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Stability of Rectangular Plates Under Shear and Bending Forces.

Die Stabilität rechteckiger Platten unter Schub﹣ und Biegebeanspruchung.

La stabilité des plaques rectangulaires soumises au cisaillement et à la flexion.

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1) Introduction.

In the design bridge, ship and aircraft structures, problems arise having to do with the stability of rectangular plates with various types of edge loading.[1] With loading higher than a certain critical value lateral deflection from the initial plane of the plate takes place. It may sometimes be permissable for a structure to carry a load higher than the critical value but a knowledge of the critical load is always desirable.
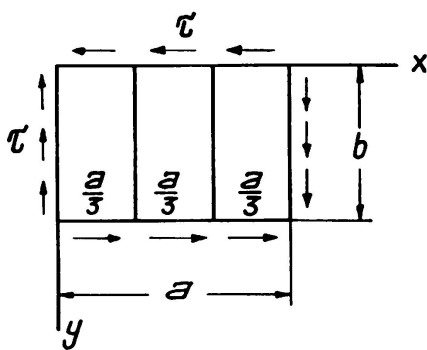
Fig. 1.
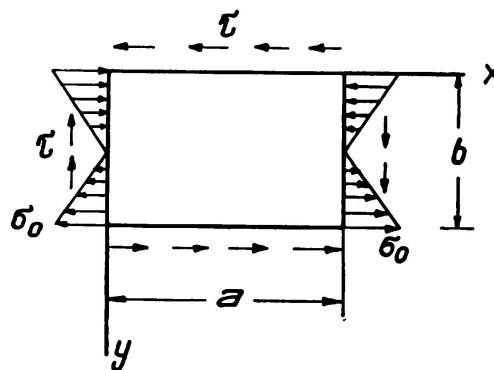
Fig. 2.

Two problems in the buckling of rectangular plates are discussed in this paper. The first, Fig. 1, is that of a plate having two stiffeners. The loading consists of uniformly distributed shearing forces on the edges. The second problem, Fig. 2, is that of a plate loaded by uniform edge shear and linearly distributed tension and compression at the ends. In both cases we assume all four edges are simply supported.

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[1] An extensive bibliography on the stability of plates has been given by *O. S. Heck* and *H. Ebner*, Luftfahrtforschung, Vol. 11, 1935, p. 211.

The method employed is the energy method, in which the critical load is calculated from the condition that the work of the edge forces during buckling is equal to the stored elastic energy. The form of the deflection must be that which makes the critical load a minimum.

2) *The Plate in Shear with Two Stiffeners.*

The problem of a rectangular plate in shear having no stiffeners and that of a plate with one stiffener have been solved by *Timoshenko*,[2] while *Southwell*[3] and *Skan* have treated the case of an infinitely long strip with edge shearing forces. Transverse stiffeners serve to increase the critical load value for a plate, and the greater the rigidity of the stiffeners the greater is this increase, in general. It is found, however, that beyond a certain point increasing the rigidity of the stiffeners in a plate with edge shear loading is useless, as the stiffeners will remain straight and only the panels of the plate will deflect.

To solve the problem of the plate shown in Fig. 1, a general expression is assumed for the deflection surface in the form of the double trigonometric series

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{\pi m x}{a} \sin \frac{\pi n y}{b} \tag{1}$$

each term of which satisfies the boundary condition of simply supported edges. Using this expression three energy quantities may be calculated, the elastic energy of the plate V, the elastic energy of the stiffeners $V_{s1} + V_{s2}$, and the work done by the edge shearing forces during buckling, V_1. The torsional rigidity of the stiffeners will be neglected. These three energy quantities in terms of the derivatives of w are:

$$V = \frac{D}{2} \int_0^a \int_0^b \{(w_{xx} + w_{yy})^2 - 2(1 - \mu)(w_{xx} w_{yy} - w_{xy}^2)\} \, dx \, dy \tag{2}$$

$$V_{s1} + V_{s2} = \frac{B}{2} \int_0^b \left\{ \left(w_{yy}\right)^2_{x=\frac{a}{3}} + \left(w_{yy}\right)^2_{x=\frac{2a}{3}} \right\} \, dy \tag{3}$$

$$V_1 = -\tau h \int_0^a \int_0^b w_x w_y \, dx \, dy \tag{4}$$

where h is the thickness of the plate, B the flexural rigidity of the stiffeners and D the flexural rigidity of the plate,

$$D = \frac{E h^3}{12(1 - \mu^2)} \tag{5}$$

[2] *S. Timoshenko:* Eisenbau, Vol. 12 (1921), p. 147.

