Lateral stability of beams

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Lateral stability of beams

La stabilité latérale des poutres

Kippstabilität von Trägern

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INTRODUCTION

The problem of lateral stability of beams is not new: the solution for the case of elastic buckling of a beam subjected to a pure bending moment was first given more than half a century ago. This solution, however, was for a thin deep beam and Timoshenko later extended the theory to include I-sections. The mathematical solutions are, however, rather complicated and Timoshenko gave an approximate energy method for an I-girder subjected to a central load. In this theory, however, he neglected the ratio of principal moments of inertia as being small and the theory is only applicable to I-girders. In the following paper it is proposed to give approximate energy solutions for beams subjected to pure bending and to a central concentrated load and no assumption is made as to the size or shape of the member except symmetry about the major axis.

The case of lateral buckling of beams when stressed above the proportional limit has been considered very little. Timoshenko* suggests a possible method of procedure. The problem is considered in more detail in this paper and a method is suggested for calculating the critical loads when the curvature of the stress–strain relationship is taken into account.

ENERGY METHOD FOR OBTAINING THE CRITICAL MOMENT FOR LATERAL BUCKLING OF BEAMS SUBJECTED TO PURE BENDING

Consider a beam of length L subjected to a pure bending moment M about the major axis. Let the bending rigidity about the major axis be A and about the minor axis B. Then due to the bending moment M the beam will take up a curvature of M/A in the plane of bending. The stability of the beam may be considered by supposing that it undergoes some small displacement from this position of equilibrium. If consequent on this small displacement a decrease of energy take place, the beam is

* See Timoshenko, *Theory of Elastic Stability.*

unstable. The critical condition such that the beam is in neutral equilibrium may be found by finding the value of M so that there shall be no gain or loss in energy.

At a distance z from one end of the beam let the lateral deflection be u and the angle of twist θ. There is thus a lateral bending moment of $M \sin \theta$ and a bending moment of $M \cos \theta$ about the major axis. Thus for an elemental length of beam dz there is an increase of strain energy of bending of:

$$\left(\frac{M^2 \cos^2 \theta}{2A}+\frac{M^2 \sin^2 \theta}{2B}-\frac{M^2}{2A}\right)dz=\frac{M^2}{2B}\left(1-\frac{B}{A}\right) \sin^2 \theta \, . \, dz$$

If θ is small, the total increase of strain energy is:

$$\int_0^L \frac{M^2}{2B}\left(1-\frac{A}{B}\right)\theta^2 \, . \, dz \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(a)}$$

It has been shown by many writers* that the torque acting at this cross-section may be written as:

$$\frac{Cd\theta}{dz}-C_1\frac{d^3\theta}{dz^3}$$

where $C\frac{d\theta}{dz}$ is the torque according to the usual St. Venant solution and the term $C_1\frac{d^3\theta}{dz^3}$ allows for non-uniform torsion and warping of the cross-section and may be calculated according to the method given by Timoshenko.

The strain energy due to torsion is thus:

$$\int_0^L \tfrac{1}{2}\left(C\frac{d\theta}{dz}-C_1\frac{d^3\theta}{dz^3}\right)d\theta=\int_0^L \tfrac{1}{2}\left[C\left(\frac{d\theta}{dz}\right)^2-C_1\left(\frac{d^3\theta}{dz^3}\right)\frac{d\theta}{dz}\right]dz \quad . \; . \; . \quad \text{(b)}$$

Thus the sum of (a) and (b) at the critical condition will be equal to the work done by the applied moments M when the beam is allowed to deflect. The work done by M may be calculated by finding the angle through which it turns.

