

**Zeitschrift:** IABSE congress report = Rapport du congrès AIPC = IVBH  
Kongressbericht

**Band:** 2 (1936)

**Artikel:** The distribution of pressure under a uniformly loaded elastic strip  
resting on elastic-isotropic ground

**Autor:** Borowicka, H.

**DOI:** <https://doi.org/10.5169/seals-3360>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

**Download PDF:** 02.02.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## VIII 3

The Distribution of Pressure Under a Uniformly Loaded Elastic Strip Resting on Elastic-Isotropic Ground.

Druckverteilung unter einem gleichmäßig belasteten, elastischen Plattenstreifen, welcher auf der Oberfläche des elastisch-isotropen Halbraumes liegt.

Répartition des pressions sous une semelle élastique chargée uniformément et placée sur un sol élastique isotrope.

H. Borowicka,

Assistent an der Technischen Hochschule, Wien.

In calculating the distribution of pressure under an elastic slab we are compelled to make certain assumptions both regarding the slab and regarding the foundation below. The simplest assumption to make (known as *Winkler's hypothesis*) is that the settlement is proportional to the pressure, which enables all the mathematical questions to be treated in a simple way. This hypothesis, however, is criticised particularly by soil physicists, since it implies that the amount of settlement is independent of the loaded area, and that a load can be the cause of settlement only underneath the loaded area. This criticism has given rise to the necessity of investigating the distribution of pressure under an elastic slab placed on the surface of an elastic isotropic body. Even though such a condition (associated with the name of *Boussinesq*) can represent the properties of the natural ground only to a limited extent, it nevertheless leads to results which are unobjectionable both from a physical and qualitative point of view. The treatment of the problem by means of finite functions gives rise to extremely complicated problems, which admit of solution only in the relatively simplest cases, and recourse must therefore be had to the development of series.

In the present work only the distribution of pressure underneath a uniformly loaded elastic strip of a slab will be determined, the latter being assumed to rest on the surface of an elastic isotropic body, and it must further be assumed that no frictional forces exist between the slab and the foundation. The corresponding work in relation to a circular slab was presented by the author before the Conference on Soil Mechanics at Cambridge.

- Let  $E_p$  be the modulus of elasticity of the slab.  
 $E_B$  the modulus of elasticity of the elastic isotropic body.  
 $m_p$  Poisson's constant for the slab.  
 $m_B$  Poisson's constant for the ground below.  
 $a$  the half width of the strip.  
 $h$  the thickness of the slab.  
 $N$  the resistance of the slab to bending.

If  $\omega$  denotes the deformation suffered by the surface of the ground measured in a vertical direction, then within the limit  $x < a$  (Fig. 1)  $\omega$  must correspond to the deflection of the bent slab, and the following relationships must be satisfied at the upper surface of the elastic isotropic ground:

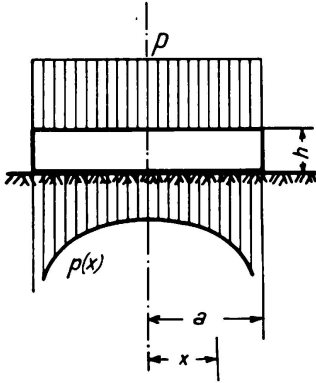


Fig. 1.

$$\left. \begin{aligned} x > a \quad \sigma_z = 0 \quad \omega \text{ any value} \\ x < a \quad \sigma_z = p(x) \quad \Delta\Delta\omega = \frac{1}{N} [p - p(x)] \end{aligned} \right\} \quad (1)$$

In the theory of slabs the differential equation

$$\Delta\Delta\omega = \frac{1}{N} [p - p(x)]$$

is usually designated as the slab equation. Laplace's operator applied twice to  $\omega$  gives rise in this case to  $\frac{d^4\omega}{dx^4}$ .

