Zeitschrift: IABSE congress report = Rapport du congrès AIPC = IVBH Kongressbericht

Band: 3 (1948)

Artikel: Influence functions for the angular deviation correction in suspension bridges

Autor: Asplund, Sven Olof

DOI: https://doi.org/10.5169/seals-4029

IIIb1

Fonctions d'influence pour la correction des déviations angulaires dans les ponts suspendus

Einflussfunktionen für die Berücksichtigung der Winkelabweichung bei Hängebrücken

Influence functions for the angular deviation correction in suspension bridges

Dr SVEN OLOF ASPLUND
Örebro

In the classical deflection theory of suspension bridges the differential equation (1)

$$(EI\eta'')'' - (H_w + H)\eta'' = Hy'' + p(x)$$

of the elastic line η of the stiffening truss is established under the assumption

---

(1) This equation is usually termed Melan's equation but was earlier published and solved by A. Ritter (1877) and H. Müller-Breslau (1881).

Notations : All notation, except that which is specific to this paper, is in conformity with common usage in most American treatises on the theory of suspension bridges. Reference to the numbered equation where the magnitude is first defined or used is given in parentheses.

A_{xk}'', a_{xk}'' = angular deviation correction influence functions (10), (12)
c = incidental flexibility of one-span bridge (10)
E = modulus of elasticity of truss (1)
f = cable sag (11), $f(x)$ = angular correction load (4)
H_w = horizontal force of dead load (1), H = increment of horizontal force due to other causes (1)
I = moment of inertia of truss (1)
I_{xk} = influence functions (2), i_{xk} = influence functions for one-span bridge of unit length (13)
J_x'', j_x'' = influence functions (2)
k = abscissa of live load elements measured from the left end (2)
l = span length (11)
M = moment in stiffening truss at section x (6)
$p(k)$ = distributed live load at abscissa k (2)
t = temporarily used abscissa (10). t_1 = abscissa measured from low-point of cable (center of span) (12)

that the cable-points move along fixed verticals. Rode ([2]) seems to have first pointed out that the differential equation may be established without resorting to that assumption. Rode gave his equation for a constant stiffness EI of the truss, but it can easily be generalized to apply to variable truss stiffness and rearranged to manifest more plainly its interesting mathematical character ([3]) :

$$(\mathrm{EI}\eta'')'' - (\mathrm{H}_w + \mathrm{H})\,[(1 + y'^2)\eta']' = \mathrm{H}y'' + p(x) \tag{1}$$

Melan's equation as well as (1) are restricted by homogeneous and linear boundary conditions namely the restraint conditions at the supports of the stiffening truss and the cable condition.

The exact solution of (1) or of Melan's equation is very complicated, since by the cable condition the live load horizontal force H is a function of all η. In fact, no solution by finite methods has yet been given. If, however, the variable H is treated as a constant, the equation (1) together with its boundary conditions may be recognized as a self-adjoint linear boundary problem with variable coefficients in the differential equation. Accordingly one enters a plausible value of H in the problem. By its solution a new value of H is calculated from the cable condition. This value is again entered in the boundary problem. By iteration of the solution with successively corrected values of H the exact solution of the fundamental equation (1) and its boundary conditions may thus be approached to any desired accuracy. In practical bridge problems the convergence of this process is rapid. More than one or two iterations are seldom required.

The general form of the solution of the *linear* boundary problem ([1]) (with an assumed constant value of H) may be obtained by an expansion of the solution according to the (orthogonal) « eigenfunctions » of the homogeneous differential equation associated to (1) (left member = 0) ([4]). It is :

$$\eta\,(x) = \frac{1}{\mathrm{H}_w + \mathrm{H}}\,[\textstyle\int_s \mathrm{I}_{xk}\, p\,(k)\, dk + \delta \mathrm{J}_x] \tag{2}$$

This equation indicates that influence lines may be suitably applied in the solution. The « deflection influence function » I_{xk} is a function of H (and of the section x considered and of the load position k). It is independent of the cable *yield*, because all cable yield effects are comprised in a factor δ and carried to the correction term $\delta \mathrm{J}_x$, J_x being a function of H (and of x). But I_{xk} cannot be made to be independent of the *form*, y, of the cable (sag-ratio, etc.). Therefore, if a set of influence functions had been computed and tabulated it could only be used for bridges with the same relative stiff-

---

w = dead load per unit length of bridge (13)
x = abscissa measured from the left support to the section to be investigated (1)
y = ordinate measured downwards from a horizontal line to the cable under dead load at section x (1)
δ = cable yield constant (2)
η = vertical displacement of truss at section x due to live load, temperature, and anchorage displacement (1)
ξ = horizontal displacement of the cable point originally at x due to same causes (8)