The lateral bending moment $M \sin \theta$ causes the ends of element dz to rotate through an angle $\frac{M \sin \theta}{B} \, . \, dz$ relative to each other. This occurs in a plane at an angle θ to the horizontal and the relative rotation in a vertical plane is $\frac{M \sin^2 \theta}{B} \, . \, dz$. Due to bending about the major axis the ends of the element dz rotate by an amount $-\frac{M}{A} \, . \, dz+\frac{M \cos^2 \theta}{A} \, . \, dz$ relative to each other, the first term being the angle before the deflection θ was given and the second term after. Thus the total relative rotation of the ends is:

$$\int_0^L \left(\frac{M \sin^2 \theta}{B} \, . \, dz-\frac{M}{A} \, . \, dz+\frac{M \cos^2 \theta}{A} \, . \, dz\right)$$

The work done by M is therefore, for small θ:

$$\int_0^L \frac{M^2}{B}\left(1-\frac{B}{A}\right)\theta^2 \, . \, dz=\text{(a)}+\text{(b)}$$

Substituting for (a) and (b) and noting that the beam is symmetrical about the centre, the equation from which the critical moment may be obtained is:

$$\int_0^{L/2} \frac{M^2}{2B}\left(1-\frac{B}{A}\right)\theta^2 \, . \, dz=\int_0^{L/2} C\left(\frac{d\theta}{dz}\right)^2 \, . \, dz-\int_0^{L/2} C_1\frac{d\theta}{dz} \cdot \frac{d^3\theta}{dz^3} \, . \, dz \quad . \; . \quad (1)$$

* See Timoshenko, *Journal of the Franklin Institute*, March, April, May, 1945.

If the relationship between θ and z were known exactly the equation (1) would give an exact value for M_{cr}, the critical moment causing lateral instability. Usually an exact relationship is not known, but if a relationship satisfying the end conditions is assumed, then an approximation to the answer is obtained.

Thus when the ends of the beam are held in such a manner that they are free to warp, $\theta=a_1 \sin\frac{\pi z}{L}$ satisfies the end conditions that $\theta=\frac{d^2\theta}{dz^2}=0$ at both ends of the beam. Substitution of this in equation (a) gives a value for the critical moment with ends free to warp of:

$$M_{cr_1}=\frac{\pi}{L}\sqrt{\frac{BC}{(1-B/A)}}\cdot\sqrt{1+\frac{\pi^2}{L^2}\cdot\frac{C_1}{C}} \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

This agrees with Timoshenko's solution for an I-girder when the value of C_1 for an I-girder is substituted and the value of B/A is neglected. The ratio of B/A may be as high as 0·4 in practice and in those cases its neglect would give appreciable error. The value of the critical moment given in equation (2) is exact because in this case the value of θ assumed is exact.

In a practical case it is almost impossible to apply a moment at the ends without preventing warping and so the case when the ends are completely restrained against warping will now be considered. In this case $\theta=b_1\left(1-\cos\frac{2\pi z}{L}\right)$ satisfies the end conditions that $\theta=\frac{d\theta}{dz}=0$ at both ends. When this value of θ is substituted in equation (1) it is found that the value of the critical moment M_{cr_2} is given by:

$$M_{cr_2}=1{\cdot}15\frac{\pi}{L}\sqrt{\frac{BC}{1-B/A}}\cdot\sqrt{1+\frac{4\pi^2}{L^2}\cdot\frac{C_1}{C}} \quad . \; . \; . \; . \; . \; . \quad (3)$$

This solution is not exact due to inaccuracies in the assumed value of θ. By taking θ of the form (see Timoshenko, *Theory of Elastic Stability*):

$$\theta=b_1\left(1-\cos\frac{2\pi z}{L}\right)+b_2\left(1-\cos\frac{4\pi z}{L}\right)+b_3\left(1-\cos\frac{6\pi z}{L}\right)+\;.\;.\;.$$

a more accurate answer may be obtained. It can be shown that equation (3) is in error by the order of 2%, negligible for all practical purposes. One noticeable point about (3) compared with (2) is that complete restraint against warping increases the critical moment by more than 15%.

Energy solution for a beam subjected to a central concentrated load through the shear centre

Suppose that a central load P is applied at a distance y above the shear centre so as to produce no twist. The stability is considered as for the case of pure bending by assuming the beam to deflect. Let θ_m be the angle of twist at the centre.

