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The Calculation of Stresses and Displacements in a Cylindrical Shell Roof Using an Electronic Digital Computer

Calcul des contraintes et des déformations dans une voûte cylindrique mince, à l'aide d'une calculatrice digitale électronique

Die Berechnung der Spannungen und Deformationen in einem zylindrischen Schalendach mit Hilfe eines elektronischen Digitalrechengeräts

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Mathematical Background

For the small displacement elastic analysis of cylindrical shells of constant thickness Jenkins [1] has shown that a particularly useful compatibility equation may be derived in terms of the ring tension T_2 as follows.

$$\nabla^8 T_2 + \frac{t}{IR^2}\frac{\partial^4 T_2}{\partial x^4} = \frac{t}{IR}\frac{\partial^4 Z}{\partial x^4} + \nabla^4\left(\frac{\partial^2}{\partial y^2}\right)\frac{\partial X}{\partial x} - \nabla^4\left(2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial Y}{\partial y}, \quad (1)$$

where the symbols are defined in figs. 2 and 3.

A complete analytical solution to this eighth order partial differential equation cannot be obtained and recourse must be made to the method of constructing solutions of the form $T_2 = \Phi(x).\ \Psi(y)$ where $\Phi(x)$ is a function of x alone and $\Psi(y)$ is a function of y alone. As is usual in the case of simply supported shells and plates $\Phi(x)$ is expressed as an infinite series of harmonic functions. In most engineering calculations it is sufficiently accurate to use simply the first term of this series, further terms, of course, giving increased accuracy. The result of this technique is to produce an ordinary linear eighth order differential equation in y with constant coefficients.

The complete solution of the compatibility equation is given by the sum of the Complementary Function and the Particular Integral. The Complementary Function is the solution of the equation when the right hand side is put equal to zero, and is thus independent of the loads X, Y and Z (fig. 2). This solution contains eight arbitrary constants which are found by inserting four conditions at each of the left hand and right hand edges of the shell (fig. 1). The Particular Integral introduces the loading X, Y and Z.

The complementary function solution for the shell equation (1) can be shown to be

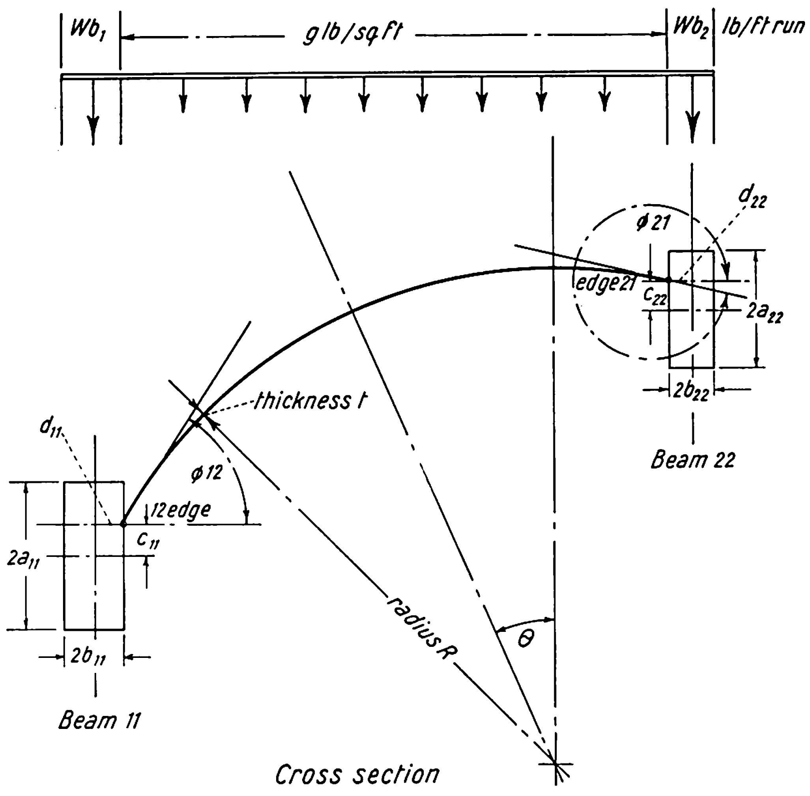

Fig. 1. Shell and Edge Beam. Dimensions and Loadings.

