

Zeitschrift: IABSE congress report = Rapport du congrès AIPC = IVBH
Kongressbericht

Band: 6 (1960)

Artikel: A plastic theory for the design of reinforced concrete slabs

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DOI: <https://doi.org/10.5169/seals-6955>

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A Plastic Theory for the Design of Reinforced Concrete Slabs*Théorie pour le calcul plastique des dalles en béton armé**Eine Theorie für die plastische Berechnung von Eisenbetonplatten*

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1. Comparison Between the Elastic Theory and the Plastic Theory

The elastic theory and the plastic theory serve two different purposes. The elastic theory indicates what happens when a slab is loaded within the elastic region, that is to say, with such small loads, that the stresses are proportional to the strains. The plastic theory, on the other hand, indicates what occurs when a slab is loaded to its maximum carrying capacity.

The advantage of the elastic theory is that it provides information regarding what occurs under the action of permissible loads. It may thus be used for calculating deflections and distributions of stresses. It must be borne in mind however, that the elastic theory, when applied to a homogenous, isotropic plate, does not give correct information regarding, for example, the stresses in the steel reinforcement of an ordinary reinforced concrete slab because after the formation of cracks, the slab is neither homogenous nor isotropic. In design procedure it is often assumed that all the reinforcement bars in the critical sections are equally stressed, although the theoretical strains are different. For these and other reasons the elastic theory does not provide accurate information regarding moments, stresses and the amount of reinforcement required, but only a roughly approximate estimate.

The reinforcement in a slab is usually placed in two directions perpendicular to one another. The reinforcement must resist the bending moments m_x and m_y and the torsional moment m_{xy} . It must be pointed out that the torsional moment cannot be assumed to be resisted without the aid of rein-

forcement, since it is converted into a bending moment if the directions of the coordinate axes are altered.

With coordinate axes parallel to the reinforcement bars, the following formulas for design moments have been proposed by the Author [1] and adopted in the Swedish Code [2]

$$m'_x = m_x + k |m_{xy}|, \quad m'_y = m_y + \frac{1}{k} |m_{xy}|,$$

where k is an arbitrary positive constant, which is usually chosen near unity. If k is altered so that m'_x is increased, m'_y will decrease and vice versa.

A complete calculation of a slab according to the elastic theory necessarily involves a study of bending moments and torsional moments at many points in the slab, because the greatest design moments often do not occur at lines of symmetry. A Swedish investigation has shown that engineers have frequently been unaware of this fact and have therefore sometimes used design moments that were too small when calculating by means of the elastic theory. When employed in this incorrect manner the elastic theory is unreliable. On the other hand, if used correctly it is very laborious. For slabs of irregular shape it is often impossible in practice to use the elastic theory in a correct manner.

The plastic theory provides a relatively simple means for calculating the carrying capacity and thus for determining design moments giving a suitable factor of safety against failure. It does not give a single solution, but an infinite number of solutions, that is to say, it is possible to decrease the amount of reinforcement at one section if it is increased at another section. All these solutions are not equivalent for design purposes, because they lead to differences in deflections, crack widths, etc. and also to differences in costs. When the plastic theory is accepted in the Building Code it therefore appears to be necessary to place some restrictions on its application in order to prevent unsuitable solutions from being chosen.

2. Rules for the Application of the Plastic Theory According to the Swedish Building Code

In the Swedish Building Code of 1957 [2] the plastic theory is accepted as a method for calculating bending moments in slabs. Somewhat detailed rules are given for its application, the most important of which are as follows:

1. Slabs in structures which must be watertight, or where limited crack widths are essential, and slabs mainly supporting moving loads must not be designed by means of the plastic theory.

2. The plastic properties of the slab must be adequate. This may be assumed to be the case if the following condition is fulfilled

$$h \geq 100 \sqrt{\frac{\sigma_y m_{el}}{\sigma_{cube} E_e}},$$

where

- h = effective depth of the slab,
 σ_y = yield point of the reinforcement,
 σ_{cube} = cube strength of the concrete,
 E_e = modulus of elasticity of the reinforcement,
 m_{el} = maximum bending moment per unit of width according to the elastic theory.

For moments along supports the mean value may be taken, and for moments on columns the value $\frac{P}{6}$ where P is the reaction at the column. As a rule it is sufficient to use approximate values of m_{el} , often even roughly approximate values (estimated to provide a large factor of safety).