Then in the manner already given, the strain energy due to lateral bending is:

$$\int_0^L \tfrac{1}{2}\cdot\frac{P}{4B}\left(1-\frac{B}{A}\right)\theta^2 z^2 dz$$

and the strain energy due to torsion is:

$$\int_0^L \tfrac{1}{2}\left[C\left(\frac{d\theta}{dz}\right)^2-C_1\frac{d\theta}{dz}\cdot\frac{d^3\theta}{dz^3}\right]dz$$

The work done by the central load may be found by considering it in two parts. If the load is applied at the shear centre, the work done by it may be found in a manner similar to that already described for the pure moment. Due to lateral bending the ends of element dz rotate through an angle $\frac{Pz}{2}\sin\theta\frac{dz}{B}$ relative to each other. Since this bending occurs in a plane at angle θ to the horizontal it causes a lowering of the load of $z\sin\theta\,.\,\frac{Pz\sin\theta}{2B}\,.\,dz$. Similarly due to bending about the major axis the load rises by an amount $\frac{P}{2A}z^2\sin^2\theta\,.\,dz$. Thus, if θ is small, total work done by P is:

$$\int_0^{L/2}\frac{P^2}{2B}\left(1-\frac{B}{A}\right)\theta^2z^2dz$$

Due to the load being applied at a distance y above the shear centre there is an additional work done of $Py(1-\cos\theta_m)=Py\theta_m{}^2/2$ approximately. Remembering the symmetry about the centre of the beam, the energy equation then becomes:

$$\int_0^{L/2}\frac{P^2}{4B}\left(1-\frac{B}{A}\right)\theta^2\,.\,dz+\frac{Py\theta_m{}^2}{2}=\int_0^{L/2}C\left(\frac{d\theta}{dz}\right)^2.\,dz-\int_0^{L/2}C_1\frac{d\theta}{dz}\cdot\frac{d^3\theta}{dz^3}\cdot dz\quad . \quad (4)$$

The solutions of this may be found as for the pure bending case and are given below:

When $y=0$ and ends free to warp: $$P_{cr1}=\frac{17{\cdot}1}{L^2}\sqrt{\frac{BC}{1-B/A}\left(1+\frac{\pi^2}{L^2}\cdot\frac{C_1}{C}\right)}\quad . \; . \; . \quad (5)$$

When $y=0$ and ends fixed: $$P_{cr2}=\frac{18{\cdot}3}{L^2}\sqrt{\frac{BC}{1-B/A}\left(1+\frac{4\pi^2}{L^2}\cdot\frac{C_1}{C}\right)}\quad . \; . \; . \quad (6)$$

When the load is applied distance y above shear centre the critical load of equation (5) is reduced to:

$$P_{cr1}{}^1=P_{cr1}\left(1-X+\frac{X^2}{2}\right)\text{ approx.}\quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

where

$$X=\frac{30By}{P_{cr1}\,L^3(1-B/A)}$$

The theory considered so far has been concerned with beams of material which behaved elastically. For beams of, say, aluminium alloy the range of elastic behaviour is small and so the elastic critical loads will not give a good approximation to the failing loads of the beams. Attempts have been made in an empirical fashion to allow for this effect, among others, by assuming some initial imperfection for the beam or some eccentricity of loading. The effect of this is that lateral deflections of the beam occur from the first application of the load becoming infinite, theoretically, near the critical load. The failing load is then determined as that load which causes the stress in the beam to exceed the yield stress of the material or some other predetermined value. A value of the initial eccentricity is then chosen to give good agreement with experiment. This method, whilst giving reasonable agreement between calculated and actual failing load, covers up the essential fact that much of the reduction in failing below the elastic critical load is due to the relationship between stress and strain being non-linear. In this paper it is proposed to give an approach which is dependent only on this fact.

The method follows that originally proposed by Engesser for struts in which the curved stress–strain relationship may be allowed for by an effective modulus of elasticity. For the problem of lateral stability of beams, the method is more complicated due to the fact that there are four factors, A, B, C and C_1, in which the modulus of elasticity plays a part. Since I-beams are most frequently used in practice and are also the simpler to deal with theoretically, the following discussion will be restricted to beams of I-section. The usual proportions of I-section will be taken, so that it is possible to assume that the web has a small effect on the bending and that in bending about the major axis there is a uniform stress in the flanges.