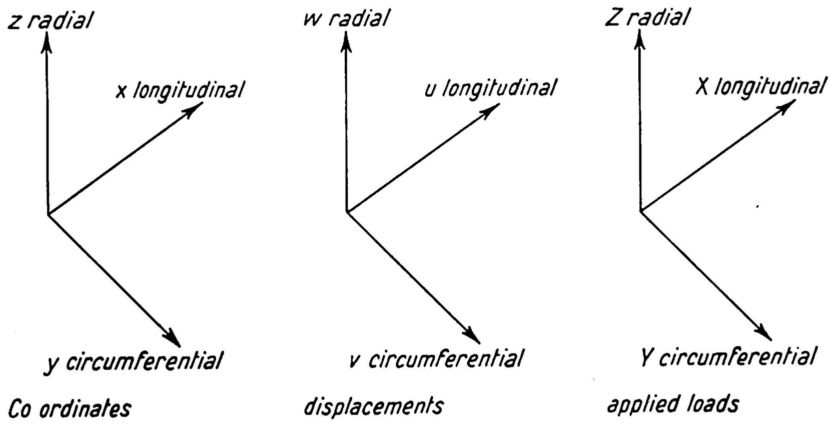

Fig. 2. Shell Co-ordinates; Displacements and Applied Loads.

$$
\begin{aligned}
T_2 &= \Phi(x)\Psi(y), \\
&= \Phi(x)\,[e^{-j\lambda y}(a_1\cos k\lambda y + a_2 \sin k\lambda y) + e^{-j_1\lambda y}(a_3\cos k_1\lambda y + a_4\sin k_1\lambda y) \\
&\qquad + e^{-j\lambda z}(b_1\cos k\lambda z + b_2\sin k\lambda z) + e^{-j_1\lambda z}(b_3\cos k_1\lambda z + b_4\sin k_1\lambda z)], \\
&= \Phi(x)\,[f_1(y) + f_2(z)],
\end{aligned} \tag{2}
$$

where $z = s - y$, s being the arc length.

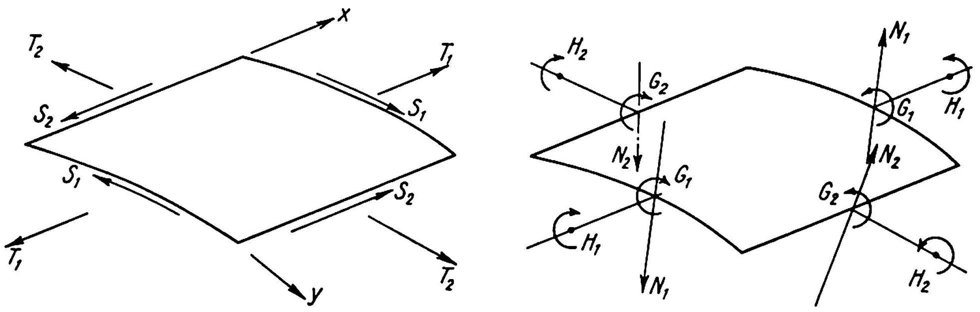

Fig. 3. Tangential and Normal Group Stress Resultants.

Here $j\ k\ j_1$ and k_1 are the roots of the auxiliary equation to (1) and $a_1\ a_2\ a_3\ a_4\ b_1\ b_2\ b_3\ b_4$ are the eight arbitrary constants. Further, $\lambda^4 = \frac{n^2\pi^2}{l^2}\frac{\sqrt{3}}{Rt}$, $\nu^2 = \frac{n^2\pi^2}{l^2}\frac{Rt}{\sqrt{3}}$, n being the harmonic number.

A convenient Particular Integral solution is that formed by considering the shell as a portion of a loaded complete tube simply supported at its ends. To obtain the shell edge conditions the tube is cut along the appropriate generators and its state maintained by applying forces equivalent to the internal forces set up in the complete tube by the external loading. The transverse loading is expressed as a whole number of sinusoidal waves on the circumference, appropriate combinations of normal and tangential loads leading to any desired transverse loading distribution.