([2]) *Proceedings Am. Soc. of Civ. Eng.*, 1928, p. 393.
([3]) S. O. Asplund, *On the Deflection Theory of Suspension Bridges* (*Ingenjörsvetenskaps-akademiens Handlingar* 184, Stockholm, 1945, p. 23).
([4]) See ([3]), p. 31-37.

ness variation *and* cable sag ratios in the different spans. Moreover, although the general form (2) of the solution is known, influence functions have not yet been published for any specific bridge, however simple. A mathematically interesting « exact » (proviso constant H) solution of special cases of (1) has been undertaken but it is not possible to see how it could be advantageously applied.

For practical reasons the author considers that the mathematically concise form (1) should be given up in favor of

$$(\mathrm{EI}\eta'')'' - (\mathrm{H}_w + \mathrm{H})\eta'' = \mathrm{H}y'' + p(x) + f(x) \tag{3}$$

$$f(x) = (\mathrm{H}_w + \mathrm{H})(y'^2\eta')' \tag{4}$$

in suspension bridge problems. The solution of this form still obtains the same form (2) with the exceptions that I_{xk} is computed from the homogeneous part (left hand side) of (3) only and therefore *no* function of the cable form y, and that instead of $p(x)$ the « load » $p(x) + f(x)$ should be applied under the integral sign of (2). In actual bridge problems the « correction load » $f(x)$ is much smaller than $p(x)$. If $f(x)$ is first neglected, η can be solved by (2) and η' and $(y'^2\eta')' = f(x)/(\mathrm{H}_w + \mathrm{H})$ computed. When this function is entered together with $p(x)$ in (2) an improved solution is obtained which is acceptable for practical purposes even if it can be made more precise by iteration.

This procedure makes possible the treatment of suspension bridges by much larger classes, inasmuch as the influence functions I_{xk} will be entirely independent of the cables (size, yield *and* form): *All* bridges with the same relative stiffness variation of the truss may be treated by the same set of tables I_{xk}.

If $f(x)$ is omitted the solution will be the same as that of Melan's equation. The application of $f(x)$ is equivalent to a consideration of changes ξ' in the horizontal projections of the cable elements and can suitably be termed the cable angular deviation correction load.

If the solution according to Melan's equation is first established, the addition of an angular deviation correction obviously yields the solution according to (1).

By differentiation of (2) the change of grade of the truss becomes

$$\eta'(x) = \frac{1}{\mathrm{H}_w + \mathrm{H}} \left\{ \int_s \mathrm{I}'_{xk} [p(k) + f(k)]\, dk + \delta \mathrm{J}_x' \right\} \tag{5}$$

and for instance the moment in the truss

$$\mathrm{M} = \int_s \mathrm{I}''_{xk} [p(k) + f(k)]\, dk + \delta \mathrm{J}_x'' \tag{6}$$

where

$$\mathrm{I}'_{xk} = \frac{\partial \mathrm{I}_{xk}}{\partial x}, \qquad \mathrm{J}_x' = \frac{d\mathrm{J}_x}{dx}, \qquad \mathrm{I}''_{xk} = -\frac{\mathrm{EI}(x)}{\mathrm{H}_w + \mathrm{H}} \frac{\partial \mathrm{I}'_{xk}}{\partial x},$$

$$\text{and} \quad \mathrm{J}_x' = -\frac{\mathrm{EI}(x)}{\mathrm{H}_w + \mathrm{H}} \frac{d\mathrm{J}_x}{dx}.$$

Instead of by the laborious computation of $f(x)$ just explained and application to (2), (5), (6), etc., the angular deviation corrections may be evaluated separately and directly from the load $p(x)$ by the use of influence functions. Entering (4) in (6) yields

Row	t/l		0	0.05	0.1	0.15	0.2
1	$2\,t/l - 1 = 2\,t_1/l$		— 1.0	— 0.9	— 0.8	— 0.7	— 0.6
2	$-(t_1/l)^2\,c^2$		— 25	— 20.3	— 16	— 12.3	— 9
3	i'_{tk}	$10^{-3}\times$	327	285	146	— 6	— 88
4	i''_{k}	$10^{-3}\times$	0	17.2	40.0	22.2	11.5
5	$i'_{tk}\cdot 2\,t_1/l$	$10^{-3}\times$	— 327	— 257	— 117	4	53
6	$-i''_{tk}\cdot(t_1/l)^2\,c^2$	$10^{-3}\times$	0	— 347	— 640	— 272	— 103
7	Sum of rows 6,7	$10^{-3}\times$	— 327	— 604	— 757	— 268	— 50
8	i''_{xt}	$10^{-3}\times$	0	4.8	11.5	22.4	40.9
9	Integrand of a''_{xk}	$10^{-3}\times$	0	— 2.89	— 8.68	— 6.01	— 2.00

