Investigation of the buckling of a parabolic arch in a Vierendeel girder under compression

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Investigation of the Buckling of a Parabolic Arch in a Vierendeel Girder under Compression.

Untersuchung über das Ausknicken des parabelförmigen Druckgurtes eines Vierendeel-Trägers.

Etude du flambage d'ensemble de l'arc parabolique comprimé d'une poutre Vierendeel.

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The parabolic arch of a Vierendeel girder, like that of a bowstring girder, is subject to conditions as regards buckling which are difficult to define exactly, for not only is it built in at its ends which form part of the lower (or tensile) boom and of the roadway structure, but at the same time it is rigidly fixed to the verticals which are themselves very stiff and which connect with the roadway structure in the form of inverted portals.

It may nevertheless be assumed, at any rate in the central portion of the arch where the height of the verticals varies only slightly, that the arch acts like a straight boom carried on verticals of constant height. By treating the arch within this zone as if it were an ordinary straight boom, and by calculating the load and the buckling length according to the well known methods of *Engesser*, *Timoshenko*, *Pigeaud*, etc., it will thus be possible to arrive at a limiting stress.

With a view, however, to more exact understanding of the problem, attempts have been made to apply *Timoshenko*'s method taking due account of the end fixations and assuming a shape of the deformed axis which agrees well enough with the truth but is also simple enough to avoid unduly complicating the calculations. Moreover, since the phenomenon of buckling will tend to make its appearance in the central zone where the verticals are highest, it has been assumed as a further simplification that the vertical members in question are of constant height.

The numerical results show that the lengths of waves caculated in this way justify the assumptions made. In studying ordinary buckling effects in a straight piece, intermediately between joints, the deformed axis is compared to a simple sinusoid; here, in order to take account of the conditions of end fixation, the shape of the deformed axis was chosen in accordance with the equation

$$y = f\left(\sin \pi \frac{x}{L} - K \sin 3\pi \frac{x}{L}\right)$$

wherein the arch, supposed to be straightened along the median plane of the beam, is taken as the x-axis; the origin is taken at one of the ends, and the y-axis (parallel to which the deflections are measured) is taken at right angles to the plane of the girder. It is further assumed that the thrust is constant trough-out the arch between the two supports.

The principle underlying the approximate method of *Timoshenko* is that of equating the external work performed by the thrust with the work performed by the ends of the verticals and by the internal stresses.

Work due to thrust.

The displacement of the thrust is equal to the difference in length between the deformed arch and its projection:

$$\Delta x = \int_0^L (ds - dx), \quad ds = \left(1 + y'^2\right)^{\frac{1}{2}} dx$$

where, approximately, $ds = \left(1 + \frac{1}{2} y'^2\right) dx$

$$ds - dx = \frac{1}{2} y'^2 \, dx$$

$$\int_0^L (ds - dx) = \Delta x = \frac{1}{2} \int_0^L y'^2 \, dx$$

$$T_Q = Q \cdot \Delta x.$$

Resisting work performed by the ends of the uprights.

It will be assumed that this thrust is continuous along the arch (*Engesser*). It should be noted that this approximation has much better justification in the Vierendeel girder than in the lattice girder, for in the former such continuity is largely realised by the enlargement of the verticals at top and bottom, and the assumption made is more likely to be true than that of isolated reactions. The thrust at the end of a vertical is conditioned by two terms which depend on the stiffness of the vertical member itself and on the stiffness of the cross girder into which it frames.

If no account is taken of the reinforcing effect of the connecting gusset and h is the height of the vertical member; p the span of the cross girder; I_h and I_p the moments of inertia of these respective members ($\varepsilon = EI$), and Σf is the deflection of the head of the vertical member under a load of one tonne, we obtain

$$\Sigma f = \frac{h^2 \cdot p}{2 \, \varepsilon_p} + \frac{h^3}{3 \, \varepsilon_h}.$$

Moreover, the unit reaction continuously distributed is Cy where y is the deformation at a given point and λ is the width of the panel, whence

$$\lambda C \Sigma f = 1 \quad \text{and} \quad C = \frac{1}{\lambda \Sigma f}.$$

The work done by the sides will be

$$\int_0^y\int_0^L C\,y\,dy\,dx = \frac{C}{2}\int_0^L y^2 dx.$$

Work due to internal stresses.