[3] *R. V. Southwell* and *S. W. Skan*, Proc. Royal Society, London, Series A, Vol. 105 (1924), p. 582.

The condition for buckling is that

$$V + V_{s1} + V_{s2} = V_1. \tag{6}$$

Equation (6) leads to an expression for τ_{cr}, the critical shearing stress, in terms of the constants A_{mn}. The next step is to minimize τ_{cr} by differentiation with respect to the constants. The equations $\delta\tau_{cr}/\delta A_{mn} = 0$ lead to a set of linear equations for the constants A_{mn}. The value of τ_{cr} is finally obtained by equating to zero the determinant of this system. The greater the number of terms considered in the infinite order determinant, the greater will be the accuracy of the τ_{cr} value calculated.

It happens that the linear equations obtained by minimizing τ_{cr} with respect to the constants A_{mn} consist of two groups, one group containing terms with $m + n$ odd and the other group containing terms with $m + n$ even. That group of equations should be used which leads to the lowest value of τ_{cr}.

It is convenient to measure the critical load by the ratio τ_{cr}/σ_e, where σ_e is the Euler stress $D\pi^2/hb^2$. In Table I values of τ_{cr}/σ_e are given as found by calculation from sixth order determinants for the two groups of equations, for the particular case when $B = 0^4$. The ratio $\dfrac{a}{b}$ is designated by β

Table I.

τ_{cr}/σ_e Values for Various β Values. $B = 0$.

β	1	1.2	1.5	2	2.5	3
τ_{cr} / σ_e $m + n$ even	9.42	8.06	7.14	6.59	6.32	6.14
τ_{cr} / σ_e $m + n$ odd	11.55		8.09	6.74	6.21	6.04

Values of τ_{cr}/σ_e for β less than unity can easily be derived from the above values and are given in Table II. The lowest τ_{cr}/σ_e value of the two is given.

Table II.

τ_{cr}/σ_e for $\beta < 1$. $B = 0$.

β	1	0.833	0.667	0.500	0.400	0.333
τ_{cr} / σ_e	9.42	11.60	16.06	26.40	38.80	54.40

For any appreciable amount of stiffening the system with $m + n$ odd leads to the least values of τ_{cr}. We equate to zero the determinant of the coefficients of the constants

$$A_{21}, \; A_{12}, \; A_{32}, \; A_{23}, \; A_{41}, \; A_{14}$$

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[4] The values for $m + n$ even agree with those obtained by *Timoshenko* (note 2). The terms used for $m + n$ even were A_{11}, A_{22}, A_{33}, A_{13}, A_{31} and A_{42}, and for $m + n$ odd A_{21}, A_{12}, A_{32}, A_{23}, A_{41}, A_{14}.

and the resulting equation enables us to calculate τ_{cr}/σ_e directly when β and the stiffener rigidity are given. The stiffener rigidity is, for convenience, measured by the ratio $\gamma = B/aD$. Values of τ_{cr}/σ_e are given in Table III. It will be noted that increasing the stiffener rigidity increases the load necessary to buckle the plate.

Table III.

τ_{cr}/σ_e Values for Various β and γ Values.