Next the deflection of the slab is to be considered. The solution of the homogeneous differential equation  $\frac{d^4\omega}{dx^4} = 0$  may be written like that for a beam

$$\omega_1 = C_0 + C_2 \left(\frac{x}{a}\right)^2 \quad (2)$$

The particular integral in the non-homogeneous differential equation is taken to be a series of powers with unknown coefficients

$$\omega_2 = \sum_{l=2}^{\infty} A_{2l} \left(\frac{x}{a}\right)^{2l} \quad (3)$$

The unknown function  $p(x)$  is likewise represented by an exponential series

$$p(x) = \sum_{n=0}^{\infty} B_{2n} \left(\frac{x}{a}\right)^{2n} \quad (4)$$

The slab equation enables the relationship between the coefficients  $A_{2l}$  and  $A_{2n}$  to be determined from the equation of coefficients

$$\begin{aligned} A_4 &= \frac{a^4}{24N} (p - B_0) \\ A_{2n+4} &= -\frac{a^4}{N} \frac{2n!}{(2n+4)!} B_{2n} \end{aligned} \quad (5)$$

The deflection of the slab then takes the following form

$$\omega = C_0 + C_2 \left(\frac{x}{a}\right)^2 + \frac{pa^4}{24N} \left(\frac{x}{a}\right)^4 - \sum_{n=0}^{\infty} \frac{a^4}{N} \frac{2n!}{(2n+4)!} B_{2n} \left(\frac{x}{a}\right)^{2n+4} \quad (6)$$

Since no fixing moment operates at the point  $x = a$  it follows that  $\left(\frac{d^2\omega}{dx^2}\right)_{x=a} = 0$  and from this condition we obtain

$$C_2 = -\frac{pa^4}{4N} + \frac{a^4}{2N} \sum_{n=0}^{\infty} \frac{1}{(2n+2)(2n+1)} B_{2n} \quad (7)$$

In Equation (6) the deflection  $\omega$  is given as a function of  $x$  or of the unknown coefficient  $\Lambda_{2n}$ . As already explained, within the limit  $x < a$  it also corresponds to the elastic deformation of the upper surface of the ground. If the variables are represented by  $\xi$  and  $\zeta$  (Fig. 2) we obtain for the point  $x$ :

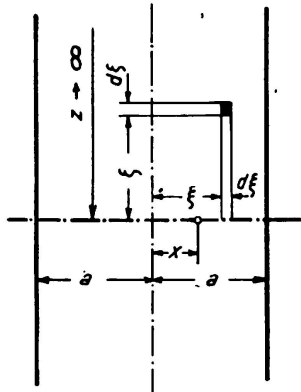


Fig. 2.

$$d\omega = 2 \int_0^{\infty} \frac{1}{2\pi r} p(\xi) \cdot 2 \frac{m^2_B - 1}{m^2_B \cdot E_B} d\xi d\zeta$$

$$\frac{m^2_B E_B}{m^2_B - 1} = M$$

$$r = \sqrt{(\xi - x)^2 + \zeta^2}$$

$$d\omega = \frac{2}{\pi M} p(\xi) d\xi \int_0^z \frac{d\zeta}{\sqrt{(\xi - x)^2 + \zeta^2}}$$

$$= \frac{2}{\pi M} p(\xi) d\xi (\ln [z + \sqrt{(\xi - x)^2 + z^2}] - \ln |\xi - x|)$$

Taking  $z \rightarrow \infty$ , infinitely great values are obtained for  $\omega$  thus determining the differences from  $\omega_0$  corresponding to the deflection at the centre of the slab.

$$d\omega_0 - d\omega = \frac{2}{\pi M} p(\xi) d\xi \left\{ \ln \frac{z + \sqrt{\xi^2 + z^2}}{z + \sqrt{(\xi - x)^2 + z^2}} + \ln |\xi - x| - \ln |\xi| \right\}$$

$$\lim_{z \rightarrow \infty} \ln \frac{z + \sqrt{\xi^2 + z^2}}{z + \sqrt{(\xi - x)^2 + z^2}} = \lim_{z \rightarrow \infty} \ln \frac{1 + \sqrt{\left(\frac{\xi}{z}\right)^2 + 1}}{1 + \sqrt{\left(\frac{\xi - x}{z}\right)^2 + 1}} = \ln 1 = 0$$

We then obtain

$$\omega_0 - \omega = \frac{2}{\pi M} \left\{ \int_x^a p(\xi) \ln(\xi - x) d\xi + \int_{-a}^x p(\xi) \ln(x - \xi) d\xi - 2 \int_0^a p(\xi) \ln \xi d\xi \right\}$$

