# Method of elastic compatibility in the solution of beams of finite length on elastic foundations

# AII 3

## Method of elastic compatibility in the solution of beams of finite length on elastic foundations

## Méthode de calcul élastique appliquée au calcul des poutres de longueur finie reposant sur des bases élastiques

## Methode zur Berechnung von endlichen Balken auf elastischer Unterlage

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### I. Beams and foundation pressures

#### 1. *Introduction*

When a "rigid" beam carrying loads rests on elastic material, it develops pressure underneath, which is uniform throughout when centrally loaded or uniformly varying in a straight line if eccentrically loaded. If, on the other hand, the beam is "semi-rigid," i.e. one capable of resisting bending with certain amount of deflections, the pressure is proportional to the deflection occurring at each point. This is because the supporting soil below beams carrying engineering structures is considered to behave elastically, which tends to recover from the relative settlements when the superimposed loads on the beams are removed.

If the soil proves to be flowing plastically under loading, as may be the case with very soft clay, the beam necessitates designing as "rigid" as if floating on liquid of heavy density. On similar arguments an absolutely "flexible" member may be sufficient to bear loads lying on rather rigid supporting medium, such as rock. The appropriate stiffness required for a beam therefore depends upon the nature of the soil below. The theory also gives easy means of determining the correct value of stiffness required for a beam (Section V, examples 2 and 3).

#### 2. *Elastic line of a semi-rigid beam and the soil pressure*

Fig. 1(*b*) shows the pressure distribution under a rigid beam *LR* loaded non-centrally as in (*a*), the straight-line variation being represented by *cd* from the average line *LCR*. If, instead, the beam is semi-rigid and rests on elastic material such that the loaded points are made to remain in one plane (not necessarily horizontal), the

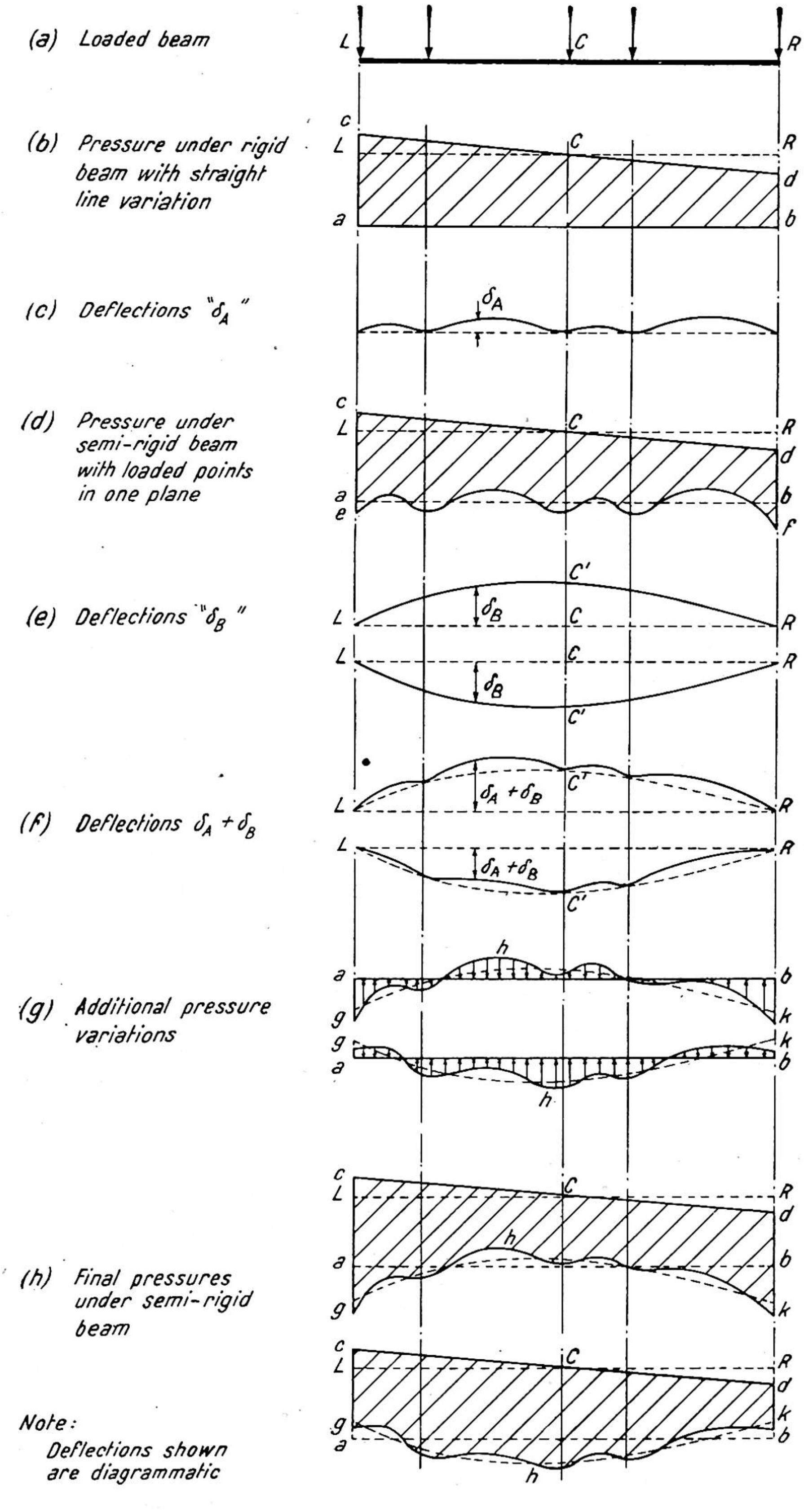

(a) Loaded beam
(b) Pressure under rigid beam with straight line variation
(c) Deflections "$\delta_A$"
(d) Pressure under semi-rigid beam with loaded points in one plane
(e) Deflections "$\delta_B$"
(f) Deflections $\delta_A + \delta_B$
(g) Additional pressure variations
(h) Final pressures under semi-rigid beam
Note: Deflections shown are diagrammatic
L
C
R
c
d
a
b
e
f
C'
$\delta_A$
$\delta_B$
$\delta_A + \delta_B$
g
h
k

Fig. 1

beam would produce deflections between the points as in (*c*) denoted by $\delta_A$ (termed "local deflections") and the pressure would vary as shown in (*d*), there being relief between the loads and increase under.

If it is now considered that according to the loading the loaded points move out of the plane so as to take different levels, the axis *LCR* of the beam would deflect to take the form *LC'R* similar to a bow of some shape either indicating "hog" or "sag" shown in (*e*). These deflections, represented by $\delta_B$ (termed "bow deflections"), are measured from a line connecting the ends of the beam. The deflections at various points along the beam would therefore be the algebraic sum of $\delta_A$ and $\delta_B$, as in (*f*). It will be noticed that the values of $\delta_A$ are negligible as compared with $\delta_B$.

With these deflections taking place throughout the beam, additional variation in earth pressure below comes into effect such that the lowest point in the beam exerts the highest upward pressure and the highest point has the maximum relief or reduction in upward pressure. These pressures would have at the same time the effect of reducing the deflections $\delta_A+\delta_B$ by a certain amount and adjusting themselves accordingly. The variations from the straight line *ab* of pressure distribution, which may take the two possible forms corresponding to the two deflection forms in (*f*), are indicated in (*g*).

Finally, these additional pressure variations *ghk* due to beam deflections, when superimposed on the average straight line *ab* of pressure distribution in (*b*), would give the two possible pressure diagrams shown in (*h*)—one giving maximum pressure at the ends and the other in the middle. It is therefore considered sufficient to check up pressures at the ends and at the section of maximum deflection in the middle of a beam. It should be realised, however, that the deflections referred to are only relative and are additional to the general settlement of the beam as a whole.

## II. Forces acting on a beam and the principle of analysis

### 3. *Forces acting on a beam in equilibrium*

The forces are considered to be divided into two systems:

(*a*) System 1

From the superimposed loads on a beam and its bearing area the average earth pressure $w_0$ per unit area is obtained. The pressure $w$ per unit run of the beam is uniform for a beam of constant width or varying accordingly. Only the prismatic beams would be dealt with at present. Cases with non-prismatic sections will be considered in Section V, para. 13.

Consider the forces acting on a beam, as if rigid, comprising the superimposed loading above and $w$ per unit run of earth pressure below as represented by *LRba* in fig. 1(*b*). If the beam is centrally loaded, this would be in equilibrium or else these forces would have an unbalanced resulting moment. This has to be balanced by an assumed straight-line variation of earth pressure from positive (acting upward) at one end to negative (acting downward) at the other, similar to that represented by line *cd* in fig. 1(*b*). These pressures are termed "balancing pressures" (B.P.).

The system of forces comprising these, such as would occur on a loaded beam if it were perfectly rigid, is termed $F_r$. The moments produced by $F_r$ throughout the beam are $M_r$ and the deflections measured from a line connecting the ends $\delta_r$, which are approximately equal to $\delta_A+\delta_B$ referred to in fig. 1(*f*). The maximum deflection occurring in the middle of the beam in particular is termed $Y_r$.

(*b*) System 2

Due to the deflections throughout a semi-rigid beam, deviations from the straight-line distribution of pressure, referred to in System 1, come to operate, having increased values at the lower points and relieved at the higher, such that the straight line representing $w$ indicates the average of the deviations as in fig. 1($h$), wherein $ghk$ was the deviated form from line $ab$.