Let us assume the theoretical approach of a beam which remains straight until buckling and then fails by bending laterally and twisting. Before buckling the stress distribution in the I-girder may be considered to be very nearly a uniform compressive stress in one flange and an equal tensile stress in the other. The strain of the flanges will be that corresponding to the stress for the material concerned, and the curvature of the beam will be the strain divided by the distance to the centre of the section. If the stress is greater than the limit of proportionality this curvature is greater than the elastic value given by M/A. It is fairly easy to see that the curvature is increased in the ratio E/E_s where E_s, the secant modulus, is the actual ratio of stress to strain. As will be seen from equation (a), it is the curvature in the plane of bending which introduces the factor A, and it is therefore proposed to allow for this by assuming A to be factored in the ratio E_s/E. This, of course, has no effect when the stress is below the limit of proportionality.

At the critical load, when the beam suddenly deflects laterally and twists, the direct stresses due to lateral bending and the shear stresses due to twist both increase rapidly, whilst the mean direct stress due to the applied moment remains constant. Thus at some points in the beam the direct stress will decrease below that caused by the applied moment, and if the mean direct stress is above the elastic limit, then the reduction in stress will occur as an unloading from the plastic region. Thus the stress

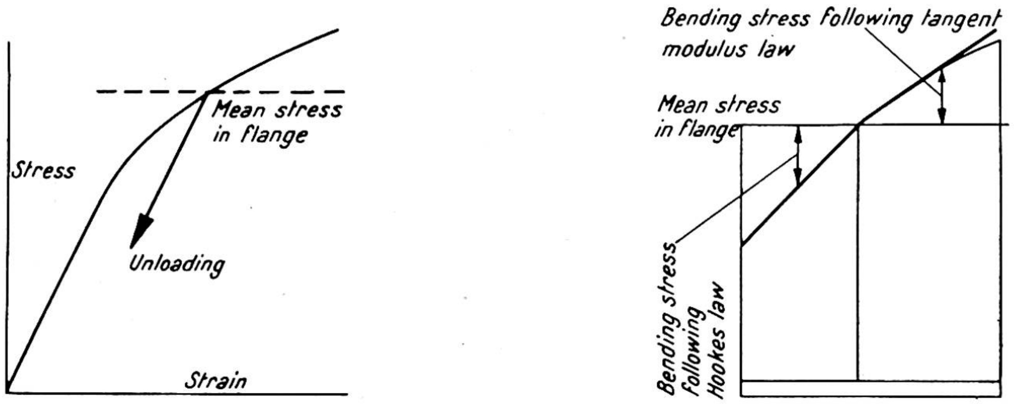

Fig. 1

distribution across a flange will be somewhat as shown in fig. 1, where the increase of loading follows the usual stress–strain curve, but unloading from the plastic region follows the usual Hooke's law.

For small lateral bending moments the increase of stress can be approximated to by a straight line whose gradient E_t is that of a tangent to the stress–strain curve at the point considered. E_t is called the tangent modulus. This effect was first mentioned by Engesser for struts, and it has been suggested that for small lateral bending stresses the effective lateral bending modulus may be taken as a reduced modulus E_r* where:

$$\frac{E_r}{E}=\frac{4E_t/E}{(1+\sqrt{E_t/E})^2}>\frac{E_t}{E}$$

The effective value of B to be used in formulae is then the elastic value factored by E_r/E.

In this purely theoretical case of a beam which remains straight until buckling, the shear stress due to twisting increases rapidly as the direct stress remains constant. There is a certain amount of evidence† that for this case the shear modulus is unchanged and the value of C remains unaltered.

