2 = 10.48$, and, from Equation 5, the coefficient of bulging $k' = 8.83$; which gives us $\sigma'_k = k' \cdot \sigma'_e = 4 \cdot k' \cdot \sigma_e = 35.32 \cdot \sigma_e$ and $\bar{\sigma}_k = 8\ k' \cdot \sigma_e = 70.64 \cdot \sigma_e$. This computed quantity would have to be reduced by 13.8 % to agree with the true value of the critical fibre bending stress $\sigma_k = 60.86 \cdot \sigma_e$ (cf. Section III.). The solution curve obtained in this way with the help of our substitute plate is shown as a dash-line in Fig. 8. For very small values of i/t, a bulge surface with $m = 2$ half-waves is applicable to this approximate solution, so that the approximation curve is composed of two branches. Where, for instance, $m = 1$, we get for $i/t = 0$; 0.847; 2.00; 2.78 and 3.00, $k' = 3.34$, 4.34, 8.83, 13.71 and 15.43 respectively; whereas, for $m = 2$, we get for $i/t = 0$; 0.2; 0.4 and 0.6 the values $k' = 2.82$, 3.05, 3.72 and 4.83 respectively.

The critical fibre bending stress in terms of 'Bulge Form II', and the limiting value $(i/t)_{I, II}$ defined in Section III., can also be computed approximately by the method outlined. The horizontal 'nodal line' exhibited by Bulge Form II. at locus $y = 0.75\ h$ splits up the original plate into two partial plates, the top one of which has a height $h' = 0.25\ h$, and is subjected to a trapezoidally distributed compressive stress of the average value $\sigma' = 0.75 \cdot \sigma$. The top part of this plate, which we shall introduce as a 'substitute plate', is connected at its lower edge with the bottom half of the plate (which is mainly stressed under tension) and is subjected there to a kind of elastic restraint. We shall consider the effect of this elastic restraint approximately by assuming in our computation (1) an unrestrained supporting, and (2) a rigid supporting of the lower edge of the substitute plate, and take the arithmetical mean of the two bulging coefficients thus obtained. If, for example, the original plate has the side ratio $a/h = 0.8$ and a stiffener with $F = 0.12\ th$ (Fig. 8), then, according to the accurate method of solving outlined in Section III., we get the form of bulge shown in Fig. 10 with $m = 3$ half-waves in the lengthwise direction, and a critical fibre bending stress of $\sigma_k = 101.85 \cdot \sigma_e$, though the bulging strength of the plate for a bulge with $m = 4$ half-waves would only be slightly higher ($\sigma_k = 103.49 \cdot \sigma_e$) than this minimum. For the substitute plate of side ratio $a'/h' = 3.20$, and assuming the edges to be supported without restraint, $k' = 4.04$ and $m = 3$ (cf. Curve 'a' in Fig. 4), we should get $\sigma'_k = k' \cdot \sigma'_e = 16\,k' \cdot \sigma_e$ and $\bar{\sigma}_k = \dfrac{16}{0.75} k' \cdot \sigma_e = 86.19 \cdot \sigma_e$. Assuming the other edge of the substitute plate is rigidly supported, then $k' = 5.41$ and $m = 4$ (cf. Curve 'b' in Fig. 5), and we should get $\bar{\sigma}_k = \dfrac{16}{0.75} k' \cdot \sigma_e = 115.41 \cdot \sigma_e$. The arithmetical mean of the two extreme values is $\bar{\sigma}_k = 100.80\ \sigma_e$, and corresponds to a form of bulge with $m = 3$—4 half-waves in the lengthwise direction — a result agreeing satisfactorily with the above-mentioned strict solution. Branch (II) of the solution curve governed by the bulging coefficient $k = 100.80$, and shown by a dash-line in Fig. 8, intersects the dash curve (I) at the point $(i/t)_{I, II} = 2.62$, and this approximation result also agrees fairly satisfactorily with the true result of the solution $(i/t)_{I, II} = 2.780$.