$$\mathrm{M} = \int_s \mathrm{I}''_{xk}\,p\,(k)\,dk + \int_s \mathrm{I}''_{xk}\,(\mathrm{H}_w + \mathrm{H})\,[(y'^2\,\eta')']_{x=k}\,dk + \delta \mathrm{J}_x'' \,. \qquad (7)$$

The middle term on the right hand side is $\mathrm{M}_{\xi'}$, the angular deviation correction for moment,

$$\mathrm{M}_{\xi'} = \int_s \mathrm{I}''_{xk}\,(\mathrm{H}_w + \mathrm{H})(2\,y'y''\eta' + y'^2\,\eta'')_{x=k}\,dk \,. \qquad (8)$$

Substituting (5) in (8) yields

$$\mathrm{M}_{\xi'} = \int_s \mathrm{I}''_{xk}\,\Big[(2\,y'y'')_{x=k} \int_s \mathrm{I}_{kt}\,p\,(t) + f\,(t)]\,dt$$

$$- (y'^2)_{x=k} \int_s \frac{\mathrm{H}_w + \mathrm{H}}{\mathrm{EI}\,(k)}\,\mathrm{I}''_{kt}\,[p\,(t) + f\,(t)]\,dt\Big\}\,dk + \delta \mathrm{J}_x'\,[y'^2\,\eta']_s \,.$$

The relatively small influence of the correction load $f(t)$ and the small δ-term could be accounted for by this formula. If they are dropped and k and t interchanged the comparatively simple formula

$$\mathrm{M}_{\xi'} = \int_s \mathrm{A}''_{xk}\,p\,(k)\,dk \qquad (9)$$

remains with

$$\mathrm{A}''_{xt} = \int_s \mathrm{I}''_{xt}\left[(2\,y'y'')_{x=t}\,\mathrm{I}'_{tk} - (y'^2)_{x=t}\,\frac{\mathrm{H}_w + \mathrm{H}}{\mathrm{EI}\,(t)}\,\mathrm{I}''_{tk}\right]dt \qquad (10)$$

as an influence function for the angular deviation correction for moment.

Such influence functions shall now be numerically evaluated for the case of a symmetrical one-span bridge with constant stiffness EI. If the span-length is l, the cable sag f, and the uniform dead load w one has

$$y'' = -\frac{w}{\mathrm{H}_w} = -\frac{8\,f}{l}\,, \qquad y' = -\frac{8\,f}{l^2}\,x_1\,, \qquad 2\,y'y'' = \frac{128\,f^2}{l^4}\,x_1\,,$$

$$y'^2 = \frac{64\,f^2}{l^4}\,x_1^{\,2}$$

0.3	0.4	0.5	0.6	0.8	1.0		t/l	Row
— 0.4	— 0.2	0	0.2	0.6	1.0		$2\,t/l - 1 = 2\,t_1/l$	1
— 4	— 1	0	— 1	— 9	— 25		$-(t_1/l)^2\,c^2$	2
— 142	— 129	— 92				$10^{-3} \times$	i'_{tk}	3
1.0	— 2.9	— 4.3				$10^{-3} \times$	i''_{ik}	4
57	26					$10^{-3} \times$	$i'_{tk} \cdot 2\,t_1/l$	5
— 4	3					$10^{-3} \times$	$-\,i''_{tk} \cdot (t_1/l)^2\,c^2$	6
53	29	0	— 4	68	113	$10^{-3} \times$	Sum of rows 6,7	7
7.1	— 6.0	— 10.8	— 11.7	— 8.1	0	$10^{-3} \times$	i''_{xt}	8
0.38	— 0.17	0	0.65	— 0.55	0	$10^{-3} \times$	Integrand of a''_{xk}	9

x_1 (and t_1) being measured from the low point of the cable at the center of the span. The influence functions $i_{xk}' = I_{xk}'$ and $i_{xk}'' = I_{xk}''/l$ are previously tabulated ([5]) for different values of

$$c = l\sqrt{\frac{H_w + H}{EI}}$$

Equation (10) becomes

$$A''_{xk} = \int_0^l l i''_{xt}\left(\frac{64\,f^2}{l^4}\,2\,t_1 i'_{tk'} - \frac{64\,f^2}{l^4}\,t_1^2\,\frac{c^2}{l^2}\,l i''_{tk}\right)dt$$

$$A''_{xk} = l\,\frac{64\,f^2}{l^2}\,a''_{xk} \tag{11}$$

$$a''_{xk} = \int i''_{xt}\left[\frac{2\,t_1}{l}\,i'_{tk} - \left(\frac{t_1}{l}\right)^2 c^2\,i''_{tk}\right] d\,\frac{t}{l}\;. \tag{12}$$

