Zeitschrift: IABSE congress report = Rapport du congrès AIPC = IVBH Kongressbericht

Band: 3 (1948)

Artikel: On integration of the differential equation for thin shells without bending

Autor: Johansen, K.W.

DOI: https://doi.org/10.5169/seals-4125

IVc4

Contribution à l'intégration de l'équation différentielle pour voiles minces non fléchis

Ueber die Integration der Differentialgleichung für dünne Schalen ohne Biegung

On integration of the differential equation for thin shells without bending

K. W. JOHANSEN
Dr techn. Copenhague

For thin shells without bending it is known (1) that the horizontal components of the membrane forces may be given by a stress function :

$$N_{xx} = \frac{\partial^2 F}{\partial y_2} + \int X dx \, ; \quad N_{yy} = \frac{\partial^2 F}{\partial x^2} + \int Y dy \, ; \quad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \, ;$$

which is determined by the differential equation

$$\frac{\partial^2 F}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \cdot \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2} \cdot \frac{\partial^2 z}{\partial x^2} = Z - \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x} \int X dx\right] - \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y} \int Y dy\right]$$

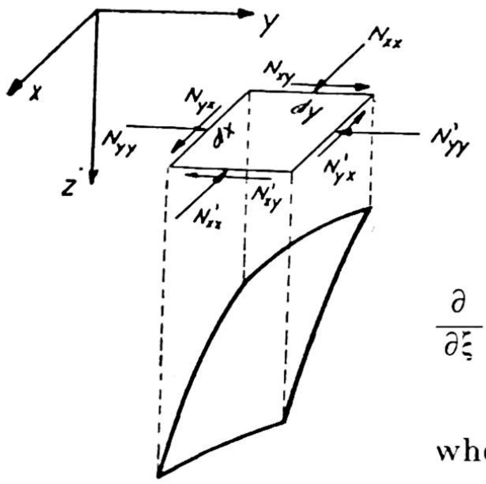
X and Y are the horizontal components of the load, Z the vertical, and z the ordinate of the middle surface of the shell. In well-known notation it may be written :

$$t \frac{\partial^2 F}{\partial x^2} - 2 s \frac{\partial^2 F}{\partial x \partial y} + r \frac{\partial^2 F}{\partial y^2} = G \, . \tag{1}$$

We now use Laplace's method for transformation of linear equations by changing the independent variables x and y to ξ and η, where ξ and η are the solutions of

$$t \, dy^2 + 2 \, s \, dy \, dx + r \, dx^2 = 0 \tag{2}$$

(1) L. Beschkine, *Comptes rendus de l'Académie des Sciences de Paris*, t. 201, p. 935, 1935.
B. Laffaille, *Mémoire sur l'étude générale des surfaces gauches minces* (*Publications IABSE*, vol. 3, p. 295, 1935).



or

$$\frac{dy}{dx} = -\frac{s \pm \sqrt{s^2 - rt}}{t} . \tag{3}$$

We then get

$$\frac{\partial}{\partial \xi}\left(\sqrt{s^2 - rt}\,\frac{\partial F}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\sqrt{s^2 - rt}\,\frac{\partial F}{\partial \xi}\right) = GD \tag{4}$$

where D is the functional determinant.

Substituting $H = \sqrt[4]{s^2 - rt}$ and $u = HF$, we get

$$2\,H\,\frac{\partial^2 u}{\partial \xi \partial \eta} - 2\,u\,\frac{\partial^2 H}{\partial \xi \partial \eta} = G \cdot D . \tag{6}$$

This equation may be integrated at once, when $\frac{\partial^2 H}{\partial \xi \partial \eta} = 0$, and that is the case when

$$H^4 = s^2 - rt = a^2 = \text{constant} . \tag{7}$$

The solution of this differential equation is obtained by eliminating q from

$$\left.\begin{aligned} z &= qy + \Phi\left(q - \frac{x}{a}\right) + \Psi\left(q + \frac{x}{a}\right) \\ 0 &= \ \ y + \Phi'\left(q - \frac{x}{a}\right) + \Psi'\left(q + \frac{x}{a}\right) , \end{aligned}\right\} \tag{8}$$

where Φ or Ψ are arbitrary functions.

We now have

$$\xi = q - \frac{x}{a} ; \qquad \eta = q + \frac{x}{a} \tag{9}$$

and

$$\left.\begin{aligned} x &= \frac{a}{2}(\eta - \xi) ; \qquad y = -\Phi'(\xi) - \Psi'(\eta) ; \\ z &= \Phi(\xi) + \Psi(\eta) - \frac{1}{2}(\xi + \eta)[\Phi'(\xi) + \Psi'(\eta)] . \end{aligned}\right\} \tag{10}$$

The differential equation (1) then becomes

$$\frac{\partial^2 F}{\partial \xi \partial \eta} = \frac{a^2}{4}[\Phi''(\xi) + \Psi''(\eta)]\,G . \tag{11}$$

which is solved by two quadratures.

When a is real, the shell surface has hyperbolic curvature. When a is imaginary, $a = ib$, the shell surface has elliptic curvature. The equation is more convenient put into real form, and we get the equation of the potential

$$b\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) = \mathrm{GD} \tag{12}$$

where ξ and η are the real and imaginary parts, respectively, of the solution $\zeta = \xi + i\eta$ to (2) or (3).