On completing the integration by substituting  $p(\xi) = \sum_{n=0}^{\infty} B_{2n} \left(\frac{\xi}{a}\right)^{2n}$  according to Equation (4), and by substituting the resulting logarithmic function by the corresponding exponential series we obtain

$$\omega_0 - \omega = \frac{4}{\pi M} \sum_{n=0}^{\infty} B_{2n} \frac{a}{2n+1} \left\{ \left(\frac{x}{a}\right)^{2n} \sum_{r=1}^{\infty} \frac{1}{2r-1} \left(\frac{x}{a}\right)^{2r} - \sum_{r=1}^{\infty} \frac{1}{2r} \left(\frac{x}{a}\right)^{2r} - \sum_{r=1}^n \frac{1}{2n-2r+1} \left(\frac{x}{a}\right)^{2r} \right\} \tag{8}$$

Equation (8) can still be somewhat transformed by altering the signs as follows:

$$\omega_0 - \omega = \frac{4a}{\pi M} \left\{ \sum_{r=1}^{\infty} \frac{1}{2r-1} \sum_{n=0}^{\infty} \frac{B_{2n}}{2n+1} \left(\frac{x}{a}\right)^{2n+2r} - \sum_{r=1}^{\infty} \frac{1}{2r} \left(\frac{x}{a}\right)^{2r} \sum_{n=0}^{\infty} \frac{B_{2n}}{2n+1} - \sum_{r=1}^n \left(\frac{x}{a}\right)^{2r} \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n-2r+1)(2n+1)} \right\} \tag{9}$$

Now Equation (6) gives for  $x = 0$ ,  $\omega_0 = C_0$  and we have

$$\omega_0 - \omega = -C_2 \left(\frac{x}{a}\right)^2 - \frac{pa^4}{24N} \left(\frac{x}{a}\right)^4 + \sum_{n=0}^{\infty} \frac{a^4}{N} \frac{2n!}{(2n+4)!} B_{2n} \left(\frac{x}{a}\right)^{2n+4} \quad (10)$$

The condition for equilibrium is

$$p = \frac{1}{a} \sum_{n=0}^{\infty} \int_0^a B_{2n} \left(\frac{x}{a}\right)^{2n} dx = \sum_{n=0}^{\infty} \frac{B_{2n}}{2n+1} \quad (11)$$

Equations (9) and (10) must be identical, and this condition is fulfilled when the coefficients are made equal to every power of  $\left(\frac{x}{a}\right)$  in them. In this way an infinite number of equations is obtained, containing an infinitely large number of unknowns  $B_{2n}$ . If, for instance, the  $m$ -th equation is to be written down, all terms containing the exponent  $\left(\frac{x}{a}\right)^{2m}$  must be picked out, and in Equation (10)  $2n+4 = 2m$  or  $n = m-2$  must then be inserted, whereupon the coefficient becomes  $\sum_{n=0}^{m-1} \frac{B_{2n}}{(2n+1)(2m-2n-1)}$ . In the second and third bracketed terms of Equations (9) we may then insert  $r = m$  giving, as the coefficient of  $\left(\frac{x}{a}\right)^{2m}$  the expression  $\frac{1}{2m} \sum_{n=0}^{\infty} \frac{B_{2n}}{2n+1} = \frac{p}{2m}$  (according to Equation 11) and  $\sum_{n=m}^{\infty} \frac{B_{2n}}{(2n-2m+1)(2n+1)}$ .

Writing

$$K = \frac{2N}{Ma^3} = \frac{1}{6} \frac{m^2_B - 1}{m^2_P - 1} \frac{E_P}{E_B} \left(\frac{h}{a}\right)^3 \quad (12)$$

and substituting for  $C_2$  from Equation (7), we obtain

$$\left. \begin{aligned} p &= \sum_{n=0}^{\infty} \frac{B_{2n}}{2n+1} \\ \frac{p}{2} + \frac{p\pi}{8K} &= \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n+1)(1-2n)} + \frac{\pi}{4K} \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n+2)(2n+1)} \\ \frac{p}{4} - \frac{p\pi}{48K} &= \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n+1)(3-2n)} - \frac{\pi B_0}{48K} \\ &\vdots \\ \frac{p}{2m} &= \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n+1)(2m-1-2n)} - B_{2m-4} \frac{\pi (2m-4)!}{2K \cdot 2m!} \end{aligned} \right\} \quad (13)$$