The effect of additional shearing stresses can also be considered in a simple way by our approximation method. If no stiffening is present (Fig. 2b), we can estimate the effect of the additional shearing stresses on the limit of stability directly with the aid of the approximation method (approximation of the solution curves by a quarter-circle or a three-sided polygon) mentioned in Section IV. If the original plate is stiffened horizontally (Fig. 2c), the approximate method of investigation is based on a substitute plate of height $h' = h/2$

(Fig. 2f), thus arriving at the solution curves outlined in Section VII. (Fig. 17), and these can for a first approximation be replaced by a suitably selected parabola or a three-sided polygon.

IX. Bulging in the 'inelastic' Region.

If the maximum local comparison stress $(\sigma_V)_k$ present in the critical state of equilibrium (the stress calculated without regard to the weakening of the rivet holes) is higher than the limit of elasticity and proportionality σ_P of the structural steel used (the German buckling specifications provide for the figure $\sigma_P = 2.073$ t/cm² both for Steel 37 and Steel 52), the buckling conditions lose their validity in terms of Hooke's law. Assuming that the quasi-isotropy of the structural steel to be maintained even in the region between the elastic limit and the yield point, all we need to do is to replace the quotient $E/(1-\mu^2)$ in Equation (1) by a quantity T' which is smaller than $E/(1-\mu^2)$ and which we may assume to depend exclusively on the local comparison stress $(\sigma_V)_k$ present at the critical state of equilibrium. If the plate is under bending stress in its plane, then $(\sigma_V)_k$, and, hence, T' as well, becomes a function of the locus x, y (non-homogeneous stress state), and this puts considerable difficulties in the way of theoretically determining the limit of stability. If, however, we ascertain the limit of stability in terms of our approximation method with the aid of the substitute plate uniformly compressed and stressed additionally by pure bending, we then arrive at a homogeneous stress state, and T' becomes independent of the locus x, y. The functional relation between T' and the comparison stress $(\sigma_V)_k$ obtaining at the limit of stability can in this case be determined by a suitable law of approximation, bearing in mind, of course, that, for $(\sigma_V)_k = \sigma_P$, *Hooke*'s law is still applicable to the condition of bulging, and that the bulging stress for $(\sigma_V)_k = \sigma_F$ is infinitely small in view of the extensive plastification of the material. Hence, for $(\sigma_V)_k = \sigma_P$, $T' = E/(1-\mu^2)$, and for $(\sigma_V)_k = \sigma_F$, $T' = 0$.

The similarity of these limiting conditions to the limiting conditions underlying the 'buckling modulus' of a structural steel member under compression leads to the idea of reducing the determination of the critical comparison stress $(\sigma_V)_k$ of a *homogeneously* stressed rectangular plate which bulges in the 'inelastic' zone, to the *determination of a buckling stress* s_k *(Schleicher*[15]). We calculate the bulging stress of the plate investigated by using the bulging conditions outlined in the previous sections of this paper, i. e., assuming an ideal material which unreservedly obeys *Hooke*'s law of deformation, thus obtaining the ideal bulging stresses $\sigma_{k,\,id}$, $\tau_{k,\,id}$, and with the help of these values, we now work out ideal critical comparison stress $(\sigma_V)_{k,\,id}$ applying to the locus x, y. If, now, we investigate a straight, double-hinged centrically compressed member and consisting of the same structural steel as the plate, and take as the ratio of slenderness of this bar, the 'ideal ratio of

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[15] *F. Schleicher:* Final Report of the Int. Congress, Paris, 1932, p. 129 and Der Bauingenieur, Vol. 15, 1934, p. 505; cf. also *E. Chwalla:* Bericht über die II. Int. Tagung f. Brückenbau und Hochbau, Vienna, 1928, p. 321, also *M. Roš* and *A. Eichinger:* Final Rep. of Int. Congress, Paris, 1932, p. 144.