3. The design moment values may not, as a rule, be chosen less than a certain percentage below the corresponding values according to the elastic theory. For slabs protected from the weather this percentage is

$$\frac{\text{dead load}}{\text{dead load} + \text{live load}} \cdot 50 a \text{ per cent},$$

where a is 1.0 for reinforcements which have a marked influence on the deflection of the slab and 2.0 for reinforcements where this influence is small.

Values between 1.0 and 2.0 may be taken for reinforcements which have a moderate influence.

For slabs exposed to the weather half the above values are permissible.

This rule may be neglected for floor slabs in ordinary houses if the permissible stresses are reduced by 10 per cent.

4. The effect of the unfavourable application of live loads has to be taken into account in a similar manner to that employed when the calculation is made by means of the elastic theory.

3. Principles of the Equilibrium Theory. Comparison with the Yield Line Theory

For plates it is impossible, as a rule, to find an exact value of the carrying capacity according to the plastic theory. Approximate values have then to be used, according to one or other of the following principles (cf. [3]):

1. A load that is sufficiently great to cause failure of the plate through the formation of plastic hinges may be found by means of the *yield line theory* [4]. The values found in this manner are theoretically *unsafe*, since we know that the load is sufficiently great to cause failure along certain yield lines, but

we do not know whether failure may not be caused by a smaller load through the formation of other yield lines.

2. A load that can be supported by the plate may be found by means of a design procedure based on a theory which is known as the *equilibrium* theory [5], and which states:

If a distribution of moments can be found which satisfies the equilibrium equation and the boundary conditions for the plate under the action of a given load, and if these moments do not exceed the yield moments at any section of the plate, the plate is capable of carrying that load.

The values found by means of the equilibrium theory are *safe*, since we know that the load can be carried by the plate, but we do not know whether the plate can carry a greater load, corresponding to another distribution of moments, which also satisfies the above conditions.

We thus have one theory giving an unsafe value and another giving a safe value. The exact value is somewhere between these two values. With both theories the calculations can be so refined as to give values closer to the exact solution.

If the yield line theory is used an incorrect determination of the system of yield lines leads to a reduction in the safety factor. It is therefore essential that the calculations should only be made by thoroughly qualified and experienced engineers. In many cases met with in practice it has proved very difficult to determine a sufficiently accurate system of yield lines.

On the other hand, when the equilibrium theory is employed a poor solution leads to an increase in the safety factor. This theory may therefore be used with safety by any engineer. An experienced and well qualified engineer would be able to make the structures more economical by using refined solutions.

From the standpoint of safety, therefore, the equilibrium theory has certain advantages over the yield line theory.

4. Application of the Equilibrium Theory. Strip Method

In orthogonal coordinates the equilibrium equation for a plate is

$$\frac{\partial^2 m_x}{\partial x^2} + \frac{\partial^2 m_y}{\partial y^2} - 2 \frac{\partial^2 m_{xy}}{\partial x \partial y} = -q.$$

Every solution to that equation which satisfies the boundary conditions may be used for the design of reinforced concrete slabs according to the equilibrium theory.

As an illustration of different methods for solving the equation let us consider a simply supported square plate with a uniform load, fig. 1, which may for example, be treated by one of the following procedures:

$$\begin{aligned}
 1. \quad \frac{\partial^2 m_x}{\partial x^2} &= -\frac{q}{2}, & m_x &= \frac{q}{4}(a^2 - x^2), \\
 \frac{\partial^2 m_y}{\partial y^2} &= -\frac{q}{2}, & m_y &= \frac{q}{4}(a^2 - y^2), \\
 \frac{\partial^2 m_{xy}}{\partial x \partial y} &= 0, & m_{xy} &= 0. \\
 2. \quad \frac{\partial^2 m_x}{\partial x^2} &= -\frac{q}{3}, & m_x &= \frac{q}{6}(a^2 - x^2), \\
 \frac{\partial^2 m_y}{\partial y^2} &= -\frac{q}{3}, & m_y &= \frac{q}{6}(a^2 - y^2), \\
 \frac{\partial^2 m_{xy}}{\partial x \partial y} &= \frac{q}{6}, & m_{xy} &= \frac{q}{6}xy.
 \end{aligned}$$

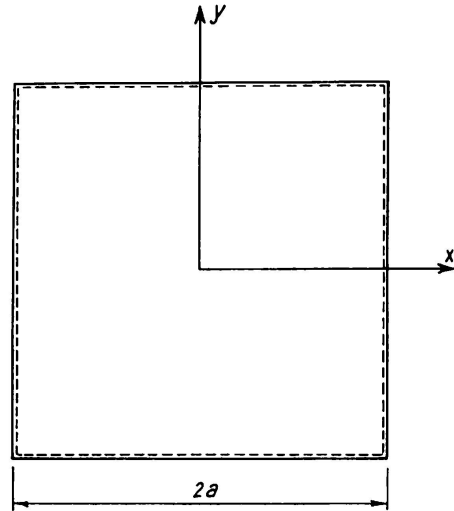


Fig. 1. Simply Supported Square Plate.

