Cantilever triangular slabs

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Cantilever triangular slabs

Dreieckförmige, längs einer Seite eingespannte Platten

Lages triangulares em consola

Dalles triangulaires en console

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Introduction

In previous publications (¹) the author had the opportunity to present a method to solve partial differential equations, by using the properties of orthogonal functions, conveniently chosen for each case. In those publications, the application of the method referred to was restricted to problems in which the equation to be solved was of 2nd order, having only one boundary condition.

It is intended now to extend that application to the plates problem, characterized by differential equations of 4th order, with two boundary conditions, using the method of calculation of isosceles triangular plates with clamped base, having the two other sides free and symmetrically loaded.

The explanation of the calculation algorithm will not be repeated herein, since it was summarized in the first of the above mentioned papers and given in detail in the second.

(¹) «L'emploi de fonctions orthogonales spéciales pour la solution du problème de la torsion», IABSE, 4th Congress, 1952, p. 189.
«Funções ortogonais na resolução de problemas da teoria da elasticidade». Associação Brasileira de Cimento Portland, São Paulo, 1952.

The plate equation

The plate equation is (2):

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \qquad (1)$$

where

x and y are the coordinates of the points of the plate,
w is the deflection of the plate,
q is the intensity of the lateral load (function of x and y),
D is the flexural rigidity of the plate equal to $Eh^3/12\ (1-\nu^2)$,
h is the thickness of the plate supposed to be constant,
E is the modulus of elasticity of the material and
ν is its Poisson's ratio.

If w_1 is a particular solution of equation (1) and w_0 is its general solution when its 2nd member is zero, we have:

$$w = w_0 + w_1. \qquad (2)$$

It is known also that:

$$w_0 = f_1\ (x + iy) + x\ f_2\ (x + iy) + f_3\ (x - iy) + x\, f_4\ (x - iy)$$

or developping the functions in power series:

$$w_0 = \sum_{m=0}^{\infty} (A'_m + C'_m\, x)\ (x + iy)^m + \sum_{m=0}^{\infty} (B'_m + D'_m\, x)\ (x - iy)^m$$

or combining the terms of the two series in order to obtain the real solution

$$w_0 = \sum_{m=0}^{\infty} (A_m + C_m\, x)\, U_m + \sum_{m=0}^{\infty} (B_m + D_m\, x)\, V_m\,,$$

where $U_m = \frac{1}{2}[(x + iy)^m + (x - iy)^m]$ and $V_m = \frac{1}{2i}[x + iy)^m - (x - iy)^m]$

are the functions already used in the previous papers:

$$\begin{array}{ll} U_0 = 1 & V_0 = 0 \\ U_1 = x & V_1 = y \\ U_2 = x^2 - y^2 & V_2 = 2xy \\ \vdots & \vdots \end{array}$$

The constants A_m, B_m, C_m and D_m must be chosen in order to satisfy the boundary conditions. Taking as x-axis the clamped base of the triangle and as y-axis its axis of symmetry, some of these constants

(2) The derivation of this equation and all the following formulas referring to moments, shearing forces, etc., can be found in specialized books as, for instance, in S. Timoshenko — «Theory of Plates and Shells», Mc. Graw Hill B. Co., 1940, p. 85/94, which notation is used herein.

are eliminated, since functions that contain odd powers of x (condition of simmetry) and functions that do not contain y^2 as a factor (clamped base condition) must not be considered.

This is obtained when the relative values of the constants are such that we have:

$$w_0 = \sum_{m=0}^{\infty} c_m W_m$$

with $(n \geq 1)$

$$b^{2n-1} W_{2n-2} = \frac{1}{2n-1} (x\, U_{2n-1} - U_{2n}) = \frac{y\, V_{2n-1}}{2n-1}$$

$$b^{2n} W_{2n-1} = \frac{x\, V_{2n}}{2n} - \frac{V_{2n+1}}{2n+1}$$

(the factor b^m is introduced in order to make homogeneous the expression of w_o, corresponding to c_m abstract values), i. e.:

$$b\, W_0 = y^2$$

$$b^2 W_1 = \frac{y^3}{3}$$

$$b^3 W_2 = x^2 y^2 - \frac{y^4}{3}$$

$$b^4 W_3 = x^2 y^3 - \frac{y^5}{5}$$

$$b^5 W_4 = x^4 y^2 - 2x^2 y^4 + \frac{y^6}{5}$$

$$b^6 W_5 = \frac{5}{3} x^4 y^3 - 2x^2 y^5 + \frac{y^7}{7}$$

$$b^7 W_6 = x^6 y^2 - 5x^4 y^4 + 3x^2 y^6 - \frac{y^8}{7}$$

$$\vdots$$

The constants c_m must still be determined so as to satisfy the boundary conditions along the side AC of the triangle (since the boundary conditions along the side AB will be automatically satisfied by the condition of symmetry and those along the side BC have already been considered when choosing W_m).

The functions W_m will in turn be grouped so as to obtain functions Z_m which satisfy the demanded conditions of orthogonality:

$$Z_m = W_m + \sum_{n=0}^{m-1} c_{m,n} Z_n \tag{3}$$

with

$$w_0 = \sum_{m=0}^{\infty} a_m Z_m \tag{4}$$

where $c_{m,n}$ and a_m are found with the calculation algorithm described in the mentioned papers and shown in Table I herein.

Boundary conditions

Along the side AC, the bending moments per unit length, which act on planes perpendicular to it, are (positive when they produce tension on the lower face of the plate):

$$M_n = -D\left[(\cos^2\alpha + \nu \operatorname{sen}^2\alpha)\frac{\partial^2 w}{\partial x^2} + (1-\nu)\operatorname{sen} 2\alpha \frac{\partial^2 w}{\partial x \partial y} + \right.$$

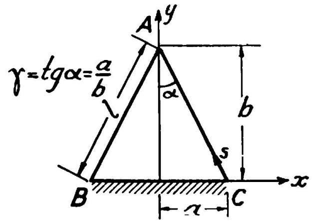

FIG. 1

$$\left. + (\operatorname{sen}^2\alpha + \nu\cos^2\alpha)\frac{\partial^2 w}{\partial y^2}\right] \qquad (5)$$

with

$$\frac{x}{a} = 1 - \frac{y}{b}$$

where α, a and b have the meaning given in Fig. 1.

The external forces which are distributed along AB are, per unit length (positive when they act upwards):

$$V = D\left\{\cos\alpha[1+(1-\nu)\operatorname{sen}^2\alpha]\frac{\partial^3 w}{\partial x^3} + \operatorname{sen}\alpha[1+(1-\nu)(\operatorname{sen}^2\alpha - 2\cos^2\alpha)\frac{\partial^3 w}{\partial^2 x \partial y} + \right.$$

$$\left. + \cos\alpha[1+(1-\nu)(\cos^2\alpha - 2\operatorname{sen}^2\alpha)]\frac{\partial^3 w}{\partial x \partial y^2} + \operatorname{sen}\alpha[1+(1-\nu)\cos^2\alpha]\frac{\partial^3 w}{\partial y^3}\right\} \qquad (6)$$

with

$$\frac{x}{a} = 1 - \frac{y}{b}.$$

In addition to these moments and forces a concentrated force R acts at the triangle vertex (positive if acting upwards):

$$R = -2D(1-\nu)\left[\frac{\partial^2 w}{\partial x \partial y}\cos 2\alpha + \left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right)\operatorname{sen}\alpha\cos\alpha\right] \qquad (7)$$

with $x = 0$ and $y = 1$.

In order to avoid considering R, when discussing the boundary conditions, we can first calculate the effect of a concentrated force at A and then solve the remaining cases without taking R into account, since its effect can always be eliminated if already known.

This case may be considered putting $w = w_o$, since when $q = 0$, $w_1 = 0$ is a particular solution of (1):

$$w = w_o = Z_o + \sum_{n=1}^{\infty} a_m Z_m \qquad (8)$$

and determining the coefficients a_m so that the M_n and V which correspond to the summation $\Sigma a_m Z_m$ be equal and of opposite signs to those due to Z_o. Thus we eliminate the moments and forces along the side AC, leaving only the external force R at A.

The same solution also applies to the case when we have a constant moment along the side AC (corresponding in actual practice to the effect of a horizontal force acting on a balustrade fixed to the slab edge), since the solution $w = w_o = Z_o$ leads to the result looked for, excepting the existence of R, which can be eliminated, as has been said.