slenderness' $\lambda_{id} = \pi \cdot \sqrt{\frac{E}{(\sigma_V)_{k,\,id}}}$, the buckling stress s_k of this bar constitutes a practically applicable approximate value for the critical comparison stress $(\sigma_V)_k$ sought. From $(\sigma_V)_k$ the critical stress components σ_k and τ_k can be found by recalculation. The buckling stress s_k is determined in the form of the conventional buckling stress diagram in terms of the slenderness ratio λ, and can therefore be taken directly from these diagrams, which are officially specified for the standard structural steels. If the calculated value is $(\sigma_V)_{k,\,id} \leq \sigma_P$, then, by this method, $(\sigma_V)_k = (\sigma_V)_{k,\,id}$; and if, on the other hand, $(\sigma_V)_{k,\,id}$ increases very considerably (which it can without limits for, say, a plate under shear stress and additional tensile stress) then $(\sigma_V)_k$ approximates still more closely to the yield stress σ_F of the structural steel used. Since the bulging safety factor ν_b is different from the buckling safety factor ν_k, we ought merely to take the value s_k from the officially specified buckling stress diagrams, but not the 'permissible' compressive stress (ω - method) as well. If the stress state investigated is *unhomogeneous*, then, instead of introducing the value $(\sigma_V)_{k,\,id}$ in terms of the locus x, y into the calculation, we can introduce the maximum valve obtaining, i.e., max $(\sigma_V)_{k,\,id}$, and so we arrive at bulging stresses which are theoretically *higher* than the true values.

X. *The Dimensioning of Horizontal Stiffeners.*

The necessary cross-sectional dimensions of the horizontal stiffener have been clearly fixed by the result of the theory propounded in Section III. or the approximation investigation outlined in Section VIII., so that we can confine our further considerations to a few remarks of fundamental character. The horizontally stiffened plate secured at the edges without restraint buckles (bulges) both under a uniformly distributed compressive load, as also in the case of a pure bending stress, and, in bulging, describes a curved surface whose longitudinal section takes the form of a simple sine-line composed of m half-waves. If 'Bulge Form I.' obtains at the limit of stability investigated, then the horizontal stiffener is bent outwards in the form of a similar sine-line. Since this sine-line represents the equilibrium figure of the stiffener (imagined as being detached from the plate) under a mean compressive load of $P = \frac{m^2 \pi^2 E J}{a^2}$, we are able to advance the following argument: If the compressive force coming on to the horizontal stiffener at a limit of stability pertaining to m half-waves should happen to coincide with this value P, the stiffener *itself* will maintain the equilibrium at this limit of stability, and will therefore be unable, in this state, to affect the bulged surface of the plate in any way. The coefficient of bulging k is therefore identical with the bulging coefficient of the *unstiffened* plate in terms of the same number of half-waves m, though it is well to note that the number of half-waves developed at the *lowest* limit of stability may be influenced by the arrangement of the stiffener, and may therefore be different in the stiffened plate to what is in the unstiffened one.

In the case of a pure bending stress (Fig. 2c, = 0), the compressive stress coming on to the stiffener at the limit of stability, is equal to half the critical extreme fibre stress $\frac{1}{2}\sigma_k = \frac{1}{2} k\,\sigma_e = \frac{k\pi^2 D}{2\,h^2 t}$, so that the condition for attaining this special state where 'the stiffening has no effect' is written:

$$\frac{m^2\pi^2 EJ}{a^2 F} - \frac{k\pi^2 D}{2\,h^2 t} = 0 \tag{6a}$$

and, after inserting equation (1), assumes the form:

$$\frac{i}{t} = \left(\frac{i}{t}\right)_0 = \frac{a}{mh}\sqrt{\frac{k}{24\,(1-\mu^2)}} \tag{6b}$$

Since the limit of stability of the plate is not affected by the stiffener. the figure to be inserted for k is the coefficient of bulging applying to the *unstiffened* plate in terms of the number of half-waves m. The value $(i/t)_0$ is independent of the auxiliary quantity δ and refers to a form of bulge having m half-waves. With a plate of side ratio $a/h = 0.8$, the special state in which stiffening has no effect is attained, where $m = 1$, for $k = 24.47$ and $(i/t)_0 = 0.847$, as already mentioned in Section III. of this paper. If the stiffener is made with $i/t < (i/t)_0$: the stiffener under compression is supported by the plate in the state of critical equilibrium, thus reducing the bulging strength of the plate. If, on the contrary, the stiffener is made with $i/t > (i/t)_0$, it is capable of supporting the plate and raising the bulging strength.