The principal bending moments in this case are

$$m = \frac{m_x + m_y}{2} \pm \sqrt{\frac{(m_x - m_y)^2}{4} + m_{xy}^2} = \begin{cases} \frac{q a^2}{6}, \\ \frac{q(a^2 - x^2 - y^2)}{6}. \end{cases}$$

$$3. \quad \begin{cases} |x| > |y| \left\{ \begin{aligned} \frac{\partial^2 m_x}{\partial x^2} &= -q, & m_x &= q(a-y)(a-x) - \frac{q(a-x)^2}{2}, \\ \frac{\partial^2 m_y}{\partial y^2} &= 0, & m_y &= \frac{q(a-x)^2}{2}, \\ \frac{\partial^2 m_{xy}}{\partial x \partial y} &= 0, & m_{xy} &= 0, \end{aligned} \right. \\ |x| < |y| \left\{ \begin{aligned} \frac{\partial^2 m_x}{\partial x^2} &= 0, & m_x &= \frac{q(a-y)^2}{2}, \\ \frac{\partial^2 m_y}{\partial y^2} &= -q, & m_y &= q(a-x)(a-y) - \frac{q(a-y)^2}{2}, \\ \frac{\partial^2 m_{xy}}{\partial x \partial y} &= 0, & m_{xy} &= 0. \end{aligned} \right. \end{cases}$$

The mean value of the positive design moment for the reinforcement in one direction is $\frac{q a^2}{4}$ for solution No. 1 and $\frac{q a^2}{6}$ for Nos. 2 and 3. In this case, solution No. 1, which is the simplest solution, is not economical. Solution No. 2 requires negative reinforcement near the corners (beyond a circle of radius = a) and is therefore less economical than solution No. 3.

As a rule it is preferable to use solutions with $m_{xy} = 0$, which implies that the load is carried by means of the bending moments m_x and m_y only. The plate can then be regarded as composed of strips in both directions and the loads may be divided into two parts q_x and q_y , which are carried in the x - and

y -directions, respectively. This method of calculation has been called the *strip method* and, as a rule, it is the most suitable method of applying the equilibrium theory.

In using the strip method, the load may be divided between the x - and y -strips in different ways. In the above example, solutions Nos. 1 and 3 show the two main possibilities. In solution No. 1 a constant part of the load is carried in one direction and the remainder in the other direction. In solution No. 3, the entire load on certain parts of the plate is carried in one direction and the whole of the load on the remainder of the plate in the other direction whereby the load at each point is preferably carried by the strip in which it causes the smallest moment. The former is the simplest and the latter the most economical solution. The application of the strip-method to rectangular plates under the action of distributed loads (uniform or non-uniform) is quite simple and will not be discussed in detail in this paper.

Sometimes a plate has the bottom and top reinforcements in different directions. In that case the strips should be assumed to act in the direction of the reinforcement in tension. The direction of the strips is then changed at the line of zero moment and the shear force along this line must be taken into account in the calculation of moments. As an example, let us consider the circular plate shown in fig. 2, which is built-in along the edges and acted upon by a uniform load q . The bottom reinforcement is orthogonal and the top reinforcement is radial. Within the line of zero moment it is assumed that half the load is carried in each direction. The moment in the x -direction is then

$$m_x = \frac{q}{4} (r^2 - y^2 - x^2).$$

The reaction force from the x -strip per unit of width of the line of zero moment is

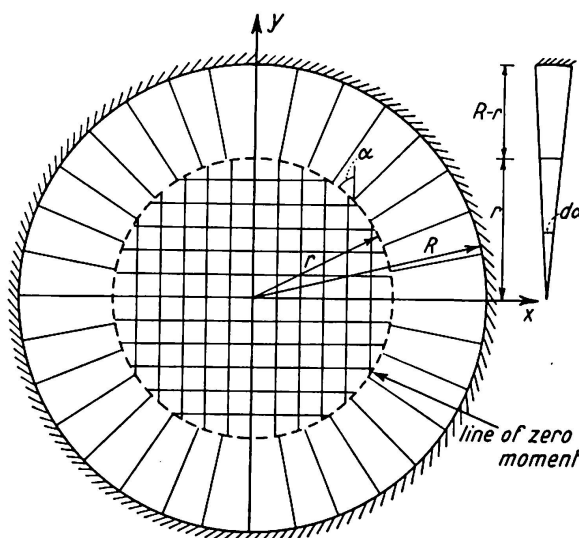


Fig. 2. Circular Plate with Orthogonal Bottom Reinforcement and Radial Top Reinforcement, Treated with Strip-method.