It will easily be seen that Equations (13) are satisfied by putting  $B_{2n} = \frac{2}{\pi} \frac{1 \cdot 3 \cdot \dots (2n-1)}{2 \cdot 4 \cdot \dots 2n}$  for  $K = \infty$ . These, however, are the coefficients of the exponential series of the function

$$p(x) = \frac{2}{\pi} \frac{p \cdot a}{\sqrt{a^2 - x^2}} \quad (14)$$

For  $K = 0$  we have  $B_0 = p$  and all the remaining coefficients become equal to zero. The uniform load is then standing directly on the elastic isotropic ground. Equation (13) has been worked out for the condition  $K = \frac{\pi}{3}$ ,  $K = \frac{\pi}{10}$  and  $K = \frac{\pi}{30}$  with  $p = 1 \text{ kg/cm}^2$ , and the results are represented in Fig. 3. In the limit the stresses corresponding to degrees of stiffness different from zero become infinite, and the correctness of this result can be visualised in the following way: — In an infinitely stiff slab the “bending surface” is a plane, and the line of deformation of the surface of the ground will therefore turn a sharp corner at the edge. If, now, the slabs is given a finite degree of stiffness this corner will

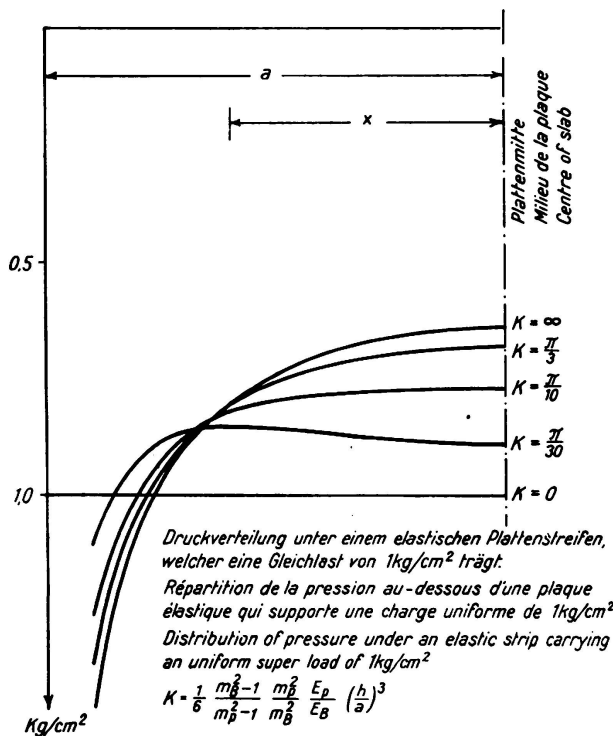


Fig. 3.

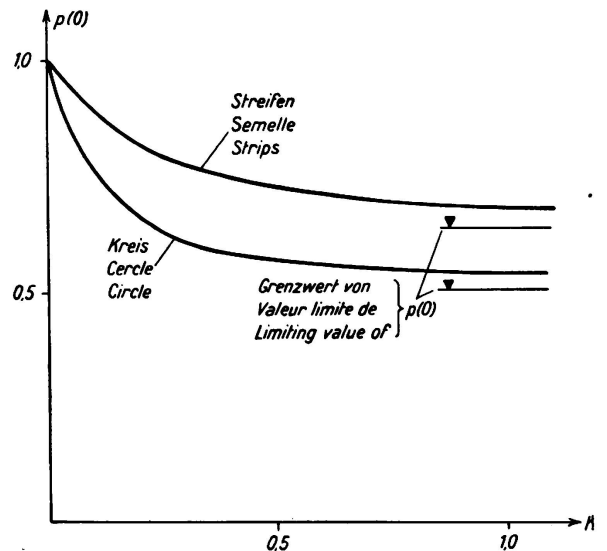


Fig. 4.

become less sharp, but it will still be present, and to correspond with this condition the pressure on the ground at the side will still become infinite, though it will increase more slowly. If the slab is loaded not with  $1 \text{ kg/cm}^2$  but with  $p \text{ kg/cm}^2$  the ordinates of the curves shown in Fig. 3 must be multiplied by  $p$ . The working out of Equation (13) is somewhat laborious, as a large number of such equations have to be solved. In Fig. 4 the value of the pressure at the centre of the slab both for strips and for a circular slab is represented in relation to  $K$ . Here again the curve corresponds to a loading of  $1 \text{ kg/cm}^2$ . The curve for the circular slab and for the strip of slab are calculated for a Poisson's ratio of  $m = 4$ .