Let us now consider the more practical case where the beam undergoes lateral deflections and rotations before the critical load is reached. These lateral deflections are due to inevitable imperfections in the beam. In this case the deflections first of all occur gradually and then more rapidly when near to the critical load. Thus the shear stress due to twisting increases gradually as the bending moment is applied. When the shear stress increases very gradually in this way while the direct stress increases more rapidly there is evidence† to show that the shear modulus is very nearly the elastic value G factored by E_s/E. Accordingly the torsional stiffness C will be factored in the ratio E_s/E. For more rapid increases of shear stress the effective modulus would be higher and closer to the elastic value which applies when the increase of stress is very rapid. In a similar manner, lateral bending occurs gradually and the direct stress distribution in a flange will change somewhat as shown

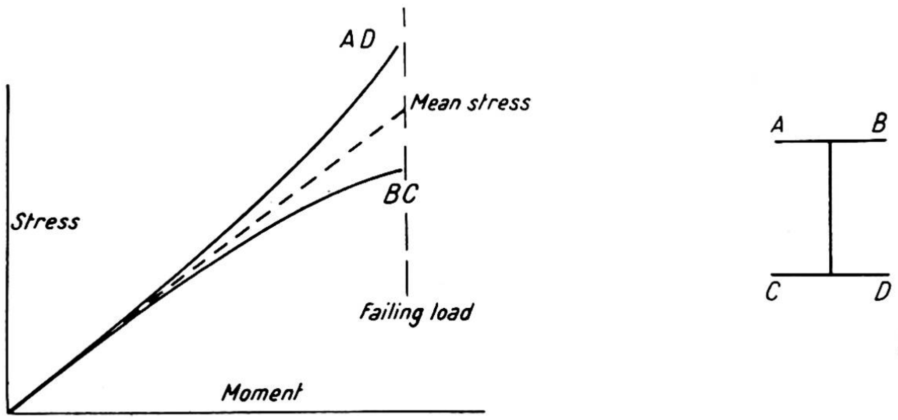

Fig. 2. Variation of stresses in beam with small eccentricity

in fig. 2. The stresses continually increase and the direct stress distribution due to lateral bending of small magnitude is such as to approximate to that given by a tangent to the stress–strain curve.

The effective value of modulus B is thus its elastic value factored by E_t/E. The effective value of the major stiffness A will be the same as that already discussed, that is, $A\times E_s/E$. Since the warping rigidity of an I-girder is provided by differential bending of the flanges, this also will be modified in the ratio E_t/E.

Thus it will be seen that in the more practical case of deflections occurring below the critical moment, the effective values of B and C are lower, giving a lower value of the critical moment. In practice therefore it is to be expected that the values of

* S. Timoshenko, *Theory of Elastic Stability*, McGraw Hill.

† S. Batdorf, "Theories of Plastic Buckling," *Journal of Aeronautical Sciences*, July 1949.

the critical moment will approximate to this lower limit. The value of the critical moment for a beam is now dependent on the material of the beam and not only on the modulus of material as given in equations (2), (3), (5) and (6).

In order to find the critical moment for a beam the stress–strain curve of the material must first be obtained and the values of E_t/E and E_s/E noted for various values of the stress. The value of the critical moment may then be most easily found from (2) and (3) by a trial and error procedure. A value for the stress in the flange caused by the critical moment is assumed so that the values of E_s/E and E_t/E are known. When these values are substituted in the equations a value of the critical moment will be obtained which will probably differ from the originally assumed value. A second approximation to the correct value can then be made until agreement is reached.

The case of the centrally applied load is rather more difficult, since the stress and therefore the effective moduli vary along the beam. Numerical methods of integration are required for the solution. With the assumptions made, the stress in the flange varies linearly from zero at the end of the beam to a maximum p at the centre. The value of P in equation (4) may thus be replaced by $4p/ZL$ where Z is the modulus of bending about the major axis. Equation (4) may then be rewritten for the case where load is applied through the shear centre:

$$\frac{4p^2}{Z^2L^2}\int_0^{L/2}\theta^2 Z^2\left(\frac{1}{B}-\frac{1}{A}\right)dz=\int_0^{L/2}C\left(\frac{d\theta}{dz}\right)^2 dz-\int_0^{L/2}C_1\frac{d\theta}{dz}\cdot\frac{d^3\theta}{dz^3}\cdot dz$$

where A, B, C and C_1 are functions of p.

Assuming some value of p, the effective values of A, B, C and C_1 may be found and each of the integrals of equation found by numerical integration. The solution gives a value of L which agrees with the chosen value of p and hence the value of the critical load for a given L. This procedure may be repeated until the relationship between P and L is found. Of course, in the above the value of Z to be used should not be the usual elastic value but one which allows for the form-factor due to the curved stress–strain relationship. For the usual I-section this correction is small.