The increase and the relief of pressure involved in the deviations comprise the "additional variation of pressure" and such a variation, similar to that in fig. 1($g$), is shown in fig. 2($a$) in typical form, in which the increase is shown at the ends and relief in the middle, consequent upon the middle of the beam deflecting upwards under force system $F_r$. The *vice-versa* would be the possible alternative.

These forces in the additional pressure variation, which tend to restore the beam from the elastic deformations or deflections due to system $F_r$, are called "elastic restoring forces" and are comprised in a system termed $F_e$. The moments produced by $F_e$ are $M_e$ and the related deflections $\delta_e$—in particular $Y_e$, the maximum in the middle.

It would be realised from fig. 1 that it is the bow deflections $\delta_B$ which are the essential factors in the development of the force system $F_e$ and the consequent deflections $\delta_e$, the influence of $\delta_A$ being negligible.

4. *Principles of analysis*

A centrally loaded beam, if rigid, would exert uniform pressure $LRba$ shown in fig. 2($b$), where $La$ equals $w$, and pressure $LRkhg$ when semi-rigid. The eccentricity of superimposed loading would only introduce the balancing pressures in addition. Since the line $ab$ in fig. 2($a$) represents the average of the forces $F_e$, the areas above and below the line should therefore be equal. To simplify calculations for moments and deflections, the variation in $F_e$ is replaced by the straight dotted lines shown and drawn symmetrically about the centre of the beam, in lieu of line $ghk$. The maximum ordinates, both above and below the line $ab$, in the variation are represented by $fw$ per unit run or $fw_0$ per unit area, $f$ being a factor or coefficient. The maximum and minimum pressures developed are therefore $w_0+fw_0$ and $w_0-fw_0$ respectively per unit area.

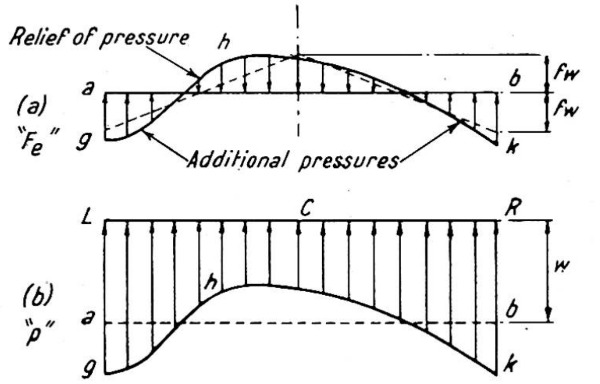


Fig. 2

It would be observed that the force system $F_e$ gives a deflection $\delta_e$ always opposite to $\delta_r$. The total deflections throughout a beam would therefore be the sum of $\delta_r$ and $\delta_e$ algebraically, and the final maximum deflection in the middle of the beam

$$Y = \Sigma Y_r + Y_e \quad \ldots \ldots \ldots \quad (4{:}1)$$

considering the maximum deflections $Y_r$ and $Y_e$ to occur approximately at the same section. (It may be worth noting that the shift of the position of the maximum deflection in a prismatic beam, simply supported at the ends with a bending moment diagram of one sign, can never exceed 1/13th of the length from the centre.) The deflections are represented in fig. 3 for the beam under the system of forces in fig. 2. The original deflection is $Y_r$ from the loading and the pressure $LRba$ of system $F_r$, which reduces to $Y$ due to the forces $F_e$ having pressure ordinates $fw$ at the centre and the ends (fig. 2($a$)).

For the purposes of analysis, it is necessary to ascertain the value of $fw$ so as to obtain the pressures and the bending moments throughout a beam. To obtain the value of $f$, the final maximum deflection $Y$ is to be considered first, which is dependent upon

(*a*) the elastic properties of the beam and

(*b*) the elastic properties of the soil,

so that the higher the "flexural rigidity" ($EI$) or the "modulus of foundation" ($k_0$), the lesser is the deflection. The value of $Y$ should be such as to be compatible with the conditions for both (*a*) and (*b*).

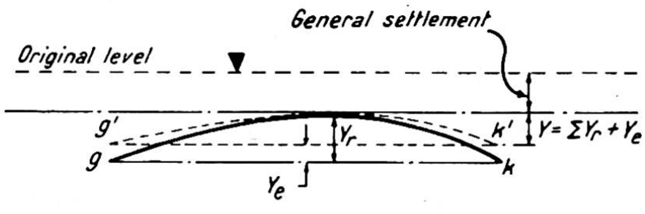



**Fig. 3**

The value of $f$, related to $Y$, having been ascertained, the bending moment diagram for system $F_e$ can be obtained with its maximum ordinate $\bar{M}_e$ at the centre, where shear is nil. The moments throughout the beam would then equal $\Sigma M_r + M_e$.

For the purposes of maximum and minimum pressures underneath, the positions of $fw$ under system $F_e$ would be considered at the ends and in the middle of the beam where maximum deflection occurs.

## III. Pressures and related deflections

### 5. *Signs*

The signs in the operations will be considered as follows:

(i) "Moments" are positive when tension is created on the underside of beams.

(ii) "Deflections" are positive given by positive moments.

(iii) "$f$-system" is positive in the positive force system $F_e$ causing positive moments $M_e$, and forces act upwards at the ends and downwards in the middle of a beam.

### 6. *Forms of pressure variation and the related deflections $Y_e$*

The value of deflection $Y_e$ for a beam is connected to the force system $F_e$, which in turn depends on the value of $f$. Therefore the equations for deflections can be expressed in terms of $f$.

(A) Form of pressure distribution in system $F_e$ with equal maximum ordinate above and below average

A positive force system $F_e$ with maximum ordinates $fw$ above and below the average line is shown in fig. 4(*a*), with consequent positive deflection $Y_e$ at (*b*). The $f$-system at (*a*) is therefore

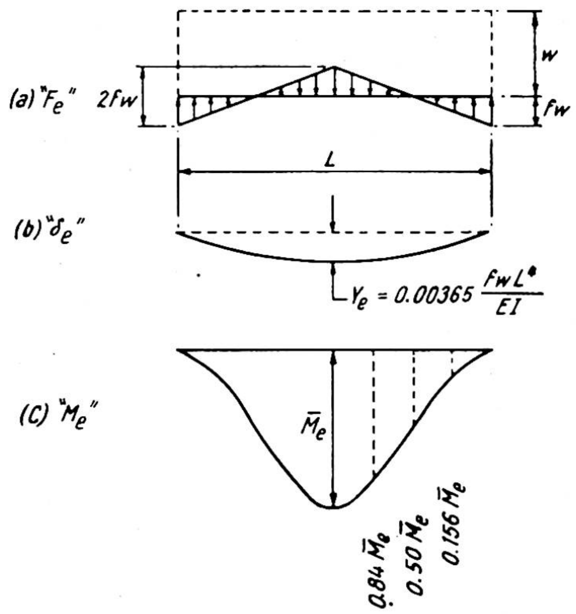



**Fig. 4**

positive. The arrangement could be of opposite kind with negative values. With these forces acting on a beam, the moments $M_e$ at any section distant $x$ from an end is given by

$$M_e=\left[\frac{x^2}{2}-\frac{2x^3}{3L}\right]fw,$$

and at centre, where $x=L/2$, the maximum value

$$\bar{M}_e=0{\cdot}0416wL^2f \quad \ldots\ldots\ldots \quad (6{:}1)$$

The deflection at any section distant $x$ from an end

$$\delta_e=\frac{wf}{EI}\left[\frac{x^4}{24}-\frac{x^5}{30L}-\frac{L^3x}{96}+\frac{7L^4}{1920}\right], \text{ where } EI=\text{flexural rigidity}$$

and the maximum deflection at centre, where $x=L/2$

$$Y_e=0{\cdot}00365\frac{wL^4}{EI}f \quad \ldots\ldots\ldots \quad (6{:}2)$$

shown at (*b*). The maximum and minimum pressures are $w+fw$ and $w-fw$ per unit run of beam respectively.

It would be observed from fig. 4 that the maximum ordinate $fw$ of pressure reduction can never exceed $w$ in value and thus also the maximum ordinate of pressure increase; in other words $f$ can never exceed 1.

(B) Other forms of pressure distribution in system $F_e$

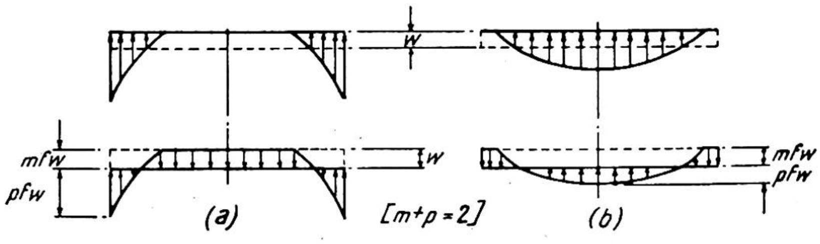



**Fig. 5**

There may be other cases of distribution such that the maximum ordinates of reduction and increase have unequal values. This would also be obvious from figs. 5(*a*) and (*b*) with positive and negative $f$-systems respectively, where some parts of the beams do not bear on the soil due to upward deflections.