*Boussinesq* shows in his “*Application des potentiels*” (Paris, 1885) that a distribution of pressures below an absolutely stiff circular slab may be calculated in the following way: — assume that the load on the slab is uniformly distributed over a hemisphere having its circular base coincident with the circumference of

the slab; then the projection of the load so distributed over the surface of the ground gives the desired distribution of stresses. The author was encouraged by *O. K. Fröhlich* to examine these questions and recommended that *Boussinesq's* construction should be adapted to the case of a strip of slab and used as a control for the limiting case of the calculation of an elastic strip of slab. As may be seen from Equation (14) the distributions of stress for the stiff strip of slab are in fact confirmed for the case in question by the use of *Boussinesq's* method.

### Bibliography.

- <sup>1</sup> *Winkler*: Die Lehre von der Elastizität und Festigkeit. Prag 1867.
- <sup>2</sup> *Zimmermann*: Berechnung des Eisenbahnoberbaues. Berlin 1888.
- <sup>3</sup> *Fröhlich*: Berechnung von Fundamenten unter Berücksichtigung der Elastizität des Baugrundes. „Beton und Eisen“ 1913, Heft 15 und 16.
- <sup>4</sup> *Hayashi*: Theorie des Trägers auf elastischer Unterlage und ihre Anwendung auf den Tiefbau. Berlin 1921.
- <sup>5</sup> *Chwalla*: Die Stabilität eines elastisch gebetteten Druckstabes. „Zeitschrift für angew. Math.“ 1927, Heft 4.
- <sup>6</sup> *Schleicher*: Über Kreisplatten auf elastischer Unterlage. Berlin 1925. Nochmals Kreisplattenfundamente. „Beton und Eisen“ 1925, Heft 12.
- <sup>7</sup> *Scheidig*: Nochmals Kreisplattenfundamente. „Beton und Eisen“ 1926, Heft 6. Die Berechnungsgrundlagen durchgehender Fundamente und die neue Baugrundforschung. „Bautechnik“ 1931, Heft 19.
- <sup>8</sup> *Wiegardt*: Über den Balken auf nachgiebiger Unterlage. „Zeitschrift für angew. Math.“ 1922.
- <sup>9</sup> *Prager*: Zur Theorie elastisch gelagerter Konstruktionen. „Zeitschrift für angew. Math.“ 1927, Heft 5.
- <sup>10</sup> *Fröhlich*: Die Bemessung von Flachgründungen und die neuere Baugrundforschung. „Beton und Eisen“ 1935, Heft 12.
- <sup>11</sup> *Melan*: Die Verteilung des Druckes durch eine elastische Schichte. „Österr. W. für den öffentl. Baudienst“ 1918, S. 394. Die Druckverteilung durch eine elastische Schichte. „Beton und Eisen“ 1919, Heft 7/8. Der Spannungszustand der durch eine Einzelkraft im Innern beanspruchten Halbscheibe. „Zeitschrift für angew. Math.“ 1932, Heft 6.
- <sup>12</sup> *Schleicher*: Zur Theorie des Baugrundes. „Bauingenieur“ 1926, S. 931 und 949.
- <sup>13</sup> *Fröhlich*: Druckverteilung im Baugrunde mit besonderer Berücksichtigung der plastischen Erscheinungen. Wien 1934.
- <sup>14</sup> *Wolf*: Ausbreitung der Kraft in der Halbebene und im Halbraum bei anisotropem Material. „Zeitschrift für angew. Math.“ 1935, Heft 5.
- <sup>15</sup> *Terzaghi* und *Fröhlich*: Theorie der Setzung von Tonschichten. Wien und Leipzig 1936.
- <sup>16</sup> *Borowicka*: Influence of Rigidity of a Circular Foundation Slab on the Distribution of Pressures over the Contact Surface. Proceedings of the International Conference on Soil Mechanics and Foundation Engineering 1936 Harvard University, Cambridge, Mass.