If the plate is under *uniformly distributed* compressive stresses (Fig. 2f with $\tau = 0$), the compressive stress coming on to the horizontal stiffener when the plate bulges is as high as the critical compressive stress $\sigma_k = k \cdot \sigma_e = \frac{k\pi^2 D}{h^2 t}$, so that the condition for attaining the special state where the stiffening has no effect is:

$$\frac{m^2\pi^2 EJ}{a^2 F} - \frac{k\pi^2 D}{h^2 t} = 0 \tag{7a}$$

The coefficient of bulging k, which again refers to an *unstiffened* plate bulging in the form of m half-waves, is $k = \frac{m^2 h^2}{a^2}\left(1 + \frac{a^2}{m^2 h^2}\right)^2$ (Curve 'a', Fig. 4), so that, after introducing Equation (1), we can write Equation (7a) in the following form as well:

$$\frac{i}{t} = \left(\frac{i}{t}\right)_0 = \frac{1 + \frac{a^2}{m^2 h^2}}{\sqrt{12\,(1-\mu^2)}} \tag{7b}$$

Here, too, the value $(i/t)_0$ is independent of the auxiliary quantity δ, and re fer to a form of bulge with the selected number of half-waves m. With a plate of side ratio $a/h = 1.60$, the special state in which the stiffening is ineffective is reached, in the case of $m = 1$, for $k = 4.9506$, $(i/t)_0 = 1.077$; and where $m = 2$, for $k = 4.2025$, $(i/t)_0 = 0.496$. The two points whose coordinates are determined by these two pairs of values, have been specially stressed in Fig. 16 (curve 'a/h = 1.60').

The special state of the 'ineffectuality of the stiffening' refers, as already pointed out, exclusively to investigated bulge form having the number of half-waves m inserted in equations (6b) and (7b). A removal of the 'ineffectual' stiffening is therefore only without influence on the critical state of equilibrium in cases where m coincides with the number of half-waves formed at the lowest limit of stability of the *unstiffened* plate. If we investigate a plate of side ratio $a/h = 1.60$, then, for the unstiffened plate, the lowest limit of stability is reached for $m = 2$ and $\sigma_k = 4.2025 \sigma_e$ (Curve 'a', Fig. 4). If, now, we reinforce this plate by a stiffener with $i/t = 0.496$, then, as may be inferred from the fact that Equation (7b) is satisfied, we arrive at a special state where the stiffening has no effect, and $k = 4.2025$, as before, but this critical state of equilibrium is no longer related to the *lowest* limit of stability. It follows from Fig. 16 that the bulging of the plate strengthened in this manner ($\delta = 0.20$, $i/t = 0.496$) occurs at the compressive load $\sigma_k = 3.832 \cdot \sigma_e$ rather than at the compressive load $\sigma_k = 4.2025 \cdot \sigma_e$, a bulged surface with only a single half-wave ($m = 1$) being formed. The stiffener is then supported by the bulging plate [the value of $i/t = 0.496$ is considerably smaller than the special value $(i/t)_o = 1.077$ than for the case of $m = 1$], so that the limit of stability is *lowered* and not raised by the arrangement of stiffener mentioned. If, instead of making the stiffener with $i/t = 1.077$ instead of with $i/t = 0.496$, then $\sigma_k = 4.9506 \cdot \sigma_e$ and we arrive at the special state of 'ineffectual stiffening' relating to *m = 1 half-waves.* Here, too, we must remember that the 'ineffectualness' relates exclusively to a type of bulging with a specific number of half-waves (here $m = 1$), and that this number of half-waves and, hence, the smallest possible coefficient of bulging as well, can be altered by removing the stiffening. The particular plate investigated would, in the unstiffened state, bulge under the compressive load $\sigma_k = 4.2025 \cdot \sigma_e$ and in this case the number of half-waves would be $m = 2$. Due to the application of the stiffener mentioned, the lowest limit of stability of the plate is appreciably raised, despite a special state being attained where the stiffening has no effect; and this increase is due simply and solely to the reduction in the number of half-waves from $m = 2$ to $m = 1$ caused by the stiffening.