For the purposes of analysis let $mfw$ and $pfw$ be the ordinates of the maximum reduction and increase respectively below and above the average, so that their sum

$$mfw+pfw=2fw \quad \ldots\ldots\ldots \quad (6{:}3a)$$

as before, or

$$m+p=2 \quad \ldots\ldots\ldots \quad (6{:}3b)$$

With such forms of pressure distribution as in fig. 5, $mfw$ would be controlled by the value of $w$, so that $mfw=w$ or $mf=1$. Then from eqn. (6:3*a*),

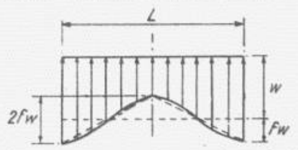
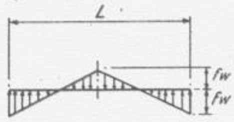
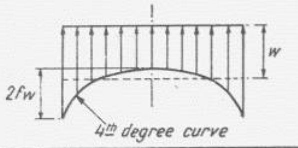
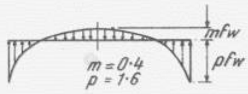
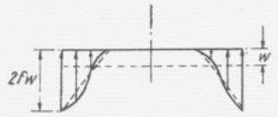
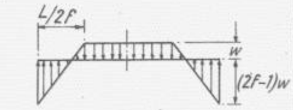
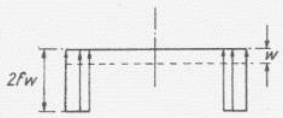
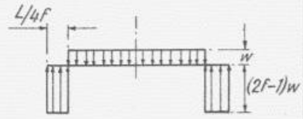
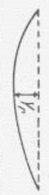
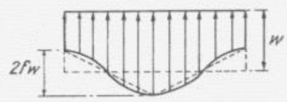
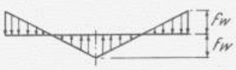
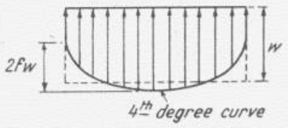
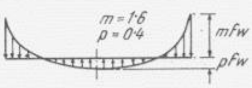
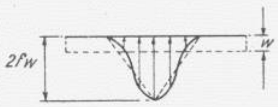
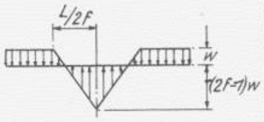
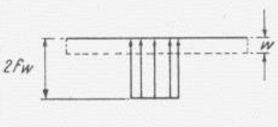
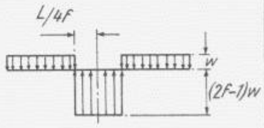
$$1+pf=2f, \text{ or } pf=(2f-1), \text{ or}$$

$$p=2-\frac{1}{f} \quad \ldots\ldots\ldots\ldots \quad (6{:}4)$$

The eqn. shows that

when $f=1$, $p=1$
$f<1$, $p<1$ and
$f>1$, $p>1$

# TABLE - I

| | Case | pressure "p" | "$F_e$" | "$Y_e$" $= F(F) \cdot \frac{wL^4}{EI} = NF \frac{wL^4}{EI}$ | "$\bar{M}_e$" $= Q \cdot wL^2$ |
|---|---|---|---|---|---|
| when "$Y_r$" is negative  | 1 |   |   | $F \leq 1$: $0.00365 F \cdot \frac{wL^4}{EI}$ | $\frac{1}{24} F \cdot wL^2$ |
| | 2 |   |   | $F \leq 2.5$: $0.0032 F \cdot \frac{wL^4}{EI}$<br>$F > 2.5$: As case – 3 (approx.) | |
| | 3 |   |   | $F \leq 1$: As case – 1<br>$F > 1$: $\left[\frac{20F^3 - 10F^2 + 1}{960F^3} - 0.0078\right] \frac{wL^4}{EI}$ | $F \leq 1$: $\frac{1}{24} F \cdot wL^2$<br>$F > 1$: $\left[\frac{3F-2}{24F}\right] wL^2$ |
| | 4 |   |   | $F \leq 1$: $0.00552 F \cdot \frac{wL^4}{EI}$<br>$F > 1$: $\left[\frac{40F^3 - 24F^2 + 1}{3072F^3}\right] \frac{wL^4}{EI}$ | $F \leq 1$: $\frac{1}{16} F \cdot wL^2$<br>$F > 1$: $\left[\frac{2F-1}{16F}\right] wL^2$ |
| When "$Y_r$" is positive  | 5 |   |   | $F \leq 1$: $0.00365 F \cdot \frac{wL^4}{EI}$ | $\frac{1}{24} F \cdot wL^2$ |
| | 6 |   |   | $F \leq 0.625$: $0.0032 F \cdot \frac{wL^4}{EI}$<br>$F > 0.625$: As case – 7 (approx.) | |
| | 7 |   |   | $F \leq 1$: As case – 5<br>$F > 1$: $\left[0.0078 - \frac{5F-1}{960F^3}\right] \frac{wL^4}{EI}$ | $F \leq 1$: $\frac{1}{24} F \cdot wL^2$<br>$F > 1$: $\left[\frac{3F-2}{24F}\right] wL^2$ |
| | 8 |   |   | $F \leq 1$: $0.00552 F \frac{wL^4}{EI}$<br>$F > 1$: $\left[\frac{24F^2 - 8F + 1}{3072F^3}\right] \frac{wL^4}{EI}$ | $F \leq 1$: $\frac{1}{16} F \cdot wL^2$<br>$F > 1$: $\left[\frac{2F-1}{16F}\right] w \cdot L^2$ |

Notes:
1. Values of "F" in the table are all absolute
2. Cases encircled are those usually covering all general practical cases
3. Intermediate values of "N" in $Y_e = NF \cdot \frac{wL^4}{EI}$ are given in graph. Fig. 6, and of "Q" in $\bar{M}_e = Q \cdot wL^2$ in graph. Fig. 12.

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Since the areas of pressures under the force system $F_e$ above and below the average lines should be equal, it is clear from the diagrams that the ordinates $pfw$ have to be greater than $mfw$, i.e. from eqn. (6:3*a*),

$$pfw > (2-p)fw, \text{ or } p>1 \quad \ldots\ldots\ldots \quad (6:5)$$

This shows therefore from eqn. (6:4) that the cases would involve values of $f>1$.

The maximum and minimum pressures developed are $2fw$ and *zero* respectively per unit run, as would be observed from fig. 5 also.

(C) Practical considerations

To serve all practical purposes, it is assumed that:

(i) when $f\leqslant 1$, the variation should be considered with equal maximum ordinates $fw$ above and below the average, and

(ii) when $f>1$, the maximum reduction $mfw$ has the limiting value $w$.

Some possible forms of pressure distribution and the connected diagrams for the force systems $F_e$ are shown in Table I, in which the deflections $Y_e$ are shown represented by the form

$$Y_e = N \cdot \frac{wL^4}{EI} \cdot f \quad \ldots\ldots\ldots \quad (6:6)$$

The "deflection coefficients" $N$ against the values of $f$ for all the cases can also be taken from fig. 6. It is to be noted that the cases 2 and 6 in Table I, having unequal ordinates $mfw$ and $pfw$, would be covered by the cases 1 and 5, since $mfw$ are not the limiting values $w$.

The foregoing assumptions give safe results, as the values of $N$ for $Y_e$ are on the higher side (see also para. 8).

When $Y_r$ is negative, $Y_e$ is positive with positive $f$-system. Cases 1 to 4 are some of the possible forms shown in Table I. Case 2 represents an ideal fourth-degree curve in view of the deflection being the fourth integral of loading and is absolutely theoretical. Under normal conditions case 1 for $f\leqslant 1$, and case 3 for $f>1$ would be apparent.

When $Y_r$ is positive, $Y_e$ is negative with negative $f$-system, such that some of the possible forms may be as shown by the cases 5 to 8. Case 5 is the case 1 inverted and case 6 represents the theoretical fourth-degree curve. Under normal conditions case 5 for $f\leqslant 1$, and case 7 for $f>1$ would be apparent, but a case with $f>1$ will not occur in practice when $Y_r$ is positive (para. 7(2)(*b*)).

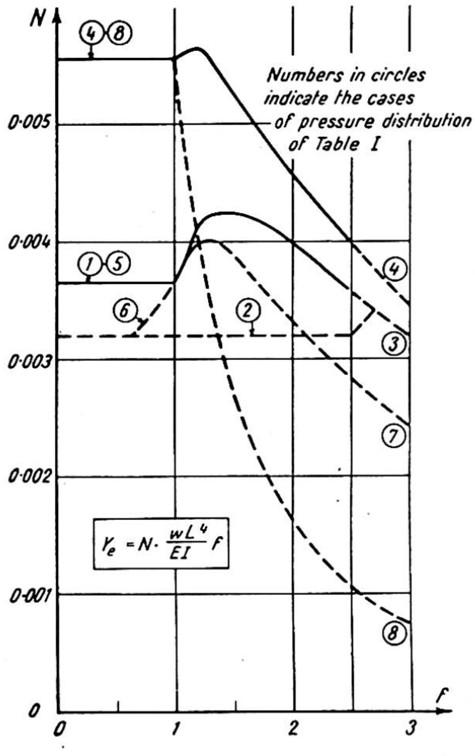



Fig. 6

7. *Factors affecting the final deflection Y in a beam*

These will be considered in the following treatment of the deflections from the elastic properties of the beam and the bearing soil (para. 4):

(1) Deflections from elastic properties of beam

From eqns. (6:6) and (4:1), $Y_e=N.\frac{wL^4}{EI}f$ and $Y=\Sigma Y_r+Y_e$, remembering that $Y_r$ and $Y_e$ are always of opposite signs.