A general stiffness matrix for one shell may be obtained by selecting the Kirchhoff group of forces and displacements which may be expressed in terms of T_2 and therefore by (2) in terms of the arbitrary constants a.

$$
\begin{bmatrix} -\frac{\partial S}{\partial x} \\ R_2 \\ T_2 \\ G_2 \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} = \frac{t^2}{2\sqrt{3}} \begin{bmatrix} \lambda & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2\lambda R} & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{2\lambda^2 R} \end{bmatrix} .
$$

$$
\cdot \begin{bmatrix} -j & k & -j_1 & k_1 \\ -j-(1-\nu)k & k-(1-\nu)j & j_1-(1+\nu)k_1 & -k_1-(1+\nu)j_1 \\ 1 & \cdot & 1 & \cdot \\ -1 & -(1+\nu) & 1 & -(1-\nu) \end{bmatrix} \begin{bmatrix} f_1 & f_2 & \cdot & \cdot \\ -f_2 & f_1 & \cdot & \cdot \\ \cdot & \cdot & f_3 & f_4 \\ \cdot & \cdot & -f_4 & f_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}
$$

$$\text{or } \bar{x} = \frac{t^2}{2\sqrt{3}} M\,A\,F^y a \tag{3}$$

$$\begin{bmatrix} E\dfrac{\partial u}{\partial x} \\ E\dfrac{\partial^2 w}{\partial x^2} \\ E\dfrac{\partial^2 v}{\partial x^2} \\ E\dfrac{\partial^2 \theta}{\partial x^2} \end{bmatrix} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2\lambda^2 R} & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \dfrac{1}{2\lambda R} & \cdot \\ \cdot & \cdot & \cdot & \lambda \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} -(1+\nu) & 1 & 1-\nu & 1 \\ \cdot & 1 & \cdot & -1 \\ k-(1-\nu)j & j+(1-\nu)k & k_1+(1+\nu)j_1 & j_1-(1+\nu)k_1 \\ -k & -j & k_1 & j_1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & \cdot & \cdot \\ -f_2 & f_1 & \cdot & \cdot \\ \cdot & \cdot & f_3 & f_4 \\ \cdot & \cdot & -f_4 & f_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

$$\text{or } \bar{u} = N\,B\,F^y a, \tag{4}$$

where $f_1 = e^{-j\lambda y}\cos k\lambda y, \quad f_3 = e^{-j_1\lambda y}\cos k_1\lambda y,$

$f_2 = e^{-j\lambda y}\sin k\lambda y, \quad f_4 = e^{-j_1\lambda y}\sin k_1\lambda y.$

Combining the conditions at both edges, eqs. (3) and (4) become

$$\bar{x} = \frac{t^2}{2\sqrt{3}} M\,A\,F^y a - J\frac{t^2}{2\sqrt{3}} M\,A\,F^z b, \tag{5}$$

$$\bar{u} = N\,B\,F^y a + J\,N\,B\,F^z b, \tag{6}$$

where $J = \operatorname{diag}\{1 \;\; 1 \;\; -1 \;\; -1\}$.

Finally eliminating the arbitrary constants a and b leads to

$$\bar{x}_{12} = \tfrac{1}{2}(\bar{G} - J\bar{G}^s)(I + J\bar{T}^s)^{-1}(\bar{u}_{12} + J\bar{u}_{21}) + \tfrac{1}{2}(\bar{G} + J\bar{G}^s)(I - J\bar{T}^s)^{-1}(\bar{u}_{12} - J\bar{u}_{21}), \tag{7}$$

$$\bar{x}_{21} = -\tfrac{1}{2}J(\bar{G} - J\bar{G}^s)(I + J\bar{T}^s)^{-1}(\bar{u}_{12} + J\bar{u}_{21}) + \tfrac{1}{2}J(\bar{G} + J\bar{G}^s)(I - J\bar{T}^s)^{-1}(\bar{u}_{12} - J\bar{u}_{21}), \tag{8}$$

where $\bar{G} = \dfrac{t^2}{2\sqrt{3}} M\,A\,B^{-1}N^{-1}, \quad \bar{G}^s = \dfrac{t^2}{2\sqrt{3}} M\,A\,F^s B^{-1}N^{-1},$

$\bar{T}^s = N\,B\,F^s B^{-1}N^{-1}.$

Both left hand and right hand edges may then be combined in one matrix equation which represents eight simultaneous equations.