From the results of all these arguments, it may be inferred that the influence of horizontal stiffening upon the lowest limit of stability in a rectangular plate subjected to uniform compression or pure bending, can usually not be deduced directly from the buckling strength of the stiffener (assumed to be detached from the plate) when it buckles at right angles to the plane of the plate. Hence it is not advisable to dimension the horizontal stiffener exclusively by regarding it as being released from the plate, and to give it a sufficient margin of safety against buckling at right angles to the plane of the plate, in terms of the official buckling specifications (ω - Method). This specification is based on the buckling of a member in the form of a single sinusoidal half-wave ($m = 1$), and on a very definite buckling safety factor, ν_k, which partly depends on the slenderness of the member; so that, depending on the side ratio of the plate and the bulging safety factor ν_b called for, very different effects can be obtained on the limit of stability and, under certain conditions, unsatisfactory dimensions for the stiffeners as well. If the buckling safety factor of the stiffener dimensioned as stated happens to be as high as the bulging safety factor of the plate, and if the side ratio of the plate is such that the *unstiffened* plate bulged in a form with $m = 1$ half-waves ($a/h = 1.41$ for a uniformly distributed compressive stress, and $a/h = 0.95$ for a pure bending stress), then the limit of stability of the plate is not the slighest bit higher than that of the unstiffened plate, despite the presence of the stiffening, which means that the efficiency of the stiffening is nil. Only when the buckling safety factor of the stiffener is higher than the required bulging safety factor of the plate, or the side ratio of the plate is large enough to permit the plate in the unstiffened state to bulge into a curved surface with *more than one* half-wave, is there a surplus of bending rigidity (stiffness against bending) which favourably affects the stability of the plate.

By putting the permissible compressive stress of the plate equal to that of the stiffener we receive a rigidity ratio of

$$\frac{i}{t} = m\sqrt{\frac{\nu_k}{\nu_b}} \cdot \left(\frac{i}{t}\right)_0$$

This ratio is higher than the limiting ratio $(i/t)_0$ and guarantees an increase of the resistance against warping (bulging), however, under certain circumstances this increase is only slight.

In conclusion, then, we find that horizontal stiffeners under compression can only be rationally dimensioned on the basis of the stability theory outlined in Section III. or the approximation method outlined in Section VIII. In this connection it will be found advisable to draw up simple approximation formulae for the extremes $(i/t)_{I,II}$ and max $(i/t)_{I,II}$ (compare also the limiting values $\gamma_{I,II}$ and max $\gamma_{I,II}$ mentioned in Section VI. in connection with Fig. 15). By dimensioning the stiffener (regarded as being separated from the plate) for buckling at right angles to the plane of the plate (for $\nu_k \geqq \nu_b$), the object in dimensioning the stiffener will be achieved inasmuch as it will be possible definitely to exclude the unfavourable case where the stiffener is supported by the bulging plate and therefore reduces the bulging strength of the plate rather than increasing it.

Summary.