Experimental results

Some experiments have been carried out at the Engineering Laboratory, Cambridge, with the support of the Aluminium Development Association to check the above theory. The beams had an I-section $2\frac{1}{2}$ in. deep, by $1\frac{1}{2}$ in. wide by $\frac{1}{8}$ in. thick

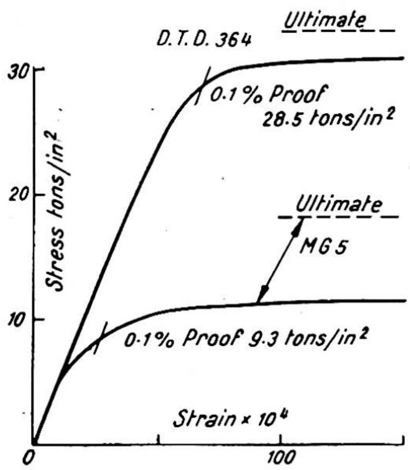

Fig. 3. Stress-strain curves

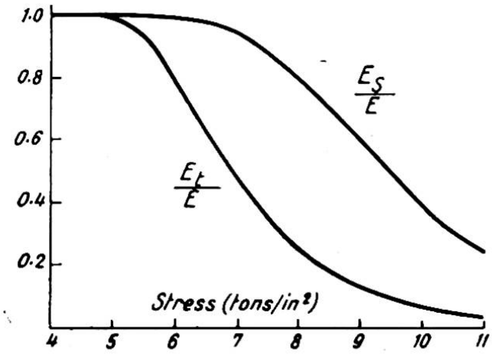

Fig. 4. Moduli for M.G.5

and two materials were used, one to specification D.T.D. 364 and the other M.G. 5, typical stress–strain curves and effective moduli being shown in figs. 3 and 4.

The specimens were supported under conditions of simply supported ends, the beam being free to deflect in vertical and horizontal planes but the ends prevented from twisting. For the case of pure moment the load could not be applied so that the ends were completely free to warp and the method of end fixing is shown in figs. 5 and 6. The blocks bolted to the flanges (fig. 5) located the specimen in the end fittings

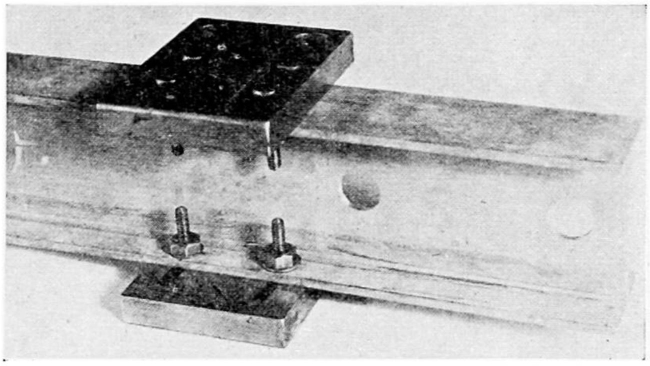

Fig. 5. Blocks locating beams in end fitting

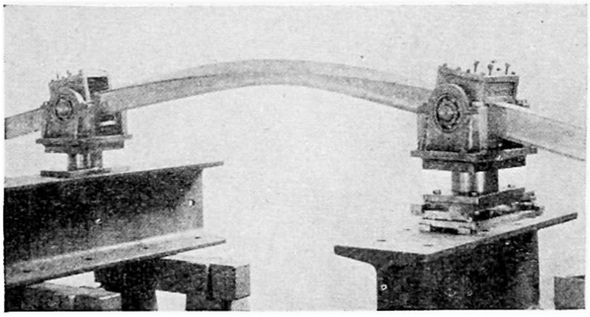

Fig. 6. 50-in. beam at failure

(fig. 6) and also provided some restraint against warping. The blocks were clamped tightly in the end fittings. End moments were applied by means of cantilevers projecting beyond the ends of the specimen.