For the case of a uniformly distributed load along the side AC (weight of a wall which rests on the slab edge) the boundary conditions are $M_n = 0$ and $V =$ constant. For the load q uniformly distributed over all the plate, the solution is obtained with $w_1 = qy^4/24D$.

The calculations needed for applying the described method will be shown in detail for the case of a plate shaped as a right triangle ($a = b$) made of a material with $\nu = 1/6$.

Composition of the orthogonal functions

Considering the case of a force applied at the vertex of a plate with $\alpha = 45^\circ$ and $\nu = 1/6$, we deduce from (8) combined with (5) and (6) (noting that $Z_o = w_o = y^2/b$):

$$M_n = -\frac{D}{12b}\left[14 + \sum_{m=1}^{\infty} a_m \left(7\frac{\partial^2 Z_m}{\partial x^2} + 10\frac{\partial^2 Z_m}{\partial x \partial y} + 7\frac{\partial^2 Z_m}{\partial y^2}\right) b\right] =$$

$$= -\frac{D}{12b}\left(14 + \sum_{m=1}^{\infty} a_m T'_m\right) \tag{9}$$

$$V = \frac{D}{4b^2\sqrt{2}} \sum_{m=1}^{\infty} a_m \frac{b^2}{3}\left(17\frac{\partial^3 Z_m}{\partial x^3} + 7\frac{\partial^3 Z_m}{\partial x^2 \partial y} + 7\frac{\partial^3 Z_m}{\partial x \partial y^2} + \right.$$

$$\left. + 17\frac{\partial^3 Z_m}{\partial y^3}\right) = \frac{D}{4b^2\sqrt{2}} \sum_{m=1}^{\infty} a_m T''_m \tag{10}$$

expressions that must vanish when $\frac{x}{a} = 1 - \frac{y}{b}$ with $0 < y < b$. The coefficients a_m of (8) must be determined so as to make the summations that appear in (9) and (10) equal to -14 and 0, respectively. In order to facilitate the seeking of its values we choose the functions Z_m, i. e., the coefficients $c_{m,n}$ of (3) in such way that the functions T'_m and T''_m (which form the terms of the summations referred to) may be orthogonal on the interval $0 < y < b$. This, however, cannot be obtained considering T'_m and T''_m separately, as in this way we should obtain different values for the same $c_{m,n}$. The functions T'_m and T''_m must be considered together as if extending over twice the interval, as it is shown for the case of T'_2 and T''_2, with $Z_2 = w_2$, in Fig. 2.

There is yet the possibility of multiplying T''_m by a constant so as to get the same convergence for the two parts of the interval. The disposition given to the coefficients that appear within and without the summations in (10) is already done with a view to that, noting that the mean square error of the orthogonal functions development is minimum when M_n and $Vb\frac{\sqrt{2}}{3}$ are put in the same scale

$I_{11} = 193,7778\, l$ $I_{21} = -137,1111\, l$ $I_{22} = 467,7037\, l$ $I_{31} = -0,266$
$I_{42} = -512,2540\, l$ $I_{43} = -826,0000\, l$ $I_{44} = 1480,1143\, l$ $I_{51} = 251,555$
$I_{55} = 1937,0133\, l$ $I_{61} = -256,2222\, l$ $I_{62} = 787,7460\, l$ $I_{63} = 992,933$
$I_1 = 98\, l$ $I_2 = 93,3333\, l$ $I_3 = 70\, l$ $I_4 = -16,8\, l$

m	l ×	l ×	l ×	l ×	l ×	
1	-	-	-	-	-	-
2	- 137,1111	-	-	-	-	0,707569
3	- 0,2667	433,4558	-	-	-	- 1,169327
4	114,7111	- 431,0880	- 321,7592	-	-	1,468100
5	251,5556	- 302,6103	- 165,7399	692,2077	-	- 1,578565
6	- 256,2222	606,4512	283,4409	- 613,0099	- 355,7693	2,309539