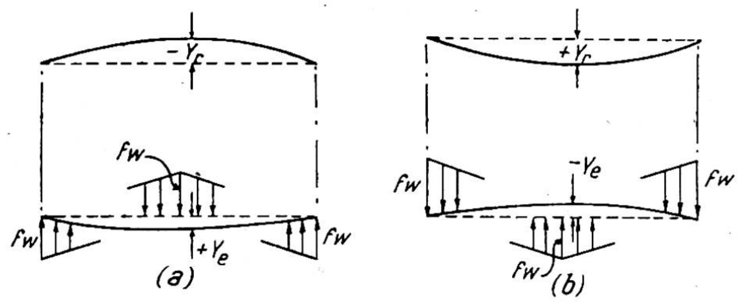



**Fig. 7**

(i) When $Y_r$ is negative, $Y_e$ is positive with positive $f$-system (fig. 7($a$)):

$$\therefore \quad Y=-Y_r+N.\frac{wL^4}{EI}(+f)=-Y_r+N.\frac{wL^4}{EI}f \quad . \quad . \quad . \quad (B_1)$$

(ii) When $Y_r$ is positive, $Y_e$ is negative with negative $f$-system (fig. 7($b$)):

$$\therefore \quad Y=+Y_r-N.\frac{wL^4}{EI}(-f)=+Y_r+N.\frac{wL^4}{EI}f \quad . \quad . \quad . \quad (B_2)$$

These equations stand for all values of $f$, whether greater, equal or less than 1.

(2) Deflections from elastic properties of soil

Since the soil reaction per unit area of foundation is assumed proportional to the settlement (para. 1), the ratio $\frac{\text{pressure per unit area } p}{\text{settlement } S}$ is a constant, termed $k_0$, which is known as the "modulus of foundation." The above relation gives

$$p=k_0S \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7:1)$$

Also

$$S=\frac{p}{k_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7:2)$$

The modulus may vary under a beam in various ways depending upon the nature of the soil and the depths to which they occur. Let the minimum value under a beam be $k_0$ and the maximum $nk_0$ per unit area, so that $n>1$. In the analysis, the variations, when taken into account, will be considered symmetrical about the centre line of the beam such that $k_0$ and $nk_0$ occur under the ends and the centre or *vice versa*, the variation being linear. Such variations are considered to cover the limits of all possible cases.*

In the derivation of the deflection equations, the distribution of pressure under force system $F_e$ will be considered under two groups as follows:

($a$) Force system $F_e$ when $f\leqslant 1$

This system includes cases 1 and 5 of Table I, and under this group the pressure variation has equal maximum ordinates $fw$ above and below the average (para. 6(C)).

* Advantage can also be taken of such variations in the moduli in an attempt to take account of the usual pressure variations experienced in cohesive and non-cohesive soils under engineering structures.

(i) When $Y_r$ is negative, $f$-system is positive (fig. 8):

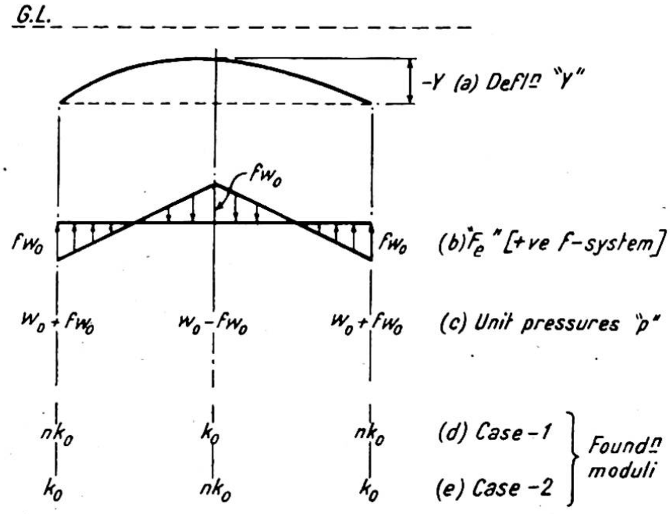



Fig. 8

In the final position of the beam the deflection (ignoring the little displacement of the position of maximum deflection from centre),

$Y$=settlement at centre minus settlement at ends

*Case 1:* if $k_0$ is the modulus at centre and $nk_0$ at ends, then from eqn. (7:2):

$$Y=\frac{w_0-fw_0}{k_0}-\frac{w_0+fw_0}{nk_0}=-\frac{w_0}{k_0}\left[1+\frac{1}{n}\right]f+\frac{w_0}{k_0}\left[1-\frac{1}{n}\right] \quad . \quad . \quad . \quad (S_1)$$

*Case 2:* if $nk_0$ is the modulus at centre and $k_0$ at ends, then:

$$Y=\frac{w_0-fw_0}{nk_0}-\frac{w_0+fw_0}{k_0}=-\frac{w_0}{k_0}\left[1+\frac{1}{n}\right]f-\frac{w_0}{k_0}\left[1-\frac{1}{n}\right] \quad . \quad . \quad . \quad (S_2)$$

(ii) When $Y_r$ is positive, $f$-system is negative (fig. 9):

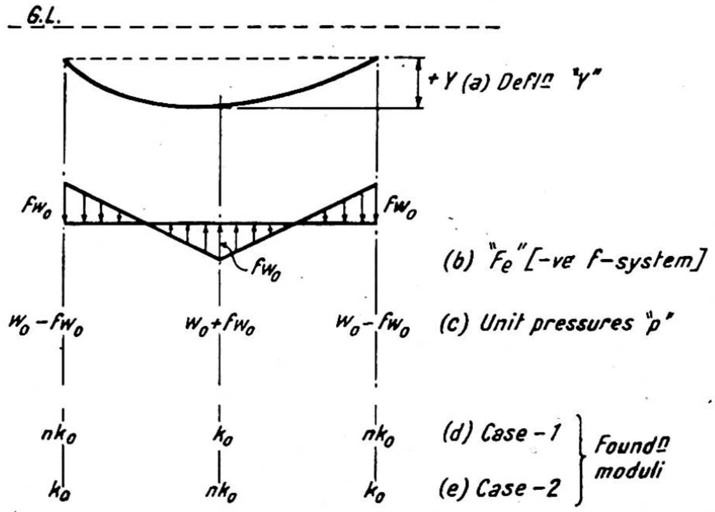



Fig. 9

In the final position the maximum deflection,

$Y$=settlement at centre minus settlement at ends

*Case 3:* if $k_0$ is the modulus at centre and $nk_0$ at ends, then:

$$Y=\frac{w_0+(-f)w_0}{k_0}-\frac{w_0-(-f)w_0}{nk_0}=-\frac{w_0}{k_0}\left[1+\frac{1}{n}\right]f+\frac{w_0}{k_0}\left[1-\frac{1}{n}\right] \quad . \quad . \quad (S_3)$$

*Case 4:* if $nk_0$ is the modulus at centre and $k_0$ at ends, then:

$$Y=\frac{w_0+(-f)w_0}{nk_0}-\frac{w_0-(-f)w_0}{k_0}=-\frac{w_0}{k_0}\left[1+\frac{1}{n}\right]f-\frac{w_0}{k_0}\left[1-\frac{1}{n}\right] \quad . \quad . \quad (S_4)$$

*Case 5:* when $k_0$ is uniform throughout, $n=1$ and all the above equations become:

$$Y=-\frac{2w_0}{k_0}f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (S_5)$$

(b) Force system $F_e$ when $f>1$

The cases 3, 4, 7 and 8 in Table I are covered by this group, where the maximum ordinates of pressure reduction and increase are $w$ and $(2f-1)w$ respectively (para. 6).

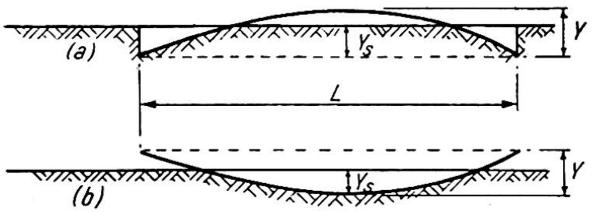


Fig. 10

It is to be realized that since some parts of the beams do not bear on the soil due to the upward deflections when $f>1$, the values of $Y$ given by the soil equations would not be the true values of the maximum deflections occurring in the beams, but would only represent the values measured up to the ground lines as shown in fig. 10 by $Y_s$. The relationship of this $Y_s$ with true $Y$ may be approximately obtained by considering the deflection curves of the beams of at least the fourth degree and are as follows when $k_0$ is uniform:

(i) when $Y_r$ is negative,

for $f=2$, $Y_s=0{\cdot}938\,Y$
$f=3$, $Y_s=0{\cdot}803\,Y$

(ii) when $Y_r$ is positive,

for $f=2$, $Y_s=0{\cdot}0625\,Y$
$f=3$, $Y_s=0{\cdot}0124\,Y$

Representing the number coefficients above by $C$, therefore, a soil equation would take the form:

$Y_s$=deflection value from derived equation$=CY$

$$\therefore \quad Y=\frac{1}{C}\text{ (deflection value from derived equation)} \quad . \quad . \quad . \quad (7{:}3)$$

The value of $C$ on soil with variable foundation modulus may be very different and difficult to judge. However, the value in a case can be ignored if the difference obtained between $Y$ and $Y_s$ is limited to, say, 10–12%, and for this purpose it is essential that for beams

(i) with negative $Y_r$, $f$ must not exceed 2·5, and
(ii) with positive $Y_r$, $f$ must not exceed 1·0.