The complete moment formula derived from (7), (9) and (11), $j_x'' = J_x''/l$, thus becomes

$$\frac{M}{wl^2} = \int i''_{xk}\,\frac{p(k)}{w}\,d\,\frac{k}{l} + \frac{64\,f^2}{l^2}\int a''_{xk}\,\frac{p(k)}{w}\,d\,\frac{k}{l} + \frac{\delta}{wl}\,j_x'' \tag{13}$$

The computation of just one value of a_{xk}'' will be explained here, namely for $c = 10$ at $x/l = 0.2$ and $k/l = 0.1$.

Intermediate values of the integrand may be determined from diagrams of rows 7 and 8. Finally the integrands of a_{xk}'' are carefully plotted and integrated by quadrature, yielding the integral $a_{xk}'' = -1.03 \times 10^{-3}$. The result of similar computations for $x/l = 0.2$, $k/l = 0.05$, 0.1, 0.15, 0.2, 0.3, 0.4, 0.5, 0.6 and 0.8, and $c = 10$ and 20 is diagrammated in fig. 1 and 2.

These a_{xk}'' influence diagrams can be used for a fairly accurate eva-

([5]) See ([3]), p. 85.

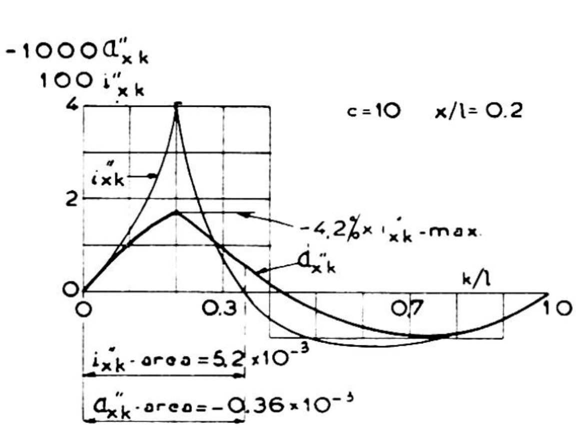

Fig. 1.

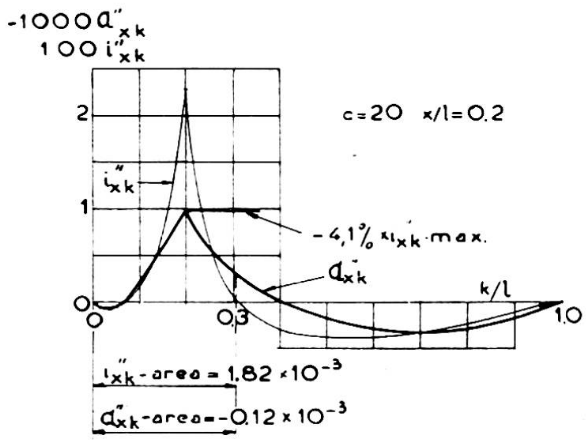

Fig. 2.

luation of the angular deviation correction for moment at any given load, for instance composed of concentrations and loaded segments.

For comparison the influence functions i_{ik}'' to be used simultaneously in (13) are shown in fig. 1 and 2. It is seen that the shape of the a_{xk}''-diagram is roughly the same as that of the i_{xk}''-diagram.

The maximum ordinate of a_{xk}'' is — 4.2 % and — 4.1 % of that of i_{xk}'' for $c = 10$ and 20, respectively, and $x = 0.2\,l$. A calculation for $x = 0.15\,l$ gives the percentages — 5.3 and — 6.2 for $c = 10$ and 15, respectively.