Calculating the functions

$bT'_1 = 14y$ $\qquad bT''_1 = \dfrac{34b}{3}$

$b^2\, T'_2 = 14x^2 + 40xy - 14y^2 \qquad b^2\, T' = \dfrac{4b}{3}(7x - 27y)$

$b^3\, T'_3 = 42x^2y + 60xy^2 - 14y^3 \qquad b^3\, T'' = 2b\,(17x^2 + 14xy - 27xy^2)$

$\vdots \qquad \vdots$

and noting that (notation of Fig. 1):

$$\int_0^l x^m\, y^n\, ds = \frac{m!\; n!}{(m+n+1)!}\, l\, b^{m+n}\, \gamma^m$$

we can perform the integrations

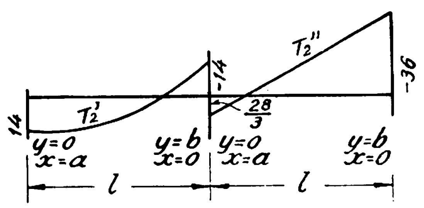

FIG. 2

$$I_{m,n} = \int_0^l T'_m\, T'_n\, ds + \int_0^l T''_m\, T''_n\, ds$$

which enable us to make the Table I in the form described in the previous papers we have already mentioned. From this Table we get coefficients $c_{m,n}$ of (3) which enable us to write:

$Z_1 = W_1$
$Z_2 = W_2 + 0{,}707.569\; Z_1$
$Z_3 = W_3 - 1{,}169.327\; Z_2 + 0{,}001.376\; Z_1$
$\vdots$

E I

$I_{32} = 433{,}6444\ l$ $I_{33} = 726{,}0190\ l$ $I_{41} = 114{,}7111\ l$

$I_{52} = -480{,}6032\ l$ $I_{53} = -519{,}9365\ l$ $I_{54} = 1436{,}3619\ l$

$I_{64} = -1886{,}0722\ l$ $I_{65} = -2345{,}4844\ l$ $I_{66} = 3519{,}9303\ l$

$I_5 = -14\ l$ $I_6 = 80\ l$ $I = 196\ l$

				1 ×	1 ×	a_m
-	-	-	-	193,7778	98,0000	0,505734
-	-	-	-	370,6882	162,6751	0,438846
0,001376	-	-	-	219,1671	- 120,0855	- 0,547918
1,162940	- 0,591972	-	-	438,5043	- 61,9296	- 0,141229
0,756226	0,816347	- 1,298165	-	145,3837	- 1,4725	- 0,010129
1,397956	- 1,293264	- 1,636014	1,322248	189,9809	8,7680	0,046152

Coefficients of the series

According to the procedure we are pursuing here for the calculations, it is necessary to calculate the integrals

$$I_n = \int_0^1 T' \, T'_n \, ds + \int_0^1 T'' \, T''_n \, ds \qquad (11)$$

in order to have $T' = -14$ and $T'' = 0$. With those integrals we calculate the two last columns of Table I, finding the constants a_m of (8), which enable us to write finally

$$w \simeq \frac{y^2}{b} - 0{,}505.734\ Z_1 - 0{,}438.846\ Z_2 + 0{,}547.918\ Z_3 +$$

$$+ 0{,}141.229\ Z_4 + 0{,}010.129\ Z_5 - 0{,}046.152\ Z_6 \qquad (12)$$

We have an idea of the accuracy gotten when we interrupt the series just after the term that contains Z_6, by comparing the sum S of the products of the figures from the last two columns of Table I, with

$$I = \int_0^1 T'^2 \, ds + \int_0^1 T''^2 \, ds = 196\ l \qquad (13)$$

towards which that sum must tend. The mean square error of all the values of 12 M_n/D and 4 Vb $\sqrt{2}/D$ along AC is given by

$$\varepsilon = \sqrt{\frac{I-S}{l}} = \sqrt{196-195{,}913} = 0{,}29$$

This error is small in comparison with the maximum moment that occurs at x = 0 and y = 0 which in the same scale is 12 M/D = 24.