$$\begin{bmatrix} \bar{x}_{12} \\ \bar{x}_{21} \end{bmatrix} = \begin{bmatrix} \bar{P} & -\bar{Q}J \\ -J\bar{Q} & J\bar{P}J \end{bmatrix} \begin{bmatrix} \bar{u}_{12} \\ \bar{u}_{21} \end{bmatrix}, \tag{9}$$

where $\left.\begin{matrix}\bar{P} \\ \bar{Q}\end{matrix}\right\} = \tfrac{1}{2}(\bar{G} + J\bar{G}^s)(I - J\bar{T}^s)^{-1} \pm \tfrac{1}{2}(\bar{G} - J\bar{G}^s)(I + J\bar{T}^s)^{-1}.$

For convenience in considering the combination of shell and edge beams, rotational and translatory transformation matrices R and C enable the shell edge force and displacement vectors to be referred to axes through the centroid of the edge beam in its principal directions.

Thus
$$\begin{bmatrix} X_{12} \\ X_{21} \end{bmatrix} = -\begin{bmatrix} P_{12} & -Q_{12} \\ -Q_{21} & P_{21} \end{bmatrix} \begin{bmatrix} U_{12} \\ U_{21} \end{bmatrix}, \tag{10}$$
where $X_{12} = -C_{12} R_{12} \bar{x}_{12}$,

and $P_{12} = (C_{12} R_{12}) \bar{P} (C_{12} R_{12})'$

the other terms having similar relationships.

Similarly the relation between the edge beam forces and displacements at its centroid in its principal directions can be put in the form
$$X = -P_{11} U.$$

Combined edge beams and shell are then specified by the equation
$$\begin{bmatrix} X_{12} \\ X_{21} \end{bmatrix} = -\begin{bmatrix} P_{11}+P_{12} & -Q_{12} \\ -Q_{21} & P_{22}+P_{21} \end{bmatrix} \begin{bmatrix} U_{12} \\ U_{21} \end{bmatrix}. \tag{11}$$

The edge displacements can now be found by taking into account the boundary stress resultants and displacements of the Particular Integral. Compatibility of the P.I. displacements of the edge beam and shell will generally have to be established, thus causing unbalanced applied forces $X^{(1)}$ at the junction. In a position of equilibrium the total applied force on the junction is zero. The junction displacements are therefore found by putting $X = -X^{(1)}$ in eq. (11).
$$\begin{bmatrix} U_{12}^{(1)} \\ U_{21}^{(1)} \end{bmatrix} = -\begin{bmatrix} P_{11}+P_{12} & -Q_{12} \\ -Q_{21} & P_{22}+P_{21} \end{bmatrix}^{-1} \begin{bmatrix} -X_{12}^{(1)} \\ -X_{21}^{(1)} \end{bmatrix}. \tag{12}$$

The junction displacements having been computed the displacements are determined at uniformly spaced intermediate points across the shell (fig. 4)

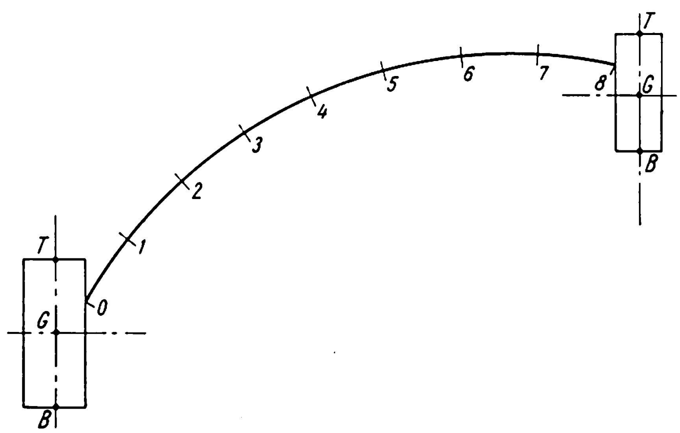

Fig. 4. Location of Output Stresses and Displacements.

by the use of a constant matrix operating on the separated edge displacements. The total displacement is the sum of the displacements caused by the edge disturbances added to the Particular Integral. The total stress resultants at the intermediate points across the shell are determined from the product of the stiffness matrix at the point and the corresponding edge disturbance displacement, added to the Particular Integral stress resultants.

Edge beam stresses and displacements are determined directly from the junction displacements.