Then the appropriate soil equations can be used without any reference to $C$.

It would normally be seen in practical problems that the above conditions are fulfilled, since the maximum pressures below would control the designs calling for the appropriate stiffnesses for the beams. If in a certain problem either of the above values of $f$ is exceeded within the limiting pressure, the beam has to be made stiffer to bring in more of the unsupported portions to bear on soil (fig. 10) and thus reduce the value of $f$. Alternatively, for beams with positive $Y_r$, an effective shorter bearing length may be considered (i.e. the portion of beam actually bearing on soil in fig. 10(*b*)) in a revised design for both beam and soil equations.

The deflection equations when $f>1$ are derived as follows, bearing in mind that $mf=1$ and $pf=2f-1$:

(i) When $Y_r$ is negative, $f$-system is positive:

*Case 1:* $k_0$ at centre and $nk_0$ at ends:

$$Y=\frac{w_0-mfw_0}{k_0}-\frac{w_0+pfw_0}{nk_0}=-\frac{w_0}{k_0}\left[m+\frac{p}{n}\right]f+\frac{w_0}{k_0}\left[1-\frac{1}{n}\right]$$

$$=-\frac{w_0}{k_0}\left[1+\frac{2f-1}{n}\right]+\frac{w_0}{k_0}\left[1-\frac{1}{n}\right]=-\frac{2w_0}{nk_0}f \quad . \quad . \quad . \quad (S_1')$$

*Case 2:* $nk_0$ at centre and $k_0$ at ends:

$$Y=\frac{w_0-mfw_0}{nk_0}-\frac{w_0+pfw_0}{k_0}=-\frac{w_0}{k_0}\left[p+\frac{m}{n}\right]f-\frac{w_0}{k_0}\left[1-\frac{1}{n}\right]$$

$$=-\frac{w_0}{k_0}\left[2f-1+\frac{1}{n}\right]-\frac{w_0}{k_0}\left[1-\frac{1}{n}\right]=-\frac{2w_0}{k_0}f \quad . \quad . \quad . \quad (S_2')$$

(ii) When $Y_r$ is positive, $f$-system is negative:

A case with $f>1$ will not occur in practice as stated before.

*Case 5:* $k_0$ is uniform, i.e. $n=1$:

The above equations also give $Y=-\dfrac{2w_0}{k_0}f$, as eqn. ($S_5$).

## 8. *Values of final deflection Y and coefficient f*

As stated in para. 4, the final deflection $Y$ should satisfy conditions for both beam and soil properties. Therefore for a particular case, a beam equation and an appropriate soil equation for deflections have to be solved simultaneously to obtain the values of $Y$ and $f$ with proper signs.

In connection with the deflection $Y_e$ in a particular beam equation, it is evident that when $f\leqslant 1$, $Y_e=0{\cdot}0037\,\dfrac{wL^4}{EI}f$. This value of $N=0{\cdot}0037$ may therefore be used in all practical cases as a trial value for solving the equations. If from the solution the absolute value obtained for $f$ is $\leqslant 1$, the result would be satisfactory; and if $>1$, a revision in the coefficient would be necessary, which can then be judged easily from fig. 6, bearing in mind the probable nature of distribution of $F_e$.

It may be worth while to note that a higher value of $N$ than anticipated for a beam, if adopted, should normally give safer results, as the solution would yield lesser values of $f$ and $Y$. In doubtful cases, however, a problem may be solved with two beam equations representing possible upper and lower limits in the values of $Y_e$, and the worse values of obtained moment and shear taken care of at each section. Similarly in a case of doubtful variation in the foundation modulus along a beam, the solution may also be carried out with two soil equations representing the upper and the lower limits.

## IV. Final moments and pressures

### 9. *Moments $M_e$*

These are obtained from force system $F_e$ when the value of $f$ is determined from the solution of the deflection equations. Referring to the Table I it would be clear that even when the value of $f$ is known, the moments $M_e$, with central ordinate $\overline{M}_e$, would depend upon the nature of distribution of force system $F_e$ in a particular case.

The $M_e$-curve was considered in fig. 4 with value of $f \leqslant 1$ and was of the third degree. With the increase in the value of $f$, the shape of the curve tends to change only slightly. For the convenience of obtaining values at intermediate points along the length of a beam, it is sufficient to consider an $M_e$-diagram as triangular with the

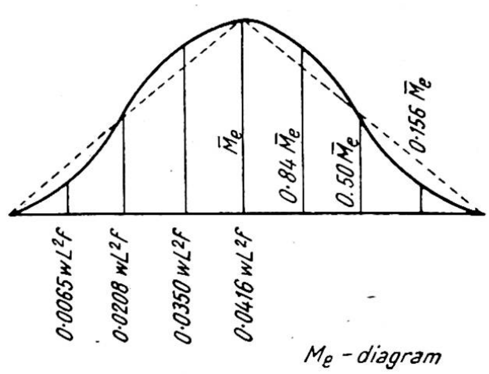



Fig. 11

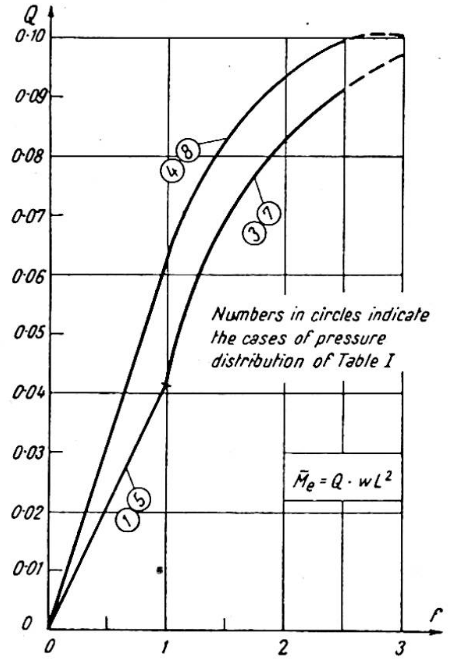



Fig. 12

ordinate $\overline{M}_e$ at centre. Such a diagram is shown in fig. 11 replacing the third-degree curve when $f \leqslant 1$. The differences in the ordinates are only little.

The values of $\overline{M}_e$ under various cases are given in Table I in the form $\overline{M}_e = Q \cdot wL^2$, where $Q$ is a function of $f$. The values of $Q$ under different cases can also be taken from fig. 12 against the values of $f$. As stated in para. 6(C), cases 1, 3 and 5 of Table I would normally cover all practical cases.

### 10. *Final moments M*

At any section of a beam, the final moment $M = \Sigma M_r + M_e$ (para. 4), $M_r$ and $M_e$ being opposite in signs. Note that $M_e$ would carry the sign of $f$.

### 11. *Final pressures under a beam and settlements*

From the value of $f$ obtained, the pressures would be as follows (para. 6):

(i) when $f \leqslant 1$, $p_{max} = w_0 + fw_0$ per unit area

$p_{min} = w_0 - fw_0$ ,, ,, ,,

(ii) when $f > 1$, $p_{max} = 2fw_0$ ,, ,, ,,

$p_{min} = 0$ ,, ,, ,,

These would be clear from the pressure distributions shown in Table I. The balancing pressures B.P. due to the eccentricity of loading on a beam from force system $F_r$ are also to be taken into account.

The settlements at various points in a beam can then be obtained from the related pressures, employing eqn. (7:2).

## V. Examples

### 12(*a*). *Beam on soil with constant foundation modulus*

*Example* 1*. A weightless beam 10 inches by 8 inches with the loading shown in fig. 13(*a*) is resting on an elastic foundation having a modulus of 200 lb./in.$^3$ The elastic modulus of the beam material is $1{\cdot}5 \,.\, 10^6$ lb./in.$^2$ Obtain the moments and pressures throughout the beam.

Thus, $L=120$ in., $I=426{\cdot}7$ in.$^4$, $E=1{\cdot}5 \,.\, 10^6$ lb./in.$^2$ and $k_0=200$ lb./in.$^3$
Total load$=P+48q=5{,}000+4{,}800=9{,}800$ lb.
Bearing area of foundation$=120 \,.\, 10=1{,}200$ in.$^2$

$$\therefore \quad w_0=\frac{9{,}800}{1{,}200}=8{\cdot}16 \text{ lb./in.}^2, \text{ and } w=8{\cdot}16 \,.\, 10=81{\cdot}6 \text{ lb./in. run.}$$

Unbalanced moment and balancing pressures B.P.:

Considering $w$ acting below and taking moments about point 6, unbalanced moment$=5{,}000 \,.\, 90+4{,}800 \,.\, 44-9{,}800 \,.\, 60=73{,}200$ in.-lb.

Section modulus of foundation area

$$Z=\frac{10 \,.\, 120^2}{6}=24{,}000 \text{ in.}^3$$

$$\therefore \quad \text{End pressures in B.P.}=\pm\frac{73{,}200}{24{,}000}=\pm 3{\cdot}05 \text{ lb./in.}^2$$
$$=\pm 30{\cdot}5 \text{ lb./in. run}$$

Moments $M_r$ :

With the superimposed load above and $w$ and B.P. below, values of moments obtained are shown in Fig. 13(*b*).