The value of R which produces the deflection surface of equation (12) is, from (7):

$$R = -3{,}07\,\frac{D}{b}\,(1-\nu) = -2{,}56\,\frac{D}{b}\,. \qquad (14)$$

Concentrated load at the vertex

For the case of unit concentrated load acting downwards at the vertex of the slab, we obtain the ordinates of the deflection surface dividing (12) by (14) with changed sign. In so doing we are able to calculate the deflection at the vertex:

$$w_A = 0{,}308\,\frac{Rb^2}{D} \qquad (15)$$

and by the application of the classical formulas we are also able to calculate, in function of the deflection surface, the bending moments (M_x and M_y acting in sections perpendicular to the directions of x and y, respectively), twisting moments (M_{xy}), shearing forces (Q_x and Q_y) and reactions ($V_{BC} = Q_y$ distributed along BC and R_c concentrated at B and C). These values are shown in Fig. 3 and contain a small error occurring because we have neglected the last terms of series (12). These results would be exact if along the edges AB and AC actuate the moments M_n and the forces V given by (9) and (10), in which series, the terms are taken until m = 6. These moments and forces are represented in Fig. 3a by dotted lines. They are drawn in the same scale of M_x, M_y and Q_y (to compare with the reactions along the clamped side BC) and of the bending moments M_t at the sections perpendicular to the edges AB and AC.

Uniformly distributed load over the edges

In this case we proceed as in the case of a concentrated load putting $T' = 0$ and $T'' =$ constant. In order to take advantage of the previous calculations in the determination of the I_n, we take for that constant the value 34 b/3. In this way we obtain:

$$w \simeq 0{,}662.844\ Z_1 - 0{,}162.476\ Z_2 + 0{,}218.720\ Z_3 + 0{,}171.969\ Z_4 -$$
$$- 0{,}131.661\ Z_5 - 0{,}182.313\ Z_6$$

to which corresponds, with the previous notation

$$I = 128{,}44\ 1$$

$$\varepsilon = \sqrt{128{,}444 - 127{,}212} = 1{,}11$$

and the external loads ($P = 2\sqrt{2}\,Vb$ is the total load):

$$R = -2{,}74\,\frac{D}{b}\,(1-\nu)$$

$$V = 2{,}40\,\frac{D}{b^2}\,(1-\nu)\,, \qquad P = 6{,}80\,\frac{D}{b}\,(1-\nu)\,.$$

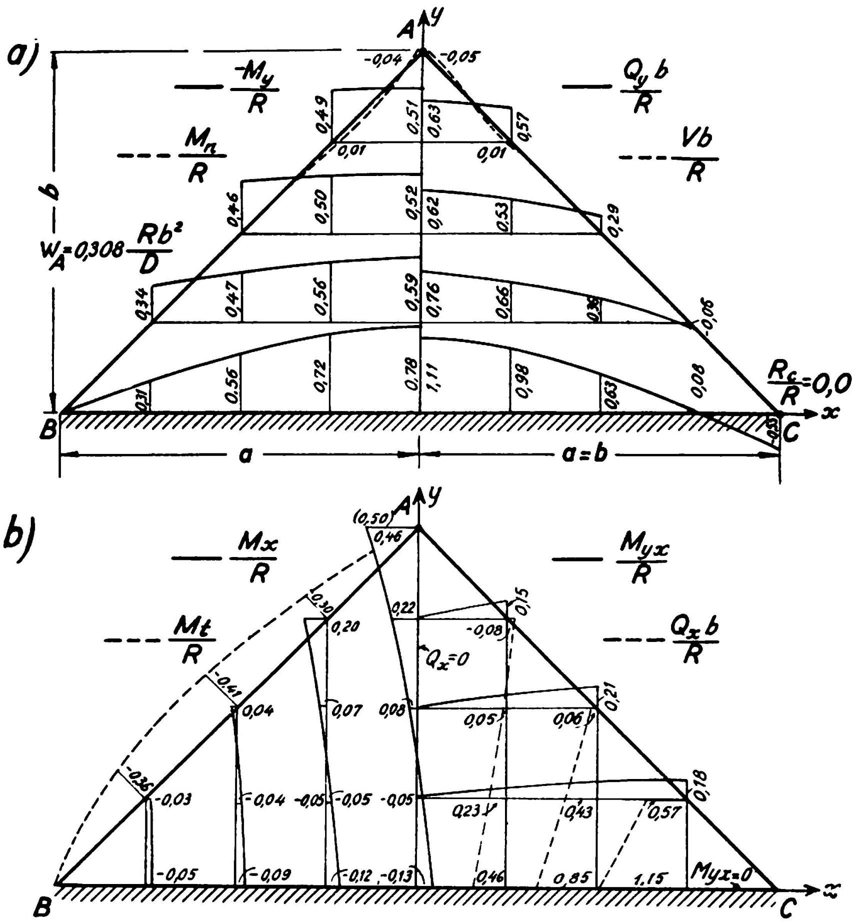

FIG. 3. Concentrated load R at the vertex A

Having subtracted the already known effect of R, we got the results of Fig. 4.