The Electronic Computer

The computer for which the shell programme has been written (the Ferranti Pegasus) is a medium sized fixed point machine with a 4,000 (or in some cases 8,000) word drum store. In view of the complexity of the calculation and arbitrary size of the data it was decided to use a floating point scheme. This results in considerable slowing down of the machine but enables very simple programming to be carried out using the "Autocode" and "Matrix Interpretive" schemes.

The Autocode converts the machine into a computer with 50 simple arithmetic and function generating orders. The function generators include such operations as exponentials, harmonics, square roots, etc.

With this scheme there are locations for up to 1,350 numbers and 210 orders. Instructions are printed as in normal algebraic format except that variables are denoted by the symbol v followed by the location number, i.e.

$$v\,100 = v\,28 \times v\,31 .$$

The Matrix Interpretive scheme converts the machine into a computer which performs matrix operations with single orders per operation. For this scheme there are locations for up to 3,070 matrix elements and 80 orders. All the standard operations of matrix algebra are possible, including such operations as transposition, inversion, etc.

Instructions are printed in the following form (multiplication)

$$(N_a, m \times n) \times (N_b, n \times l) \rightarrow N_c$$

where N_a, N_b and N_c are the addresses at the first location of each matrix and m, n and l are the matrix dimensions.

Application of the Computer

From the description of the mathematical background to the problem it will be seen that the computation falls into two stages

a) the determination of the elements of the matrices, and
b) the manipulation of matrices.

The elements of the matrices are algebraic and trigonometric functions of the geometric constants and the loading of the shell. The Pegasus Autocode is a convenient method of undertaking these computations. A typical example of part of the Autocode programme is given below, this being the determination of the auxiliary equation root j which is given by

$$j = \left(\frac{\sqrt{1 + (1+\nu)^2} + (1+\nu)}{2}\right)^{1/2}.$$

The value of ν is available in location $v\,32$

$v\,38 = v\,32+1$	Form $(1+\nu)$,
$v\,39 = v\,38 \times v\,38$	Form $(1+\nu)^2$,
$v\,39 = v\,39+1$	Form $(1+\nu)^2+1$,
$v\,39 = SQRT\,v\,39$	Form $\sqrt{(1+\nu)^2+1}$,
$v\,39 = v\,39+v\,38$	Form $\sqrt{(1+\nu)^2+1}+(1+\nu)$,
$v\,39 = v\,39/2$	Divide by 2,
$v\,40 = SQRT\,v\,39$	Form j.

Note the overwriting of elements when they are no longer required.

The application of the Matrix Interpretive Scheme to part of the shell calculation is illustrated in the following example of the formation of $\bar{G} = \frac{t^2}{2\sqrt{3}} M A B^{-1} N^{-1}$, it being assumed that these matrices are already available in the machine.

$(625,\ 4\times4) \to 900$	Copy B,
$(642) \to 916$	Copy I,
$(900,\ 4\times4),\ (916) \to 917$	Form B^{-1},
$(609,\ 4\times4)\times(917,\ 4\times4) \to 900$	Form $A B^{-1}$,
$(601,\ 4/)\times(900,\ 4\times4) \to 933$	Form $M(A B^{-1})$,
$(605,\ 4/) \to 949$	Copy N,
$(642) \to 916$	Copy I,
$(949,\ 4/),\ (916) \to 953$	Form N^{-1},
$(933, 4\times4)\times(953,\ 4/) \to 900$	Form $(M A B^{-1}) N^{-1}$,
$(641)\times(900,\ 4\times4) \to 933$	Form $\frac{t^2}{2\sqrt{3}}(M A B^{-1} N^{-1})$.

Points to note are that it has been necessary to copy B, I, and N since they are required later in the manipulation and the operation of inversion spoils them.

Input for Computer

The following nineteen numerical values which determine the geometry of the structure and the applied loads are used as the data for the programme (see also fig. 2).

l = length of shell,
R = radius of shell,
t = thickness of shell,
θ = angle of inclination of central radius,
g = load on shell,
n = harmonic number,
ϕ_{12} = inclination of shell edge 12,
ϕ_{21} = inclination of shell edge 21,
a_{11} = half depth of edge beam 11,
b_{11} = half width of edge beam 11,
a_{22} = half depth of edge beam 22,
b_{22} = half width of edge beam 22,
ρ = density of edge beam material,
c_{11}, d_{11} = co-ordinates of intersection of beam 11 and edge 12,
c_{22}, d_{22} = co-ordinates of intersection of beam 22 and edge 21,
W_{b1} = vertical load on beam 11,
W_{b2} = vertical load on beam 22.