Finally, when $rt = s^2$, the shell surface is developable. Then $\xi = x$ and η is the solution of $\frac{dy}{dx} = -\frac{s}{t}$. The equation is transformed into

$$\frac{\partial}{\partial \xi}\left(\frac{t}{\eta_y}\frac{\partial \mathrm{F}}{\partial \xi}\right) = \frac{\mathrm{G}}{\eta_y}\;; \qquad \eta_y = \frac{\partial \eta}{\partial y}\;. \tag{13}$$

This equation is solved by two quadratures.

H = constant is a very special solution of the equation $\frac{\partial^2 \mathrm{H}}{\partial \xi \partial \eta} = 0$. Transforming $\frac{\partial^2 \mathrm{H}}{\partial \xi \partial \eta} = 0$ into the original variables x and y and putting $\frac{1}{\mathrm{H}} = h$, we get the more general differential equation

$$r\frac{\partial^2 h}{\partial y^2} - 2s\frac{\partial^2 h}{\partial x \partial y} + t\frac{\partial^2 h}{\partial x^2} = 0\;; \qquad h = (s^2 - rt)^{-\frac{1}{4}} = \mathrm{H}^{-1}\;. \tag{14}$$

It is of the 4th order, and the general solution may be difficult to find, but special solutions are

$$\frac{z}{c} = \frac{x}{a}\,\mathrm{tg}^{-1}\frac{x}{a} - \frac{y}{b}\,\mathrm{tg}^{-1}\frac{y}{b}\,, \qquad \frac{z}{c} = \frac{x}{a}\,\mathrm{th}^{-1}\frac{x}{a} - \frac{y}{b}\,\mathrm{th}^{-1}\frac{y}{b}\,,$$

$$\frac{z}{c} = \frac{x}{a}\,\mathrm{tg}^{-1}\frac{x}{a} + \frac{y}{b}\,\mathrm{th}^{-1}\frac{y}{b}\,, \qquad \frac{z}{c} = \frac{a^2}{x^2} \pm \frac{b^2}{y^2}\,,$$

$$\frac{z}{c} = \log\frac{y}{x}\,, \qquad \Phi(z) = x + \mathrm{A}ye^{az}\,.$$

By the mathematical methods it may be somewhat difficult to satisfy the boundary conditions. I should therefore like to call attention to a very simple practical method. In most cases the shell is designed for one load only, dead load and snow, as the wind-pressure has little influence. We may now reverse the problem, choose the stress function, and determine the shell surface. Usually it will not be difficult to choose a stress function satisfying the boundary conditions. For the shell surface there are generally no boundary conditions at all, or only very limited ones.

This circumstance facilitates the solution of the differential equation, which may be solved by the methods of finite differences in the simplest form. In other applications of finite differences one may solve a great number of linear equations and the solution must finally be differentiated once or twice to get the stresses. This process involves much work and great accuracy. But in our case the ordinate z gives the final solution at once, and the lack of boundary conditions makes it possible to choose some ordinates arbitrarily and thus in most cases compute the rest successively. One may also start with a rough net, and then interpolate to intermediate points. In this connection it should be noted that the net need not fit the contour. when there are no boundary conditions.

Résumé

Grâce à la transformation de Laplace, on peut obtenir une forme simple (6) pour l'équation différentielle des tensions (1). Pour le type de voûte défini par l'équation (8), cette équation peut encore se simplifier pour prendre la forme (11) qui peut être résolue par deux intégrations. Pour les voûtes à courbure elliptique on obtient l'équation différentielle (12). Cette même forme simple est également obtenue par les voûtes du type (14).

L'auteur montre en outre qu'il est possible de choisir d'abord la forme de l'équation des tensions et d'en déduire la forme de la voûte, et cela d'une manière très simple par le calcul différentiel.

Zusammenfassung

Mit Hilfe der Transformation von Laplace kann die Differentialgleichung (1) der Spannungsfunktion in die einfache Form (6) gebracht werden. Für den Schalentyp, der durch (8) beschrieben wird, kann die Gleichung weiter vereinfacht werden auf (11), welche durch zwei Integrationen gelöst werden kann. Für Schalen mit elliptischer Krümmung erhält man die Potentialgleichung (12). Die gleiche einfache Form wird auch erhalten für Schalen vom Typus (14).

Ferner wird gezeigt, dass in vielen Fällen die Spannungsfunktion gewählt und die Schalenform danach bestimmt werden kann. Dies kann z.B. ganz einfach mittels Differenzenrechnung durchgeführt werden.

Summary

By Laplace's transformation the differential equation (1) for the stress function can be converted into the simple form (6). For the shell type represented by (8) the equation is further reduced to (11), which is solved by two integrations. For surfaces with elliptic curvature, the potential equation (12) is obtained. The same simple forms are also obtained for shell types which satisfy (14).

Furthermore, it is pointed out that in many cases the stress function can be chosen and the shell surface determined thereafter. This determination can, for instance, be made quite simply by the method of finite differences.