When account is taken only of the elastic energy of bending, we obtain the expression

$$T_1 = \frac{1}{2}\int_0^L \frac{M^2}{\varepsilon}\,dx$$

or expressing M in terms of y',

$$T_1 = \frac{1}{2}\int_0^L \varepsilon y''^2 dx = \frac{\varepsilon}{2}\int_0^L y''^2 dx,$$

I being assumed constant.

General equation.

As indicated above, the deformed shape of the neutral axis was taken to be in accordance with the equation

$$y = f\left(\sin\pi\frac{x}{L} - K\sin 3\pi\frac{x}{L}\right)$$

where $y = 0$ at the two ends and $y = f\ (1 + K)$ at the centre

$$y' = f\frac{\pi}{L}\left(\cos\pi\frac{x}{L} - 3\,K\cos 3\pi\frac{x}{L}\right)$$

$$\int_0^L y'^2 dx = f^2\frac{\pi^2}{L^2}\cdot\frac{L}{2}(1 + 9\,K^2)$$

$$\int_0^L y^2 dx = f^2\frac{L}{2}(1 + K^2)$$

$$y'' = -f\frac{\pi^2}{L^2}\left(\sin\pi\frac{x}{L} - 9\,K\sin 3\pi\frac{x}{L}\right)$$

$$\int_0^L y''^2 dx = f^2\frac{\pi^4}{L^4}\cdot\frac{L}{2}(1 + 81\,K^2).$$

The fundamental relationship may be written

$$P\cdot\frac{1}{4}\frac{f^2\pi^2}{L}(1 + 9\,K^2) = \frac{C f^2 L}{4}(1 + K^2) + \frac{1}{4}f^2\frac{\pi^4}{L^3}\varepsilon(1 + 81\,K^2)$$

And finally:

$$\underline{P = \frac{\varepsilon\pi^2}{L^2}\,\frac{1 + 81\,K^2}{1 + 9\,K^2} + C\frac{L^2}{\pi^2}\,\frac{1 + K^2}{1 + 9\,K^2}}.$$

For the case of simple buckling this takes the form

$$P = \frac{\varepsilon\pi^2}{L^2} + C\frac{L^2}{\pi^2}.$$

including a first general term of *Euler*'s formula and a complementary term obtained from the ends of the vertical members. On determining the minimum value of P in accordance with the rule

$$\frac{dP}{dL} = 0$$

the following results are obtained, L being the buckling length:

$$C \cdot \frac{L_1^2}{\pi^2} = \frac{\varepsilon\pi^2}{L_1^2}$$

The buckling load

$$P_1 = 2\frac{\varepsilon\pi^2}{L_1^2} = 2\frac{C \cdot L_1^2}{\pi^2}; \quad P_1 = 2\sqrt{C \cdot \varepsilon}; \quad L_1 = \pi\sqrt[4]{\frac{\varepsilon}{C}}; \quad \varepsilon = E \cdot I.$$

To fix the minimum value for P according to the new hypothesis we may write

$$P = \frac{\varepsilon\pi^2}{L^2} \cdot A + \frac{CL^2}{\pi^2} \cdot B.$$

The condition $\frac{dP}{dL} = 0$ is expressed by

$$B \cdot \frac{CL_2^2}{\pi^2} = A \cdot \frac{\varepsilon\pi^2}{L_2^2}$$

$$L_2^4 = \pi^4 \cdot \frac{A}{B} \cdot \frac{\varepsilon}{C}$$

$$L_2 = \pi \cdot \sqrt[4]{\frac{A}{B}} \cdot \sqrt[4]{\frac{\varepsilon}{C}}$$

$$P_2 = \frac{\varepsilon\pi^2}{L_2^2}\left(1 + \frac{A}{B}\right)$$

$$A = \frac{1 + 81\,K^2}{1 + 9\,K^2}$$

$$B = \frac{1 + K^2}{1 + 9\,K^2}$$

$$\frac{A}{B} = \frac{1 + 81\,K^2}{1 + K^2}.$$

Juxtaposing these results with those above it is seen that

$$L_2 = L_1\sqrt[4]{\frac{A}{B}}$$

$$P_2 = \frac{\varepsilon\pi^2}{L_1^2}\sqrt{\frac{B}{A}}\left(1 + \frac{A}{B}\right) = P_1 \cdot \frac{1}{2}\sqrt{\frac{B}{A}}\left(1 + \frac{A}{B}\right)$$

To evaluate these expressions we must choose a value of K.