Other load cases

The solution of the case of a constant moment applied along the edges AB and AC is given by the superposition of the solution

$$w = y^2$$

to which corresponds M_n = constant and $V = 0$, but $R = -2D\,(1 - \nu)$, with that of the case of a concentrated force at A, in order to eliminate R.

For a load $p = \frac{P_o}{b^2}$ uniformly distributed over all the plate we make $w_1 = py^4/24D$, to which corresponds, in (11) and (13), $T' = -3{,}5bpy^2/D$ and $T'' = -17pb^2y/3D$.

In the design of reinforced concrete slabs we must know the reactions at the supports and the bending moments. For those two load cases these

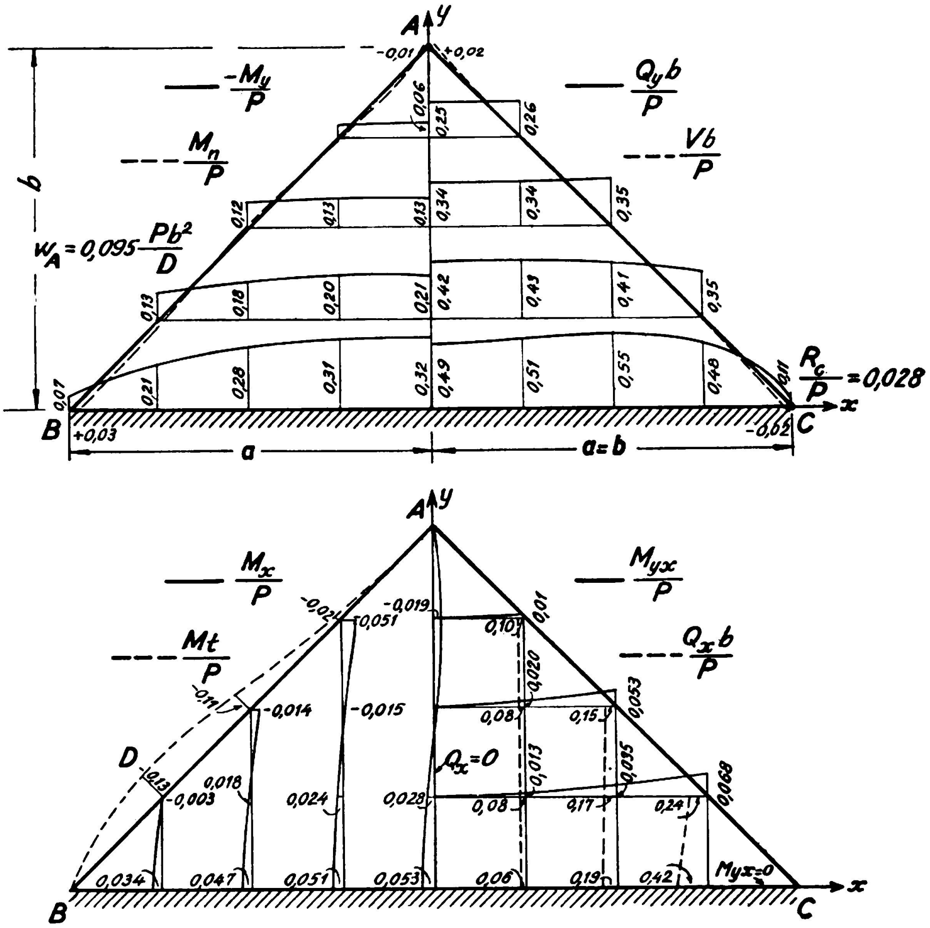

FIG. 4. Uniformly distributed load P over the edges AB and BC

reactions and moments are shown in Figs. 5 and 6. The deflection of the vertex is also given for the solution of the following problem.