$$Q_x = \frac{q}{2} r \cos \alpha \cos \alpha = \frac{q r}{2} \cos^2 \alpha.$$

The total reaction force per unit of width of the line of zero moment is

$$Q = Q_x + Q_y = \frac{q r}{2} (\cos^2 \alpha + \sin^2 \alpha) = \frac{q r}{2}.$$

The bending moment at the support of the radial strip is

$$m_{r, max} = -\frac{1}{R d \alpha} \left[\frac{q r}{2} r d \alpha (R - r) + q r d \alpha \frac{(R - r)^2}{2} + q (R - r) d \alpha \frac{(R - r)^2}{6} \right] = -\frac{q (R^3 - r^3)}{6 R}.$$

If, for example, we take $r = 0,6 R$, we then have $m_{r, max} = -0,131 q R^2$ and $m_{x, max} = 0,090 q R^2$.

In plates which are supported by columns, as shown in fig. 3, the moments under the action of a uniform load may be calculated in the following manner [6].:

The plate is divided by lines of zero shearing force according to fig. 3. In parts which are supported along an edge the entire load is assumed to be

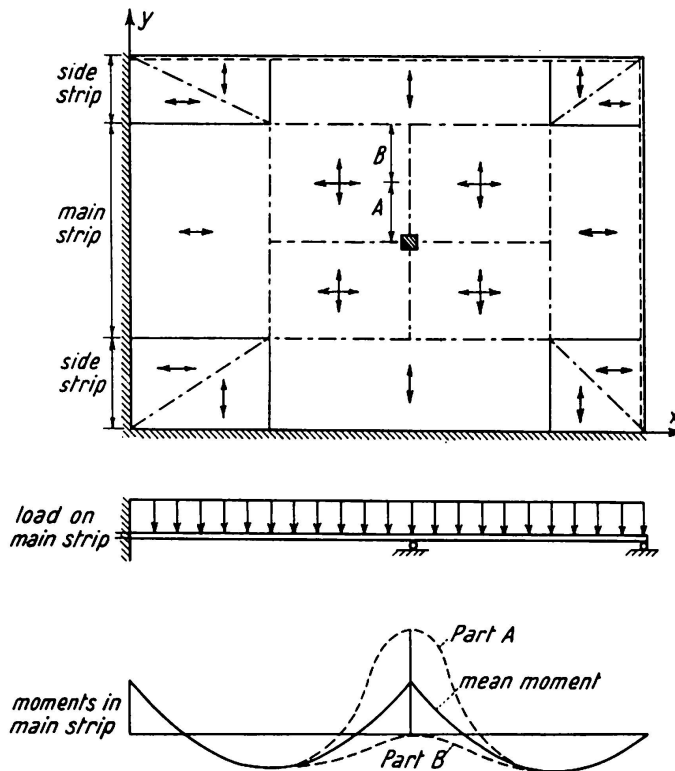


Fig. 3. Rectangular Plate Supported Along the Edges and by a Column, Treated with Strip-method.

- - - - Lines of Zero Shearing Force = Maximum Bending Moment.
- ← → Direction, in Which the Load is Assumed to be Carried.

carried perpendicularly to that edge. In those parts of the plate which are supported at a corner, the mean bending moments along two opposite edges are equal to those which would be obtained if the plate were supported along one of these edges. The calculations must therefore be made for each direction as if the plate were carrying the entire load in that direction. The actual distribution of moments within that part of the plate is somewhat complicated and will not be discussed in detail here. It has been shown in [6] that it is reasonable to assume that the maximum negative moment is uniformly distributed along one-half of the edge. The design moment at that part of the edge is thus twice the mean moment.

From fig. 3 it can be seen that we obtain fully loaded strips (= main strips) in both directions. The strip in the x -direction is shown in the figure. The points of zero moment in the strip in the x -direction give the width of the strip in the y -direction and vice versa. The mean design moments in the side strips are $1/3$ times the moments in the main strips (part B).