$\beta = 1$		$\beta = 1,2$		$\beta = 1,5$		$\beta = 2$		$\beta = 2,5$		$\beta = 3$	
γ	τ_{cr}/σ_e	γ	τ_{cr}/σ_e	γ	τ_{cr}/σ_e	γ	τ_{cr}/σ_e	γ	τ_{cr}/σ_e	γ	τ_{cr}/σ_e
0	11.55	20	36.7	0	8.09	0	6.74	0	6.21	0	6.04
10	32.75	25	40.1	5	19.43	2	13.07	1	10.3	0.2	7.29
20	41.6	30	43.2	10	25.2	5	18.2	2	13.1	0.4	8.32
30	48.5			15	29.5	10	23.8	3	15.1	0.6	9.21
40	54.4									0.7	9.59

If the stiffeners are made very rigid the calculated critical load for the plate will be greater than the critical load for each of the three panels. The situation then is such that the stiffeners will remain straight and only the panels of the plate will buckle. The condition that the stiffeners remain straight is that their rigidity be greater than that necessary to make the critical load for the plate equal that for one panel. The critical load for one panel is calculated by making the assumption that all edges are simply supported. Actually, each panel is partially constrained by the adjacent panel or panels.

We let γ_{min} be the minimum value of γ for which the stiffeners will remain straight. To illustrate the method of calculation we shall consider the case of $\beta = 1.2$. The β value for one panel of this plate is 0,400 and the corresponding critical load ratio is 38.8 as given in Table II. By plotting graphically the values given for τ_{cr}/σ_e with $\beta = 1.2$ in Table III we find that $\tau_{cr}/\sigma_e = 38.8$ when γ is 23. Hence γ_{min} is 23 for $\beta = 1.2$. In the same manner values of γ_{min} for other values of β may be found.

It is useful to present the results in terms of the dimensions of one panel instead of the dimensions of the plate. Let c be the distance between the stiffeners. We introduce the symbols β' and γ' defined by

$$\beta' = \frac{c}{b} \qquad \gamma' = \frac{B}{cD}.$$

In Table IV, values of τ_{cr}/σ_e and γ'_{min} are given for various panel ratios. These values have been plotted in the curves in Fig. 3. We note that the required stiffener rigidity for two stiffeners is not very much greater than the required rigidity for one stiffener. For three or more stiffeners, the γ'_{min} value would probably be only very slightly larger than the value for two stiffeners.

Table IV.

One Stiffener			Two Stiffeners		
β'	γ'_{min}	τ_{cr} / σ_e	β'	γ'_{min}	τ_{cr} / σ_e
0.500	30.4	26.4	0.333	120	54.4
0.625	12.6	17.9	0.400	69	38.8
0.750	5.8	13.3	0.500	34	26.4
1.000	1.66	9.42	0.667	10.8	16.06
			0.833	4.2	11.61
			1.000	2.0	9.42

8) Plates Loaded by Combined Shear and Bending Forces.

If, as shown in Fig. 2, the plate is loaded by uniformly distributed edge shearing stress, τ, and linearly distributed tension and compression at the ends, $\sigma = \sigma_0 (1-2 y/b)$, the critical bending stress $\sigma_{0\,cr}$ will depend upon the magnitude of the shearing stress. Similarly, the critical shearing stress may be said to depend upon the magnitude of σ_0. For convenience we introduce the parameters $\varkappa$ and ρ defined as follows:

$$\varkappa = \frac{\sigma_{0\,cr}}{\sigma_e} \qquad \rho = \frac{\tau}{\sigma_e}.$$

To solve the problem, the expression (1) which satisfies the boundary conditions for simply supported edges may again be assumed for the buckling deflection. The elastic energy of the plate after buckling will be given, as before, by Equation (2). The work V_1 of the edge forces during buckling will be, in this case,

$$V_1 = \frac{h}{2} \int_0^b dy \int_0^a \sigma_0 \left(1 - \frac{2 y}{b}\right) w^2_x \, dx - \tau h \int_0^a \int_0^b w_x w_y \, dx \, dy \qquad (7)$$

The condition for buckling is that $V = V_1$. This leads to an expression for $\sigma_{0\,cr}$, which is then minimized with respect to the constants A_{mn}. The equations $\delta \varkappa / \delta A_{mn} = 0$ became a linear system in A_{mn}. The magnitude of $\varkappa$ is obtained by equating to zero the determinant of this system. The order of the determinant used determines the accuracy of the result. We here use the determinant of the coefficients of the eight[5] terms A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}, A_{31}, A_{33}, and the calculated values of $\varkappa$ for various values of β and ρ are as given in Table V.