The slab supported at the vertex

If the slab besides being clamped along the side BC is also supported at A, the resulting moments, shearing forces and reactions can be obtained adding those obtained for the cantilever slab (Fig. 4, 5 and 6) with those produced by the reaction at A, made equal to – R, so as to use the data of Fig. 3.

The value of this reaction is obtained from the ratio of the deflections at A; in this way we have:

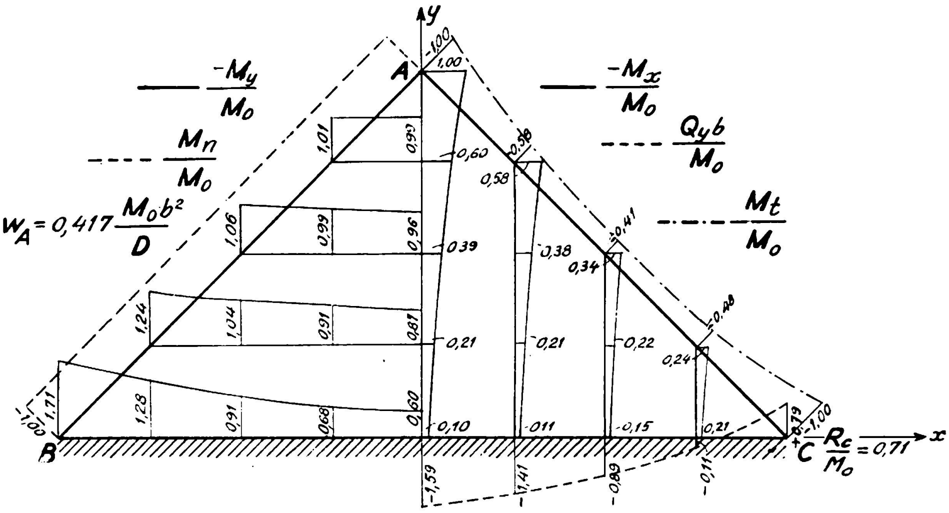

FIG. 5. Constant moment M_0 along the edges AB and AC

1) Load $P = 2\sqrt{2}\ V_b$ distributed over the edge:

$$-R\ \frac{0{,}095}{0{,}308}\ P = 0{,}31\ P;$$

2) Moment M_0 uniformly distributed over the edge:

$$-R = \frac{0{,}417}{0{,}308}\ M_0 = 1{,}35\ M_0;$$

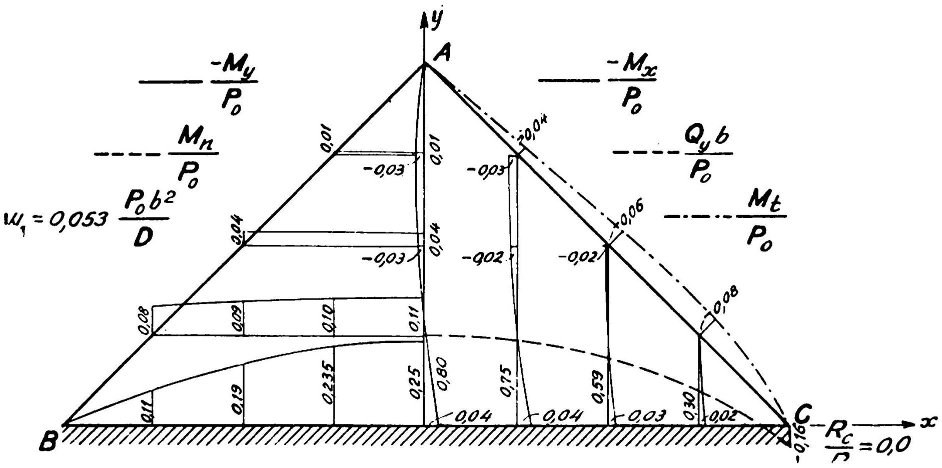

FIG. 6. Load P_0 uniformly distributed over all the plate

3) Load $P = pb^2$ uniformly distributed over the slab:

$$-R = \frac{0{,}053}{0{,}308}\ P = 0{,}17\ P.$$

SUMMARY

The author applies his method, expounded in his contribution to the IABSE 4th Congress, making use of the orthogonal functions to solve the plate equation.