With the central loading the ends were allowed to warp by supporting the I-section through the web only. With the higher strength alloy, D.T.D. 364, restraint against warping was provided for as in the pure bending case, but with the M.G.5 the ends were welded to $\frac{1}{2}$-in. thick blocks of aluminium in the hope of providing full restraint.

The results of the tests together with the calculated results are shown in figs. 7, 8, 9 and 10.

It will be seen that for long slender beams the failing load may be greater than the critical load. This is to be expected since in this region the critical load falls below the minimum strength of the beam. For the end fittings of type shown in fig. 5 the experimental results lie consistently between the two calculated curves showing approximately the same amount of restraint against warping and that full restraint was not obtained.

On the whole the experimental results seem to agree well with the theory; the largest discrepancies appear in the neighbourhood of the proportional limit, where the "elastic" curve diverges from that calculated by the use of effective moduli. It is in this region that the greatest divergence might be expected, due to the rapid change in slope of the stress–strain curve. For example, consider a practical beam in which there is inevitably some small deflection near the critical load, and let us suppose that the length is such that the critical load just produces a stress equal to the proportional

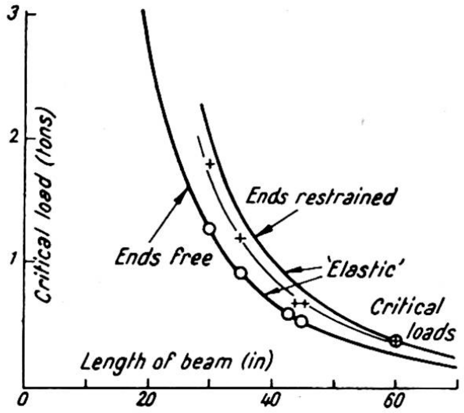

Fig. 7. I-section D.T.D. 364. Central load

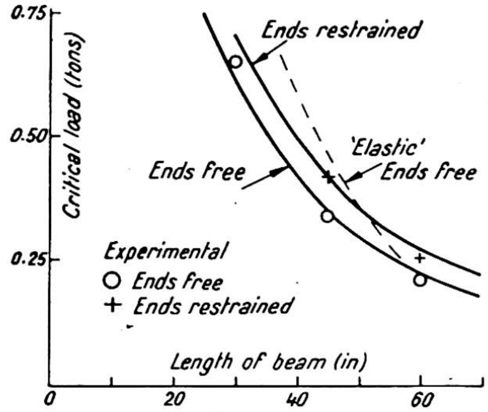

Fig. 8. I-section M.G.5. Central load

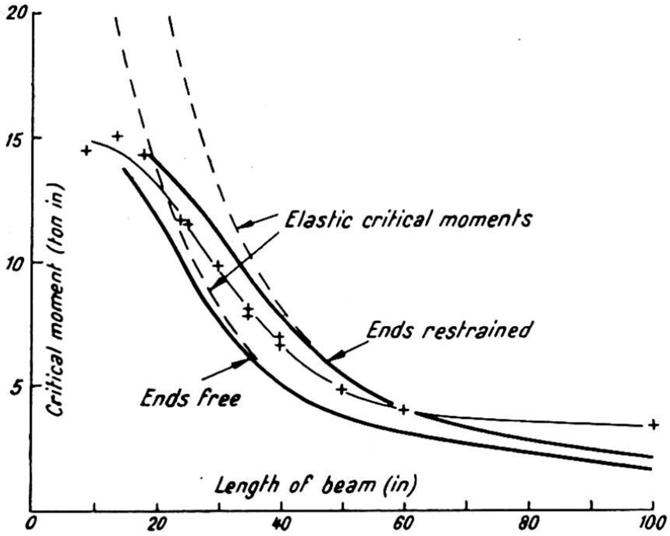

Fig. 9. I-section D.T.D. 364. Pure bending.