Deflection $Y_r$ :

From the $M_r$ diagram, the value of maximum deflection $Y_r$ is found conveniently by the "Conjugate Beam Method" at a section 54 in. from the left end as 0·0810 in., which is positive in value. (Approximation of the $M_r$ diagram by straight lines, shown dotted, is permissible for this purpose.)

Beam equation:

Since $Y_r$ is positive, eqn. ($B_2$) of para. 7 applies,

$$\therefore \quad Y=+Y_r+N\frac{wL^4}{EI}f=+0{\cdot}0810+0{\cdot}0037\frac{81{\cdot}6 \,.\, 120^4}{1{\cdot}5 \,.\, 10^6 \,.\, 426{\cdot}7}f$$
$$=+0{\cdot}0810+0{\cdot}0980f \quad \dots \dots \quad (1)$$

Soil equation:

Since $k_0$ is constant and $Y_r$ is positive, eqn. ($S_5$) of para. 7 applies,

$$\therefore \quad Y=-\frac{2w_0}{k_0}f=-\frac{2 \,.\, 8{\cdot}16}{200}f=-0{\cdot}0816f \quad \dots \quad (2)$$

Solution:

Solving eqns. (1) and (2) above, $f=-0{\cdot}45$ and $Y=+0{\cdot}0368$ in. The value of $f$

* The example is taken from *Beams on Elastic Foundation*, by M. Hetenyi, University of Michigan Press, Ann Arbor, 1946, p. 47.

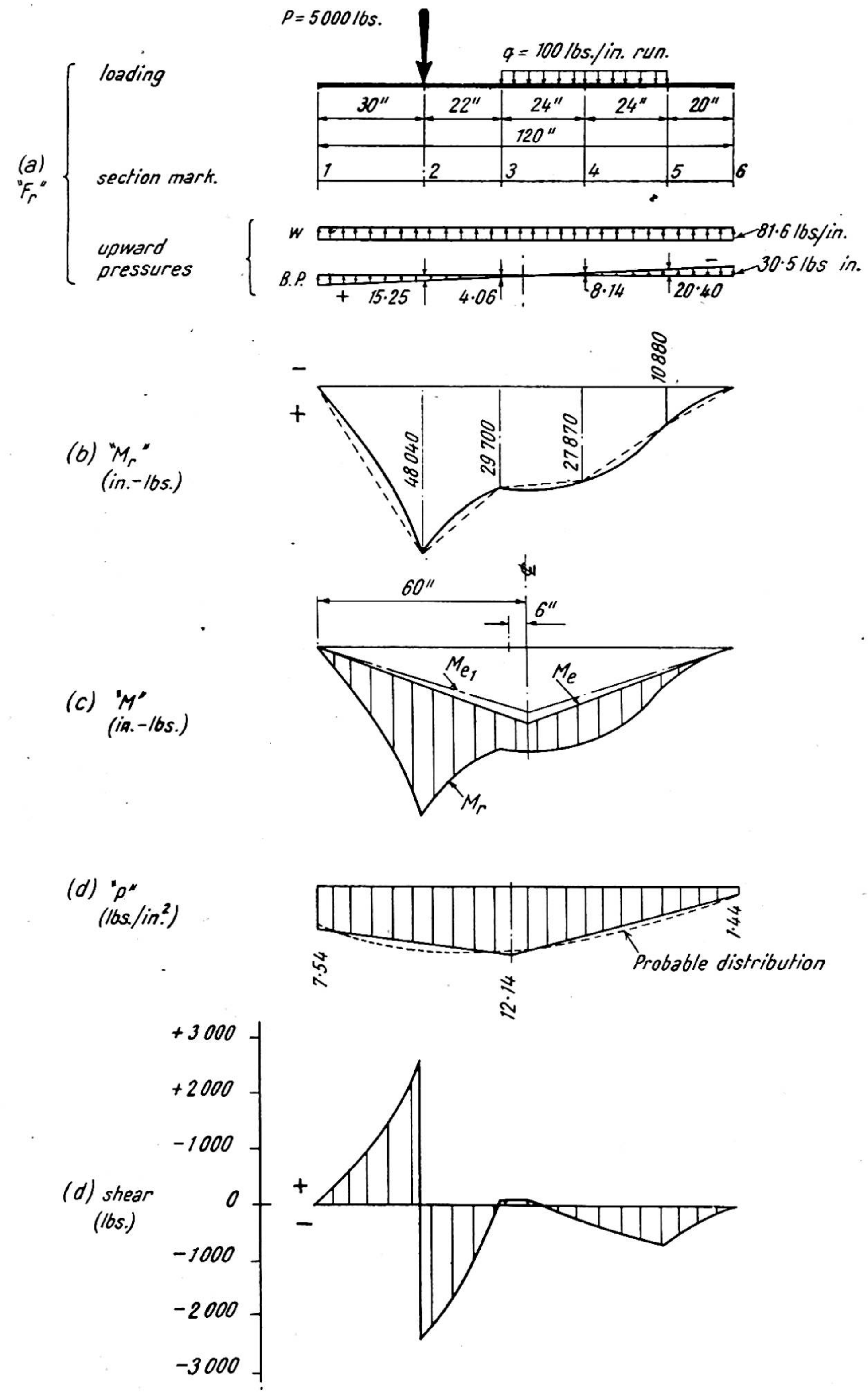

P= 5000 lbs.
q = 100 lbs./in. run.
loading
30" 22" 24" 24" 20"
120"
(a) "Fr"
section mark.
1 2 3 4 5 6
upward pressures
W
81·6 lbs/in.
B.P.
30·5 lbs in.
+ 15·25 4·06 8·14 20·40
(b) "Mr" (in.-lbs.)
10880
48040
29700
27870
(c) "M" (in.-lbs.)
60"
6"
Me1
Me
Mr
(d) "p" (lbs./in.²)
7·54
12·14
1·44
Probable distribution
(d) shear (lbs.)
+3000
+2000
-1000
0
-1000
-2000
-3000

Fig. 13

obtained is $<1$, which shows that the value of $N$ adopted in beam equation is suitable (para. 8). Note that the value is in the negative system.

Moment $\bar{M}_e$ :

Since $f<1$ and $Y_r$ is $+ve$, case 5 of Table I applies. From fig. 12, $Q=0{\cdot}0187$ against $f=0{\cdot}45$.

$$\therefore \quad \bar{M}_e=-QwL^2=-0{\cdot}0187 \,.\, 81{\cdot}6 \,.\, 120^2=-22{,}000 \text{ in.-lb.}$$

This is the central ordinate of the triangular $M_e$ diagram.

Final moments $M$ (in.-lb.):

| Section | $M_r$ | $M_e$ | $M$ | Hetenyi's values of $M$ |
|---|---|---|---|---|
| 2 | +48,040 | −11,000 | +37,040 | +35,460 (calculated) |
| 3 | +29,700 | −19,100 | +10,600 | |
| Centre | +30,000 | −22,000 | + 8,000 | + 9,623 |
| 4 | +27,870 | −16,150 | +11,720 | |
| 5 | +10,880 | − 7,340 | + 3,540 | |

These are shown in fig. 13($c$), with the $M_r$ and $M_e$ diagrams superimposed.

Final pressures $p$ (lb./in.$^2$): $fw_0=0{\cdot}45 \,.\, 8{\cdot}16=3{\cdot}67$

| Section | $w_0$ | $fw_0$ | B.P. | $p$ | Hetenyi's values of $p$ |
|---|---|---|---|---|---|
| 1 | +8·16 | −3·67 | +3·05 | + 7·54 | + 6·07 |
| Section of max. defln. | +8·16 | +3·67 | +0·31 | +12·14 | +10·39 (centre) |
| 6 | +8·16 | −3·67 | −3·05 | + 1·44 | + 1·26 |

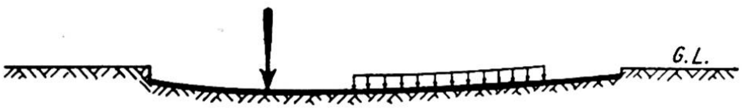
These are shown in fig. 13($d$).

Settlements (inches): From eqn. (7:2), $S=p/k_0$

| Section | $S$ | Hetenyi's values of $S$ |
|---|---|---|
| 1 | 7·54/200=0·038 | 0·03036 |
| Near centre | 12·14/200=0·061 | 0·05193 |
| 6 | 1·44/200=0·0072 | 0·00628 |

Settlements at intermediate points may be found by obtaining the relative deflections.

Fig. 14. shows the beam in its final position.




*Final position of beam*

**Fig. 14**

12($b$). *Value of I for beam to control deflection*

*Example* 2. What should be the value of $I$ for the beam in example 1 if the maximum deflection $Y$ is not to exceed 0·02 in.?

Using the soil eqn., $\qquad Y=-0{\cdot}0816f,$

$$\therefore \quad +0{\cdot}02=-0{\cdot}0816f, \qquad \therefore \quad f=-0{\cdot}245$$

Withdrawing the value of $I$, the beam eqn. is expressed as $Y=+\frac{34\cdot6}{I}+\frac{41\cdot8}{I}f$, and substituting the appropriate values

$$+0\cdot02=\frac{1}{I}[34\cdot6+41\cdot8(-0\cdot245)]=\frac{1}{I}(34\cdot6-10\cdot2)=\frac{24\cdot4}{I}$$

$$\therefore\quad I=\frac{24\cdot4}{0\cdot02}=1{,}210 \text{ in.}^4$$

With 10 in. width, depth $d=\sqrt[3]{\frac{12I}{b}}=\sqrt[3]{\frac{12\,.\,1{,}210}{10}}=11\cdot32$ in.