These values have been plotted in the curves in Fig. 4. When $\varkappa = 0$ we have the condition for the buckling of a plate in pure shear, while for $\rho = 0$ we have

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[5] Stein made calculations with four terms. Considerably lower $\varkappa$ values are obtained above by using eight terms. O. Stein: Der Stahlbau, Berlin, Vol. 7 (1934), p. 57.

Table V.

Values of ϰ and ρ For Various β Values.

β = 1		β = ⁴/₅		β = ²/₃		β = ¹/₂	
ρ	ϰ	ρ	ϰ	ρ	ϰ	ρ	ϰ
0	25.6	0	24.5	0	23.9	0	25.6
2	24.6	4	22.8	4	23.05	4	25.4
4	22.2	8	17.7	8	20.35	8	24.3
6	18.4	10	13.25	12	15.23	12	22.55
8	12.4	11	10.01	14	11.01	16	19.94
9	6.85	12	4.61	15	8.0	20	16.13
9.42	0	12.26	0	16.09	0	24	10.26
						26	5.44
						26.9	0

the condition for the buckling of a plate with bending forces at the ends. The values of ρ for ϰ = 0 agree very closely with those obtained by *Timoshenko* for the case of pure shear. The slight disagreement for small values of β is due to the fact that among the eight terms used in these calculations there are only five m + n even terms, while *Timoshenko* uses six terms. For pure shear and β = ¹/₂ the difference in the values of τ_{cr}/σ_e for five and six term calculations is only 2 %. For pure bending (ρ = 0) the values of ϰ agree with those obtained by *Timoshenko* using three terms.

4) *Numerical Examples.*

Suppose we have a plate with edge shear loading which we wish to reinforce with two stiffeners. Let a = 2000 mm, b = 1000 mm, h = 7 mm, E = 21,000 kg/mm², μ = 0.3. Let it be required to find the load causing the plate to buckle and the proper rigidity for the stiffeners.

$$D = \frac{21,000 \cdot 7^3}{12\,(1 - 0.09)} = 660,000 \text{ kg mm,}$$

$$\sigma_e = \frac{660,000 \cdot \pi^2}{1,000,000 \cdot 7} = 0.93 \text{ kg/mm}^2.$$

For β = 2 we have β' = 0.667, and we have from Table IV, γ'_{min} = 10.8 and τ_{cr}/σ_e = 16.06 from which τ_{cr} = 14.94 kg/mm² and B = 10.8 · 0.667 · 660,000 = 4750 · 10⁶ kg/mm². If one stiffener were used, the critical shearing stress would be 8.77 kg/mm², and the required stiffener rigidity 1096 · 10⁶ kg · mm².

As a second example, take the case of the end portion of the web of a plate girder. Let the depth b, be 2000 mm and the thickness h = 8 mm. Let it be required to find the proper stiffener spacing for the end of the girder and also the stiffener rigidity so that buckling will occur when τ = 10 kg/mm².

$$D = \frac{21,000 \cdot 8^3}{12 \cdot 0.41} = 985,000 \text{ kg mm}$$

$$\sigma_e = \frac{985,000 \cdot \pi^2}{4 \cdot 10^6 \cdot 8} = 0.3035 \text{ kg/mm}^2$$

$$\tau_{cr} = 10 \text{ kg/mm}^2$$

$$\frac{\tau_{cr}}{\sigma_e} = 32.9.$$

By Fig. 3 we see that $\beta' = 0.44$ so that the proper stiffener spacing is given by

$$c = 2000 \cdot 0.44 = 880 \text{ mm.}$$

To find the stiffener rigidity we assume that every third stiffener in the girder is perfectly rigid. We then use the curve for γ'_{min} in Fig. 3 for two stiffeners and find $\gamma'_{min} = 50$. The required stiffener rigidity is therefore