If the condition $y' = 0$ is strictly observed at the ends we obtain

$$f\frac{\pi}{L}(1-3K)=0; \qquad K=\frac{1}{3}.$$

This value of K will give the value 5 to the coefficient A in the term $\frac{\varepsilon\pi^2}{L^2}$ for P. It appears that in the most favourable circumstances the maximum value of A cannot exceed 4, which would correspond to perfect end fixation. This would imply $K^2 = \frac{1}{15}$ or $K = \frac{1}{4}$ approximately. On applying these values of K we obtain the following results:

$$(1) \quad K^2=\frac{1}{9}, \quad \frac{A}{B}=9, \quad \sqrt[2]{\frac{B}{A}}=\frac{1}{3}$$

$$P_2=\frac{\varepsilon\pi^2}{L_1^2}\cdot\sqrt[2]{\frac{B}{A}}\left(1+\frac{A}{B}\right)=\frac{\varepsilon\pi^2}{L_1^2}\cdot\frac{10}{3}$$

$$L_2=L_1\cdot\sqrt[4]{\frac{A}{B}}=L_1\sqrt[2]{3}=L_1\cdot 1.73$$

$$(2) \quad K^2=\frac{1}{15}, \quad \frac{A}{B}=6, \quad \sqrt[2]{\frac{B}{A}}=\sqrt{\frac{1}{6}}=\frac{1}{2.45}$$

$$P_2=\frac{\varepsilon\pi^2}{L_1^2}\cdot\frac{7}{2.45}=2.85\cdot\frac{\varepsilon\pi^2}{L_1^2}$$

$$L_2=L_1\cdot\sqrt{2.45}=L_1\cdot 1.565$$

$$(3) \quad \text{If } K^2=0, \quad L_1=L_2, \quad P_2=2\cdot\varepsilon\frac{\pi^2}{L_1^2}.$$

Numerical application.

A numerical application of these results was made for the case of a parabolic Vierendeel girder for a single track railway bridge of 100.10 m span and 14.30 m rise ($^1/_7$th), divided into eleven panels of 9.10 m width.

Here L_1 = approximately 34 m.
$P_1 = 2 \times 1500 = 3000$ tonnes.

Assuming a built-in arch with $K^2 = \frac{1}{15}$

$$L_2=L_1\cdot 1.565=53\text{ m}=L_2$$

$$P_2=\frac{\varepsilon\pi^2}{L_1^2}\cdot 2.85=4250\text{ t}$$

$$\text{If } K^2=\frac{1}{9}, \quad L_2=L_1\cdot 1.73=58.5\text{ m}=L_2$$

$$P_2=\frac{\varepsilon\pi^2}{L_1^2}\cdot\frac{10}{3}=1500\cdot\frac{10}{3}=5000\text{ t}.$$

For the arch under consideration the axial thrust due to full live load amounts to 1.035 tonnes at the crown and 1.230 tonnes at the springings, taking due account of dynamic augment.

These calculations show that in the most unfavourable case, which is the limiting condition for a girder without end fixation and with constant height, the critical buckling load of the whole structure represents a factor of safety of nearly 3. Making the more correct assumption that the ends are to a considerable extent built-in, and are firmly held by the ends of the vertical members and cross girders, with $K^2 = \frac{1}{15}$ the factor of safety will be at least 4.

It is found that the arch is of approximately the same strength whether considered as bearing only on the verticals or as being fixed at the ends so that its whole length acts as a single piece. It may also be concluded from these results that the rigidity of the vertical members confers upon the bridge a considerable margin of lateral strength; a margin which increases with the rigidity of the verticals themselves and with the continuity of their connection to the arch. These conditions are fulfilled best of all in the Vierendeel type of girder here considered.