12($c$). *Value of I for beam to control pressure*

*Example* 3. What may be the value of $I$ for the beam of example 1 if the maximum pressure underneath is not to exceed 14 lb./in.$^2$?

We have seen that in the middle of the beam

$$p_{max}=w_0+fw_0+\text{B.P.}=+8\cdot16+8\cdot16f+0\cdot31=+8\cdot47+8\cdot16f$$

$$\therefore\quad 14=+8\cdot47+8\cdot16f,\quad \therefore\quad f=\frac{5\cdot53}{8\cdot16}=0\cdot675 \text{ (in negative system).}$$

From the soil equation, therefore,

$$Y=-0\cdot0816f=-0\cdot0816(-0\cdot675)=+0\cdot055 \text{ in.}$$

Withdrawing the value of $I$ from the beam eqn.,

$$Y=\frac{1}{I}[34\cdot6+41\cdot8f]$$

and substituting the appropriate values

$$+0\cdot055=\frac{1}{I}[34\cdot6+41\cdot8(-0\cdot675)]=\frac{6\cdot4}{I}$$

$$\therefore\quad I=\frac{6\cdot4}{0\cdot055}=116 \text{ in.}^4$$

With 10 in. width, depth $d=\sqrt[3]{\frac{12\,.\,116}{10}}=5\cdot18$ in.

12($d$). *Beam on soil with variable foundation modulus*

*Example* 4. Solve the problem in example 1 assuming that the modulus varies from 200 lb./in.$^3$ at centre to 350 lb./in.$^3$ at ends. Then, the beam equation, as before

$$Y=+0\cdot0810+0\cdot0980f \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

Soil eqn.:

$$n=\frac{350}{200}=1\cdot5$$

$Y_r$ is $+ve$, and in anticipation of $f\leqslant1$, eqn. ($S_3$) applies.

$$\therefore\quad Y=-\frac{8\cdot16}{200}\left[1+\frac{1}{1\cdot5}\right]f+\frac{8\cdot16}{200}\left[1-\frac{1}{1\cdot5}\right]=-0\cdot0683f+0\cdot0135 \quad . \; . \; . \; . \; . \; . \quad (2)$$

Solving (1) and (2), $f=-0\cdot405$ and $Y=+0\cdot0413$ in. From fig. 12, case 5, $Q=0\cdot0168$.

$$\therefore\quad \bar{M}_e=-0\cdot0168\,.\,81\cdot6\,.\,120^2=-19{,}700 \text{ in.-lb.}$$

The diagram is represented by $M_{e1}$ in fig. 13.

$M_{max}$ at section $2=+48{,}040-\frac{19{,}700}{2}=+38{,}190$ in.-lb.

$p_{max}$ at middle $=+8\cdot16+(0\cdot405\,.\,8\cdot16)+0\cdot31=11\cdot77$ lb./in.$^2$

13. *Beam with non-prismatic section having constant width*

The procedure is the same as shown before except for a little adjustment involved in the value of $Y_e$. For this purpose an equivalent "constant moment of inertia" is obtained for the same amount of maximum deflection within the beam. The example, which follows, will clarify the problem.

*Example* 5*. A continuous footing 30 ft. wide, having a cross-section as shown in fig. 15, rests on soil with a modulus of 300 kips/ft.$^3$ There is a line load of 150 kips/ft. run at the centre and the elastic modulus of the material may be taken as 432,000 kips/ft.$^2$ The weight of the beam is neglected.

Thus,
$$k_0=\frac{300{,}000}{12^3}=173\text{ lb./in.}^3\text{ (uniform)},$$
$$E=\frac{432{,}000{,}000}{12^2}=3{,}000{,}000\text{ lb./in.}^2,$$
$$P=150\text{ kips/ft.}=150{,}000\text{ lb./ft.}$$

Considering 1 ft. length of footing as width of beam, bearing area=360 in. × 12 in. Also
$$w=\frac{150{,}000}{360}=416\text{ lb./in. run, and }w_0=\frac{416}{12}=34{\cdot}8\text{ lb./in.}^2$$

The system $F_r$ is shown in (*a*). The loading being symmetrical about the centre there is no B.P.

$M_r$:

With the load $P$ above and $w$ acting below, the moments developed in the beam are shown in (*c*). The variations in the moment of inertia are shown in (*b*).

$Y_r$:

To obtain the maximum deflection $Y_r$, a diagram for $M_r/I$ is obtained first as in (*d*). From this the maximum deflection at centre, $Y_r=+0{\cdot}196$ in.

$Y_e$:

Equivalent constant moment of inertia $I_c$ for the beam to give the same amount of maximum deflection in the middle under force system $F_e$ is to be considered first. For this purpose the beam is to be considered loaded at the centre with a concentrated unit load when supported at the ends. This is reasonable, since the $M_e$ diagram is nearly triangular, which is corresponding to the above condition of loading.

Let the moment diagram from the unit load be called $M_1$ and the maximum deflection $Y_1$. Then the central ordinate of $M_1$ diagram
$$\bar{M}_1=+\frac{W\,.\,L}{4}=+\frac{1\,.\,360}{4}=+90\text{ in.-lb.} \quad . \; . \; . \; . \; . \quad (13{:}1)$$
shown in (*e*). The maximum deflection with $I_c$,
$$Y_1=+\frac{1}{48}\cdot\frac{WL^3}{EI_c}=+\frac{1}{48}\cdot\frac{1\,.\,360^3}{3{,}000{,}000\,.\,I_c}=+\frac{0{\cdot}324}{I_c}\text{ in.-lb.} \quad . \quad (13{:}2)$$
With the present variable $I$, the maximum deflection $Y_1$ is found from $M_1/I$ diagram as in (*f*), and the value at centre
$$=+0{\cdot}00{,}000{,}365\text{ in.} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13{:}3)$$
From eqns. (13:2) and (13:3),
$$I_c=\frac{0{\cdot}324}{0{\cdot}00{,}000{,}365}=89{,}000\text{ in.}^4 \quad . \; . \; . \; . \; . \; . \quad (13{:}4)$$

* The example is taken from "Successive Approximation for Beams on Elastic Foundations," by E. P. Popov, *Proc.A.S.C.E.*, May, 1950, vol. 76, Separate No. 18, p. 5.