The web of a solid girder is mainly stressed by bending, and only to a slight extent by shear in the region of the middle bays. It should be dimensioned and designed so that, under the service load, *permanent deformations* (ignoring the concentrations of stress at the rivet holes) and *lateral bulging* are avoided. To obviate local plastification, the ideal 'comparison stresses' occurring in the web must be prevented from attaining the yield point stress of the structural steel used (Section I. of the paper), and to prevent the web from bulging (buckling) prematurely, it should be impossible for it to

reach its lowest limit of stability under the service load. The theoretical determination of this limit of stability is bound up with far-roaching assumptions as regards the geometrical and material properties of the web-plate, and as regards its location and loading (Section II. of the paper); so that, in order to allow for the unavoidable discrepancy between supposition and fact, we have introduced a bulging safety factor v_b. This factor may as a rule be smaller than the average buckling safety factor v_k for structural steel members under compression, since the supporting capacity of peripherally supported plates is only exhausted considerably above the lowest limit of stability, owing the considerable deformation of the middle surface when bulging occur. The bulging strength is usually increased by stiffeners. As the webs are subjected mainly to bending stresses, horizontal stiffeners applied on the bending pressure side have been found suitable from the viewpoint of theoretical stability. Since these stiffeners are rivetted or welded to the web plate, they are stressed by axial compressive forces which must be fully allowed for when ascertaining the effect of stiffening on the limit of stability of the plate.

For ascertaining the theoretical behaviour of webs of this kind, an investigation has been made into the stability of a rectangular plate with edges supported free from restraint, stressed by bending in its plane, and stiffened by horizontal stiffeners (Section III.). Since, in practice, the occurrence of additional shearing stresses must be reckoned with, the effect of these shearing stresses on the limit of stability has been briefly outlined (Section IV.). For the proximate determination of the stability of horizontally stiffened webs, it is usual to imagine a strip of plate being out on the bending pressure side of the web. The triangularly or trapezoidally distributed compressive stresses acting on this strip of plate (termed the 'substitute plate') are replaced by mean uniformly distributed compressive stresses. The bulging strength of the web is then deduced from the bulging strength of the substitute plate. The determination of the lowest limit of stability of the substitute plate requires far less work than the elucidation of the stability of the web. The method is briefly outlined in Section V. for the case of a uniformly distributed compressive and shearing load, in Section VI. for the case of a uniformly distributed compressive load and the arrangement of a horizontal middle stiffener, and, finally, in Section VII. for the case of a uniformly distributed compressive and shearing load and a horizontal middle stiffener. If the same edge conditions are laid down for the substitute plate as are prescribed for the web-plate investigated, the approximation method gives critical values of extreme fibre stress which are theoretically higher than the true values (Section VIII.). This over-estimation of the bulging strength is due to the fact that a lateral fixation is assumed along the lower edge of the substitute plate, whereas the points of the middle surface of the plate located on this edge line actually (when the web bulges) undergo considerable lateral displacements.

Section IX. is devoted to a study of 'inelastic bulging', which occurs when the maximum ideal 'comparison stress' occurring in the state of critical equilibrium exceeds the limits of proportionality and elasticity of the structural steel used. In terms of the approximation method outlined, the substitute plate is only subjected to uniformly distributed compressive and shearing stresses, so that the stress state obtaining is homogeneous and the 'comparison stress' occurring at the critical state of equilibrium is independent of the locus. Assuming the quasi-isotropic state of the steel to be maintained in the region between the elastic limit and the yield point, the determination of this 'critical comparison stress' can be reduced to a determination of the buckling stress of a *centrically compressed* bar of given "ideal" slenderness ratio.

The final Section X. contains a few fundamental remarks with regard to the problem of the design of horizontal stiffeners. The influence of stiffeners on the stability of the plate, and the dimensioning of these stiffeners can only be elucidated in a rational way in terms of the results of the stability investigation. In this connection it is found advisable to fix simple approximation formulae for the extreme values of the stiffness ratio that are of practical significance. By dimensioning the stiffener purely on the assumption that it is separated from the plate, and allowing a sufficient margin of safety against buckling (bulging) at right angles to the plane of the plate in terms of the official buckling specifications, it is possible to avoid the unfavourable case where the stiffener is supported on the bulging plate and the limit of stability is lower than the limit of stability of the *unstiffened* plate; but depending on the side ratio of the plate and the bulging safety factor required, different efficiencies are obtained for this stiffening which it is impossible to elucidate directly.

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