The reactions at the walls and the column are easily calculated, since we know the position of the lines of zero shearing force. The same method of calculation can also be used for other types of plates, for example those shown in fig. 4.

As a final example of the calculation by means of the strip method let us

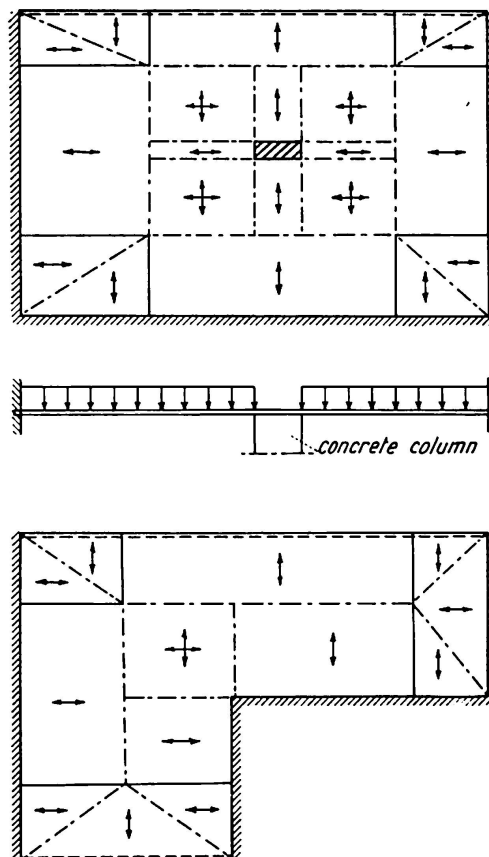


Fig. 4. Examples of Plates, Which Can be Treated with Strip-method.

consider an elongated simple-supported plate carrying a narrowly distributed load Q per unit of length across a small span (fig. 5). The distribution of the loads may be assumed to occur according to fig. 5, that is to say, the load Q is carried by strips in the x -direction, which in their turn are carried by strips in the y -direction. The maximum design moments are

$$m_x = \frac{Q a}{8}, \quad m_y = \frac{Q l^2}{8 a}.$$

The value of a may be chosen so as to give a suitable and economical construction.

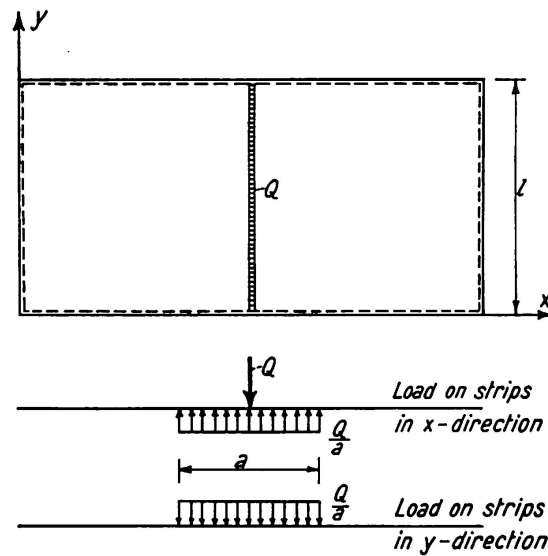


Fig. 5. Plate With a Narrow Load.

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Summary

In Sweden the plastic theory has been accepted as a means of calculating moments in reinforced concrete slabs under certain conditions, which are briefly outlined in the present paper. Two different theories may be used,

viz. the yield line theory and the equilibrium theory. The latter has the advantage of being safer. The main purpose of the present paper is to describe this theory and to give some examples of its application.

Résumé

En Suède, la théorie de la plasticité a été admise à titre de méthode pour le calcul des moments dans les dalles en béton armé, dans certaines conditions que la présente contribution expose brièvement. Deux théories différentes peuvent être employées: la théorie des lignes de rupture et la théorie dite de l'équilibre. Cette dernière présente l'avantage d'une plus grande sûreté. L'auteur traite essentiellement de cette théorie et présente quelques exemples de son application.

Zusammenfassung

In Schweden wurde die Plastizitätstheorie als Methode für die Momentenberechnung bei armierten Betonplatten unter gewissen Bedingungen anerkannt, die der vorliegende Beitrag kurz darlegt. Es können zwei verschiedene Theorien verwendet werden, und zwar die Bruchlinientheorie und die sogenannte Gleichgewichtstheorie. Die letztere hat den Vorteil größerer Sicherheit. Das Hauptanliegen der vorliegenden Arbeit ist die Beschreibung dieser Theorie und die Vorführung einiger Anwendungsbeispiele.