The data can be given in any convenient system of units provided all linear dimensions are in the same units and loads and densities are correctly related. All angles are to be given in radians.

Output of Results from Computer

Since the longitudinal distribution of forces and displacements has been specified harmonically, it is only necessary to compute their maximum values and thus obtain the values at any other transverse section by multiplying by the appropriate sin or cos function. In the case of the longitudinally symmetrical functions (w, v, θ, T_1, T_2, G_1, G_2, R_2, and N_2) the maximum value occurs at the midpoint of the span, whereas the maximum value of the longitudinally anti-symmetric functions (u, S, H, R_1 and N_1) occurs at the gables For the shell the transverse distribution is given at 9 points (see fig. 4) for the following 14 stress resultants and displacements, i. e. 126 values.

Eu, Ew, Ev, $E\theta$ — Displacement values obtained by dividing computer results by the chosen value of the elastic modulus E

S, T_1, T_2 — Tangential group forces (fig. 2)

H, G_1, G_2 — Normal group moments (fig. 2)

R_1, R_2, N_1, N_2 — Normal shear forces (fig. 2)

For the edge beams the longitudinally force and the vertical displacement are computed for points T, G and B (see fig. 4) at the mid span of the beams, the longitudinally distribution being symmetrical.

Future Extensions of the Programme

In the first place shells are most frequently used several bays side by side. This merely involves the evaluation of the stiffness matrices for the various shell segments and edge beams and their combination. This analysis has already been suggested by JENKINS [1] and more recently in connexion with computers by MORICE [2].

The second extension is the consideration of other end support conditions, apart from simple supports. The general theory has been developed by MORICE [3] and this follows quite closely the method described above although it is necessary to use SCHORER's [4] governing equation and different stiffness matrices. Broadly however, the steps of the existing programme can be used.

References

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Summary

The paper outlines the principal steps in the exact solution of a circular cylindrical shell segment with boundary members formulated in matrix terminology. The arithmetical stages in the numerical solution may be divided into two parts. Firstly, the evaluation of the matrix elements from the structural geometry and loading. Secondly, the manipulation of the matrices with numerical elements.

A short description of the electronic digital computer demonstrates that it is ideally suited to both these tasks. The method of programming for each is briefly described. The form in which the data is required and the way in which the computer produces the results is also given.

The paper concludes with a short note on the extensions to the work which are in progress.

Résumé

La présente étude expose les pas les plus importants dans la solution exacte du problème du segment d'un voile à cylindre circulaire avec conditions marginales sous forme matricielle.

Les aspects arithmétiques de la solution numérique peuvent être scindés en deux parties. Les éléments matriciels sont tout d'abord déterminés à partir des dimensions de l'ouvrage et des charges; ces matrices sont ensuite traitées avec des éléments numériques.

Une courte description de la calculatrice digitale électronique montre qu'elle constitue un instrument idéal pour ces deux groupes d'opérations. Les auteurs exposent brièvement les méthodes de programmation correspondantes. Ils indiquent également le mode d'introduction des valeurs dans les machines et le processus suivant lequel les résultats sont fournis.

Enfin, une courte notice indique les développements qui se poursuivent dans ce domaine.

Zusammenfassung

Die vorliegende Arbeit umreißt die wichtigsten Schritte in der exakten Lösung für ein Kreiszylinderschalensegment mit Randbedingungen in Matrizenform.

Die arithmetischen Abschnitte der numerischen Lösung können in zwei Teile getrennt werden. Zuerst werden die Matrix-Elemente aus den Tragwerksabmessungen und aus den Lasten bestimmt, und dann werden diese Matrizen mit numerischen Elementen bearbeitet.

Eine knappe Beschreibung des elektronischen Digitalrechners zeigt, daß er ideal für diese beiden Aufgaben geeignet ist. Die Programmierungsmethoden für jede dieser Teilaufgaben wird kurz beschrieben. Ebenso wird die Form der Eingabe der Werte und die Art, wie das Rechengerät die Resultate abgibt, erklärt.

Schließlich ist im Bericht eine kurze Notiz enthalten über die fortlaufenden Erweiterungen dieser Arbeit.