Forming orthogonal functions by the sum of biharmonic functions which are particular solutions of the plate equation, he seeks, employing the mentioned method, the coefficients of the development in series of the function which represents the deflection surface of the plate, so as to satisfy the boundary conditions.

In this way he obtains, by successive approximations and with the desired accuracy, the equation of the deflection surface and, therefore, the shearing forces, bending moments and twisting moments by making use of the partial derivatives classical formulas.

The method is applied to the case of isosceles triangular plates with one side clamped and the two other sides entirely free. A numerical solution is presented for the case of right isosceles triangle and the procedure of calculation to be used in the case when the vertex of the triangle is also supported, is also indicated.

ZUSAMMENFASSUNG

Der Verfasser wendet sein im Beitrag zum 4. Kongress erläutertes Verfahren an, indem er für die Auflösung der Plattengleichung die Orthogonal-Funktionen verwendet.

Aus der Zusammensetzung biharmonischer Funktionen, die Teillösungen der Plattengleichung darstellen, ergeben sich Orthogonalfunktionen. Mit dem erwähnten Verfahren lassen sich die Koeffizienten für die Reihenentwicklung derjenigen Funktion ermitteln, welche die Durchbiegungsfläche der Platte erfasst und damit die Randbedingungen erfüllt.

Auf diese Weise erhält man durch fortlaufende Annäherung und mit der gewünschten Genauigkeit die Gleichung der Durchbiegungsfläche und daraus mit den bekannten Formeln über die partiellen Differentialquotienten die Querkräfte, Biegungs- und Torsionsmomente.

Diese Methode wird für Platten von der Form gleichseitiger Dreiecke angewendet, wobei eine Seite eingespannt ist und die beiden anderen auskragen.

Dann folgt die numerische Behandlung eines rechtwinkliggleichschenkligen Dreiecks und das Berechnungsverfahren für den Fall, dass die Spitze des Dreiecks ebenfalls aufliegt.

RESUMO

O autor aplica o seu próprio método, já exposto quando do 4.º Congresso da Associação Internacional de Pontes e Estruturas, em que utiliza funções ortogonais, à resolução da equação das placas. Formando funções ortogonais pela soma de funções bi-harmónicas que são soluções particulares da equação das placas, procura, empregando o referido método, os coeficientes do desenvolvimento em série da função que representa a superfície flectida da placa, de modo a satisfazer as condições nos limites.

Deste modo obtém, por aproximações sucessivas com a precisão que deseja, a equação da superfície flectida e, portanto, os esforços cortantes, momentos flectores e momentos de torsão, utilizando as fórmulas clássicas de derivação parcial.

Este método é aplicado ao caso de placas triangulares isósceles encastradas num lado estando os outros dois inteiramente livres. O autor apresenta ainda uma solução numérica para o caso do triângulo rectângulo isósceles e o processo de cálculo aplicável no caso do vértice do triângulo ser simplesmente apoiado.

RÉSUMÉ

L'auteur applique sa méthode, déjà exposée lors du 4ème Congrès de l'Association Internationale de Ponts et Charpentes, et oú il utilise des fonctions orthogonales, à la résolution de l'équation des plaques.

Formant des fonctions orthogonales par l'addition de fonctions bi-harmoniques, qui ne sont autres que les solutions particulières de l'équation des plaques, il cherche, en appliquant sa méthode, les coefficients du développement en série de la fonction qui exprime la surface fléchie de la plaque, de manière à satisfaire les conditions aux limites.

Par ce procédé il obtient, par approximations successives et avec la précision désirée, l'équation de la surface flèchie, et, par conséquent, les efforts tranchants moments fléchissants et moments de torsion en utilisant les formules classiques de dérivation partielle.

L'auteur applique sa méthode au cas d'une plaque triangulaire isocéle, encastrée sur un coté, les deux autres étant entièrement libres. Il donne également une solution pour le cas du triangle rectangle isocéle et indique encore le procédé de calcul applicable au cas oú le sommet du triangle est simplement appuyé.

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