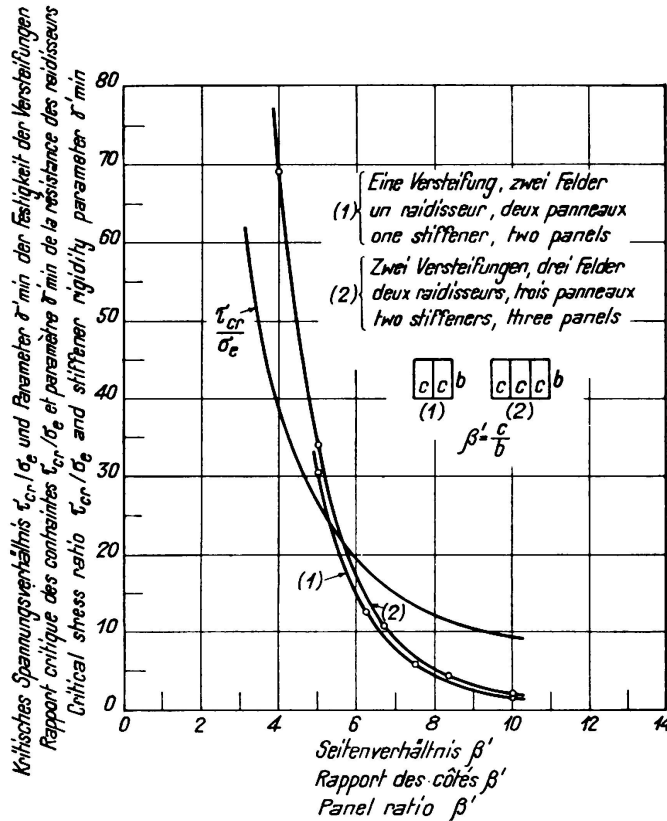$$B = 50 \cdot 880 \cdot 985,000 = 43,300 \cdot 10^6 \text{ kg} \cdot \text{mm}^2.$$

Fig. 3.

Required stiffener rigidity for given panel proportions.

As a third example, suppose we have a rectangular panel with a = 1000 mm, b = 2000 mm, h = 10 mm which is loaded by bending and shearing forces as

shown in Fig. 2. Let the bending stress, σ_o, be 10 kg/mm² and let it be required to find the shearing stress which will cause buckling. In this case $\beta = {}^1/_2$ and

$$D = \frac{21,000 \cdot 10^3}{12 \cdot 0.91} = 1,923,000 \text{ kg mm}$$

$$\sigma_e = \frac{1,923,000 \cdot \pi^2}{4 \cdot 10^6 \cdot 10} = 0.474 \text{ kg/mm}^2$$

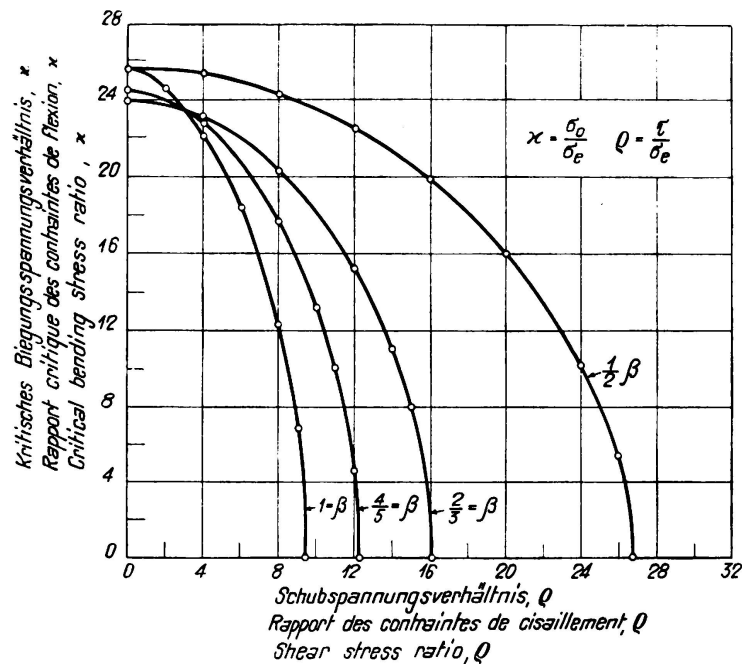$$\varkappa = \frac{10}{0.474} = 21.1.$$

Fig. 4.

Critical loads combined shear and bending.

From Fig. 4 the value of ρ for $\beta = {}^1/_2$ and $\varkappa = 21.1$ is 14.3. The corresponding value of τ is $14.3 \cdot 0.474 = 6.78$ kg/mm².