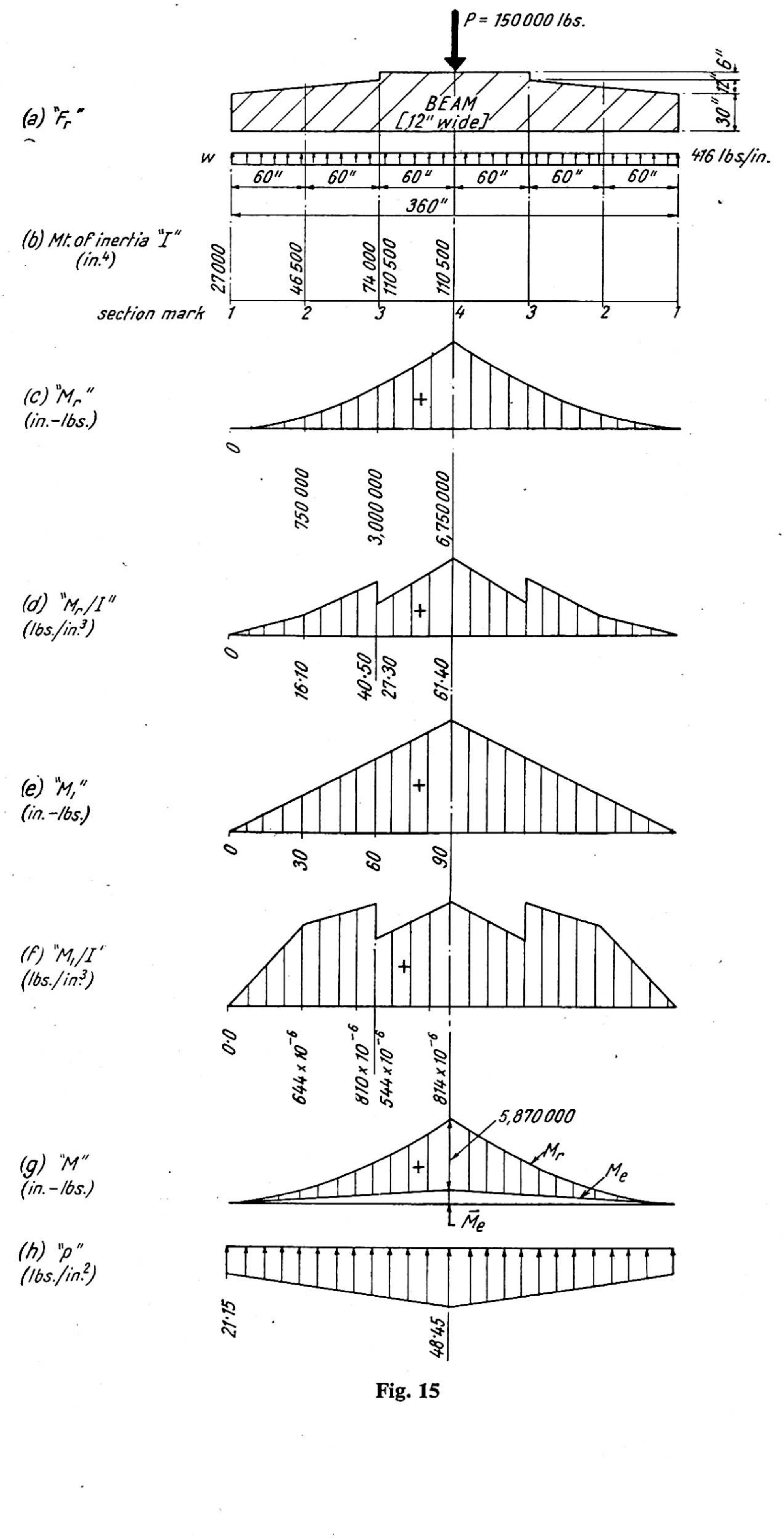

P = 150000 lbs.
(a) "$F_r$"
BEAM
[12" wide]
6"
12"
30"
w
416 lbs/in.
60"
60"
60"
60"
60"
60"
360"
(b) Mt. of inertia "I"
(in.$^4$)
27000
46500
74000
110500
110500
section mark 1 2 3 4 3 2 1
(c) "$M_r$"
(in.-lbs.)
0
750000
3,000000
6,750000
(d) "$M_r/I$"
(lbs./in.$^3$)
0
16·10
40·50
27·30
61·40
(e) "$M_1$"
(in.-lbs.)
0
30
60
90
(f) "$M_1/I$"
(lbs./in.$^3$)
0·0
644 x 10$^{-6}$
810 x 10$^{-6}$
544 x 10$^{-6}$
814 x 10$^{-6}$
(g) "M"
(in.-lbs.)
5,870000
$M_r$
$M_e$
$\bar{M}_e$
(h) "p"
(lbs./in.$^2$)
21·15
48·45

Fig. 15

The procedure hereafter is as for a prismatic beam with constant moment of inertia $I_c$.

Beam equation:

Since $Y_r$ is $+ve$, eqn. ($B_2$) applies.

$$\therefore \quad Y=+0{\cdot}196+0{\cdot}0037\,.\,\frac{416\,.\,360^4}{3{,}000{,}000\,.\,89{,}000}f=+0{\cdot}196+0{\cdot}097f \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Soil equation:

$k_0$ being uniform eqn. ($S_5$) applies.

$$\therefore \quad Y=-\frac{2\,.\,34{\cdot}8}{173}f=-0{\cdot}402f \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Solution:

From eqns. (1) and (2) above, $f=-0{\cdot}392$ and $Y=+0{\cdot}158$ in.

$\bar{M}_e$:

Since case 5 of Table I applies, from fig. 12, $Q=0{\cdot}0163$.

$$\therefore \quad \bar{M}_e=-0{\cdot}0163\,.\,416\,.\,360^2=-880{,}000 \text{ in.-lb.}$$

$M$ (in.-lb.):

These are shown in ($g$).

$p$ (lb./in.$^2$):

$fw_0=0{\cdot}392\,.\,34{\cdot}80=13{\cdot}65$

| Section | $w_0$ | $fw_0$ | $p$ | Popov's values of $p$ |
|---|---|---|---|---|
| 1 | $+34{\cdot}80$ | $-13{\cdot}65$ | $+21{\cdot}15$ | $+18{\cdot}85$ |
| 4 | $+34{\cdot}80$ | $+13{\cdot}65$ | $+48{\cdot}45$ | $+45{\cdot}00$ |

These are shown in ($h$).

## VI. Remarks

### 14. *Remarks*

Comparing the present method with that developed mathematically from differential equations for elastic lines, the solution is reliable for a beam having a value of

$\lambda l \not> 2\pi$, when $Y_r$ is negative, and

$\lambda l \not> \pi$, when $Y_r$ is positive,

where $\lambda=\sqrt[4]{\dfrac{bk_0}{4EI}}$ and $b$=width of beam.

With higher value of $\lambda l$ the pressures are in error, as the deflection curve of the beam develops reverse curvatures at distant points from the loads. The maximum possible bending moment will not, however, exceed the value obtained by this method, and in practical designs with reinforced concrete foundation beams, recourse may have to be made to nominal reinforcements in the compression faces.

**Summary**

The forces acting on a beam are considered to be divided into two systems:

System 1, comprising the superimposed loads on the beam and the pressure underneath such as would occur if the beam were perfectly rigid, due consideration being given to the eccentricity of loading, if any, involving straight-line variation of pressure, and

System 2, comprising only the additional variation of pressure under the beam due to deflections throughout from the average straight-line variation obtained in System 1.

The additional pressure variation of System 2 related to the deflections is obtained from consideration of

(*a*) the elastic properties of the beam, and
(*b*) the elastic properties of the soil.

This being known, the corresponding moment diagram is readily approximated. This diagram, when superimposed on that due to System 1, gives the final moment values throughout the beam.

The advantage of the method lies in obtaining readily

(1) the final bending moment diagram,
(2) the maximum deflection occurring in a beam, and
(3) the maximum and minimum pressures underneath.

Other advantages available from the theory include the determination of the appropriate moment of inertia of a beam to control

(*a*) maximum deflection, and
(*b*) maximum pressure underneath.

The method can be applied to beams, prismatic or non-prismatic, with any kind of loading and solutions give with comparative ease results which are reasonably close to those obtained by accurate analysis. The paper includes illustrative examples already solved by other methods.

**Résumé**

On considère que les forces agissant sur une poutre se divisent en deux systèmes:

1er Système: comprenant les charges appliquées à la poutre et la pression s'exerçant en-dessous, telles qu'elles se présenteraient si la poutre était parfaitement rigide, compte tenu éventuellement de l'excentricité de la charge, impliquant variation de pression en ligne droite.

2ème Système: comprenant uniquement la variation additionnelle de pression sous la poutre, due aux déviations d'un bout à l'autre, à partir de la variation moyenne en ligne droite obtenue dans le 1er système.

La variation additionnelle de pression du deuxième système, relative au déviations, est obtenue par la prise en considération:

(*a*) des propriétés élastiques de la poutre,
(*b*) des propriétés élastiques du sol.

Celles-ci étant connues, on obtient sans difficulté une approximation de la courbe du moment correspondant. Cette courbe, lorsqu'on la superpose à celle qui résulte du premier système, donne les valeurs définitives du moment d'un bout à l'autre de la poutre.

L'avantage de la méthode réside dans le fait qu'on obtient instantanément:

(1) la courbe définitive du moment de flexion,
(2) la déviation maximum se produisant dans une poutre,
(3) les pressions maximum et minimum en-dessous.

Parmi les autres avantages offerts par cette théorie, il fait mentionner la détermination du moment d'inertie d'une poutre permettant d'équilibrer:

(*a*) la déviation maximum,
(*b*) la pression maximum au-dessous.

La méthode peut être appliquée aux poutres prismatiques ou autres, avec n'importe quelle sorte de charge et les solutions donnent, avec une facilité relative, des résultats qui sont suffisamment proches de ceux que l'on obtient par une analyse rigoureuse. L'exposé contient des exemples explicatifs déjà résolus par d'autres méthodes.

**Zusammenfassung**

Die auf einen Balken wirkenden Kräfte werden in zwei Systeme eingeteilt:

System 1 umfasst die auf ihn wirkenden Nutzlasten sowie die auf der Unterlage entstehenden Pressungen für den Fall, dass der Balken vollkommen steif ist. Eine etwaige Exzentrizität der Belastung wird dabei im Sinne eines geradlinigen Verlaufs der Pressungen berücksichtigt.

System 2 umfasst lediglich die zusätzlichen Aenderungen dieser Pressungen entsprechend den Durchbiegungen, die von der für das System 1 gewählten mittleren geradlinigen Verteilung abweichen.

Die zusätzliche Aenderung der Pressungen im System 2 ergibt sich aus der Betrachtung

(*a*) der elastischen Eigenschaften des Balkens,
(*b*) der elastischen Eigenschaften des Untergrundes.

Diese Eigenschaften als bekannt vorausgesetzt, lässt sich die entsprechende Momentenlinie schnell und in guter Annäherung ermitteln. Sie ergibt, nach Ueberlagerung derjenigen des Systems 1 den endgültigen Momentenverlauf im Balken.

Der Vorteil der Methode besteht darin, dass

(1) der endgültige Momentenverlauf im Balken,
(2) die grösste Durchbiegung des Balkens,
(3) die grösste und kleinste Pressung der Unterlage schnell und leicht ermittelt werden kann.

Als weiterer Vorteil ergibt sich aus der Theorie die Möglichkeit, das Trägheitsmoment eines Balkens zweckmässig so festzulegen, dass

(*a*) die grösste Durchbiegung,
(*b*) die grösste Pressung im Untergrund innerhalb bestimmter Grenzen bleiben.

Das Verfahren kann auf Balken prismatischen oder nicht prismatischen Querschnitts und für jede Art von Belastungen angewandt werden. Es liefert auf verhältnismässig einfache Weise Ergebnisse, welche mit den genauen Lösungen gut übereinstimmen. Der Aufsatz enthält Beispiele, die zum Vergleich auch mit Hilfe anderer Methoden gelöst wurden.

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