The zero point of a_{xk}'' is to the right of that of i_{xk}''. The branches leading to the maximum point are more straight for a_{xk}''. When the positive area of i_{xk}'' is loaded the engaged area of a_{xk}'' is —6.9 % and —6.6 % of that of i_{xk}'' for $c = 10$ and 20, respectively, and $x = 0.2\,l$ (fig. 1 and 2). For ordinary cases of loading the maximum positive moment in a one-span suspension bridge of even stiffness occurs near the point $x = 0.2\,l$, while the larger part of the i_{xk}''-area is loaded with concentrated or uniform loads. The given percentages indicate that the angular deviation correction between $c = 10$ and 20 amounts to between — 6 % $\times$ 64 f^2/l^2 and — 7 % $\times$ f^2/l^2 of the maximum positive moment at $x = 0.2\,l$. These values can be roughly used if influence lines a_{xk}'' are lacking. In a one-span bridge of flexibility between $c = 10$ and 20 and with a sag-ratio $f/l = 0.12$ the angular deviation correction is thus about — 6 % of the maximum moment.

The completed solution, according to this paper, of the angular deviation correction can be directly applied to bridges of any sag-ratio. The direct mathematical solution of (1) mentioned above, would be even more difficult if the influence of a variable f/l had to be taken into account. When influence functions are determined by tests on model ([6]), (1) and its boundary conditions is so to speak solved directly by a model machine. Then the resulting influence functions I_{xk} of (2) will also include the angular deviation correction, the magnitude of which will not be disclosed. Therefore the set of influence lines obtained will not be exactly applicable to any other bridges than such with the *same* sag-ratios of the cable as in the model bridge used.

In a one-span bridge the maximum moment is seen to *reduce* the angular deviation correction. An omission of this correction therefore is

([6]) See ([3]). p. 110

on the safe side. It should not be overlooked that the absolute value of the minimum moment at the tower of a continuous bridge is *increased* by the angular deviation correction.

Résumé

L'équation différentielle classique et quasi linéaire des ponts suspendus ancrés (équation de Melan) peut se compléter par une expression qui tient compte des variations d'angles des éléments des câbles. Il se présente alors le problème de la valeur aux limites, dont la solution purement mathématique n'est, cependant, pas avantageuse. Si, au lieu de cela, l'on ajoute ladite expression à la partie non homogène de l'équation comme fonction d'interférence, cette interférence peut se calculer rapidement en se servant à plusieurs reprises des fonctions d'influence. L'auteur indique le calcul numérique d'une valeur de la fonction d'influence. Deux courbes complètes de l'influence, utilisables pour des ponts avec des flèches différentes, sont indiquées sur des schémas. Pour un pont ayant une ouverture et une flexibilité c comprise entre 10 et 20, la dimension de rectification de la variation d'angle pour les plus grands moments absolus, avec $x = 0{,}2\ l$ peut atteindre entre — 6 et — 7 % de 64 f^2/l^2 fois la valeur du moment, qui se calcule en se servant de la théorie classique des moments fléchissants.

Zusammenfassung

Die klassische quasilineare Differentialgleichung der verankerten Hängebrücke (Gleichung von Melan) kann durch einen die Winkelabweichungen der Kabelelemente erfassenden Ausdruck ergänzt werden. Dabei entsteht ein selbstadjungiertes Randwert-Problem, dessen direkte mathematische Lösung sich aber als unvorteilhaft erweist. Wenn statt dessen der genannte Ausdruck dem inhomogenen Teil der Gleichung als Störungsfunktion angefügt wird, kann die Störung schnell durch wiederholte Anwendung von Einflussfunktionen berechnet werden. Die numerische Berechnung eines Wertes der Einflussfunktion wird gezeigt. Zwei bestimmte vollständige Einflusslinien, anwendbar auf Brücken mit verschieden grossem Durchhang sind in Diagrammen dargestellt. Bei einer Brücke mit einer Oeffnung und einer Biegsamkeit c zwischen 10 und 20 kann die Korrekturgrösse der Winkelabweichung für die grössten positiven Momente bei $x = 0{,}2\ l$ zwischen — 6 und — 7 % von 64 f^2/l^2 mal den Wert des Moments erreichen, das mit der klassischen Durchbiegungstheorie berechnet wird.

Summary

A term accounting for the angular deviations of the cable elements may be included in the classical quasi-linear differential equation of the truss (Melan's equation). A self-adjoining boundary problem is then formed but a direct mathematical solution appears disadvantageous. If the term in question is instead carried to the non-homogeneous part of the equation as a disturbance function, the disturbance may be expediently calculated

by iterated use of influence functions. The numerical evaluation of one influence function value is demonstrated. Two significant complete influence lines, applicable to bridges of various sag ratio are given in diagrams. In a one-span bridge with flexibility c between 10 and 20 the angular deviation correction for maximum positive moments at $x = 0.2\, l$ may amount to between — 6 and — 7 % of 64 f^2/l^2 times the moment as calculated by the classical deflection theory.