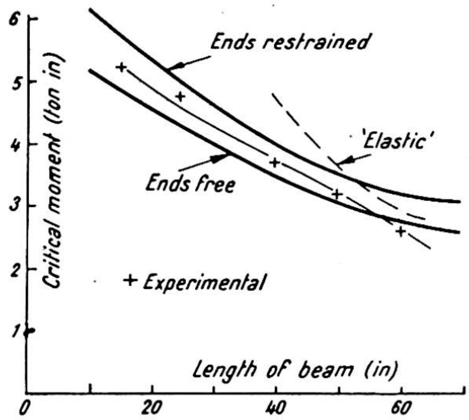

Fig. 10. I-section M.G.5. Pure Bending

limit. Any small lateral bending therefore produces stresses which extend into the region of curved stress–strain relationship and the ratio E_t/E is less than 1. The simplified theory so far considered gives the effective modulus at this point as E and hence it is to be expected that in practice the failing load will be less than that calculated. This difference will be greater, the greater the initial imperfections which produce the lateral deflection, and it is only in this region that the initial imperfections would be expected to have much effect.

Nowhere in the theory has any mention been made of the size of the initial eccentricities which must be present in any practical beam. Some eccentricity was assumed in the theory, in order to produce small deflections below the critical load, but no specific magnitude was attached to it. The basic assumption was that the lateral deflections were small, so that the bending stress distribution could be approximated to by a straight line. It was also assumed that no unloading of the fibres occurred. This second assumption is not strictly true. Measurements of deflection which were taken enabled an estimate to be made of the point at which unloading occurred and it appeared that unloading usually occurred but never below 95% of the failing load. This is sufficiently close to failure to make the assumption reasonable. In this region the lateral bending becomes so large that the first major assumption is no longer tenable and the bending stresses no longer follow a reasonably straight-line law. It can be shown that the effect of unloading and this effect tend to cancel each other and hence the reasonable agreement of the theory with experiment.

Conclusions

On the basis of the experimental data presented it seems that the calculated critical load for lateral buckling does give a good approximation to the failing load of beams in bending, even when the magnitude of the initial eccentricities is neglected.

Summary

The usual mathematical solutions for the problem of lateral stability of beams are long and complicated, particularly when allowance is made for the ratio of the maximum and minimum bending stiffnesses. An approximate energy solution is presented in this paper for the two cases of a beam in pure bending or under a central concentrated load.

The theory is extended to allow for beams fabricated from materials whose stress–strain curve is non-linear, which is the case with aluminium alloys. The method used for this follows that originally presented by Engesser for struts when the usual elastic modulus is replaced by an effective modulus. Experimental results are given for I-beams fabricated from two different aluminium alloys. These results show good agreement with the theory.

Résumé

Les solutions mathématiques habituelles du problème de la stabilité latérale des poutres sont longues et complexes, tout particulièrement lorsque le rapport entre les valeurs maximum et minimum de la rigidité à la flexion est variable. L'auteur présente une solution approchée, basée sur des considérations énergétiques, dans les deux cas de la flexion pure et de la concentration de la charge au milieu de la poutre.

La théorie est élargie aux poutres constituées en un matériau dont le diagramme d'allongement est non-linéaire, comme c'est le cas par exemple pour les alliages d'aluminium. La méthode employée suit celle qui a été indiquée initialement par Engesser, dans laquelle le module habituel d'élasticité est remplacé par un module efficace. L'auteur reproduit des résultats expérimentaux obtenus sur des poutres constituées par deux alliages légers différents. Ces résultats présentent une bonne concordance avec la théorie.

Zusammenfassung

Die üblichen mathematischen Lösungen des Problems de seitlichen Stabilität von Trägern sind lang und kompliziert, besonders bei veränderlichem Verhältnis der grössten zur kleinsten Biegesteifigkeit. Dieser Aufsatz bringt eine Näherungslösung auf Grund einer Energiebetrachtung für die beiden Fälle der reinen Biegung und der Einzellast in der Mitte des Trägers zur Darstellung.

Die Theorie wird erweitert auf Träger aus Material mit nichtlinearem Spannungs-Dehnungsdiagramm, wie zum Beispiel Aluminiumlegierungen. Die dabei verwendete Methode folgt der ursprünglich von Engesser für Streben angegebenen, bei der der übliche Elastizitätsmodul durch einen effektiven Modul ersetzt wird. Es werden Versuchsresultate für Träger aus zwei verschiedenen Aluminiumlegierungen angegeben. Diese Resultate zeigen eine gute Uebereinstimmung mit der Theorie.

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