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The Use of Stability Functions in the Analysis of Rigid Frames

Emploi de fonctions de stabilité pour l'étude des cadres rigides

Die Verwendung von Stabilitätsfunktionen zur Untersuchung von steifen Rahmen

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Stability Functions

It does not yet appear to be generally known that tables of stability functions [1] are available which considerably simplify the elastic stability analysis of rigid frames and even make practicable the solution of frames containing plastic hinges.

JAMES [2] appears to have been the first to extend the Cross method of moment distribution to include the effects of axial loads in members by introducing the ideas of non-dimensional stiffness and carry-over factors (s and c) which are not constant but depend on the axial loads in the members. Thus in fig. 1a

$$M_{AB} = s\,k\,\theta \quad \text{and} \quad M_{BA} = c\,M_{AB},$$

where

$$k = \frac{E\,I}{l}, \qquad P_E = \frac{\pi^2 E\,I}{l^2},$$

s is the stiffness factor and c the carry-over factor and both s and c are functions of $\frac{P}{P_E}$ as shown in fig. 2.

Values of s and c were tabulated in terms of

$$\alpha = \frac{\pi}{2}\sqrt{\frac{P}{P_E}}$$

by JAMES and later by LUNDQUIST and KROLL [3] and with these tables, frames, such as triangulated frames, where the sway of the members is not important, could be solved by ordinary moment distribution methods. A series of functions, usually known in England as "Berry Functions", related to s and c and from which they can be obtained have also been tabulated [4, 5, 6]. The tables of reference [1] are the first in which $\frac{P}{P_E}$ is taken as the independent variable and also cover a wider range than previous tabulations.

Where sway occurs we have to consider the shear equilibrium of a structure as well as the equilibrium of a joint and if axial load effects are important, the relation between the shear and the end moments is not the same as in the no stability case.

Thus for a simple sway displacement (fig. 1 b)

$$M_{AB} + M_{BA} + Fl + Pl\phi = 0.$$

And $$M_{AB} = M_{BA} = -s(1+c)k\phi.$$

Therefore $$Fl = +2s(1+c)k\phi - \pi^2 \rho k\phi,$$

where $$\rho = \frac{P}{P_E},$$

i.e. $$Fl = \frac{2s(1+c)k\phi}{m},$$

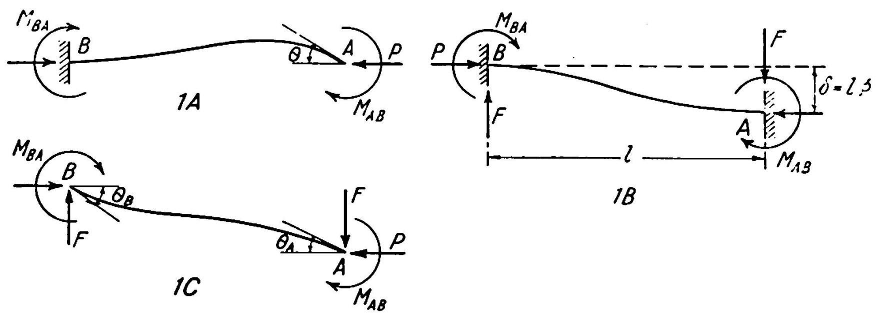

Fig. 1. Clockwise Moments and Rotations are $+VE$.

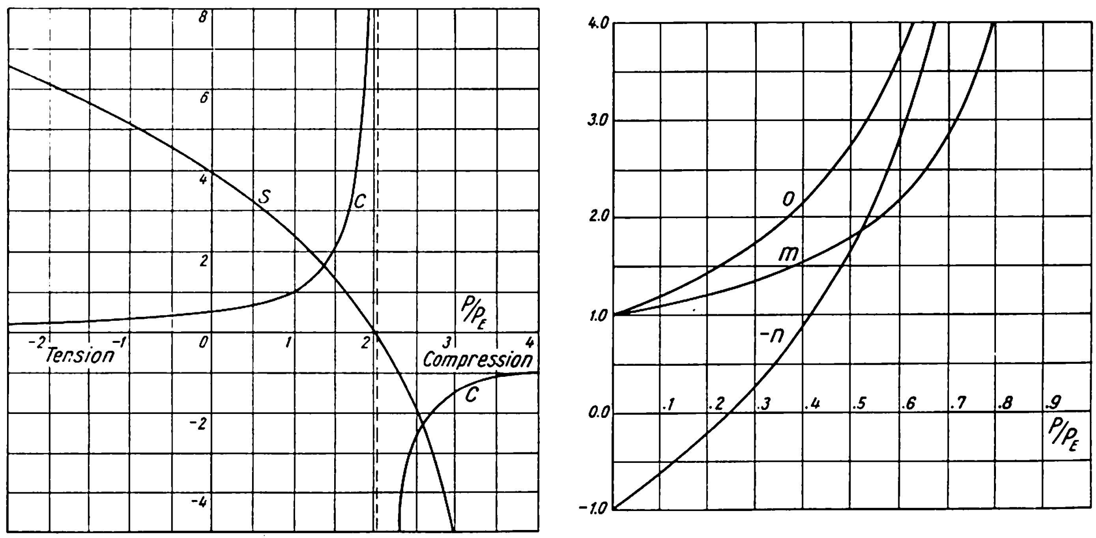

Fig. 2.

where $$m = \frac{2s(1+c)}{2s(1+c)-\pi^2\rho}.$$

For a combined displacement as in fig. 1c.

$$\begin{aligned} M_{AB} &= k[s\theta_A + cs\theta_B - s(1+c)\phi], \\ M_{BA} &= k[cs\theta_A + s\theta_B - s(1+c)\phi], \\ Fl &= s(1+c)k\left[\frac{2\phi}{m} - \theta_A - \theta_B\right], \end{aligned} \tag{I}$$

m is a magnification factor of considerable physical significance. For a simple sway ($\theta_A = \theta_B = 0$) $M_{AB} = M_{BA} = -\frac{mFl}{2}$ and thus m indicates directly how much greater the end moments due to a shear force are than when axial forces are neglected.

Of course m like s and c is a function of $\frac{P}{P_E}$. It is difficult in developing a new subject to decide how many functions to name and tabulate. Thus in trigonometry $\sin x$, $\cos x$, $\tan x$ are all treated as separate functions because of the simplicity this brings to manipulations rather than using, for example, the set $$\sin x, \quad \sqrt{1-\sin^2 x} \quad \text{and} \quad \frac{\sin x}{\sqrt{1-\sin^2 x}}.$$

Two further functions n and o are of considerable use and they can be obtained as follows.

In equations (I) put $$\phi' = \phi - \frac{m}{2}(\theta_A + \theta_B).$$

Then

$$\begin{aligned} M_{AB} &= k[n\theta_A - o\theta_B] - m\frac{Fl}{2}, \\ M_{BA} &= k[-o\theta_A + n\theta_B] - m\frac{Fl}{2}, \\ Fl &= \frac{2s(1+c)k\phi'}{m} = 2Ak\phi', \end{aligned} \tag{II}$$

where

$$\begin{aligned} \phi &= \phi' + \frac{m}{2}(\theta_A + \theta_B), \\ A &= \frac{s(1+c)}{m}, \\ n &= s\left[1 - \frac{m}{2}(1+c)\right], \\ 0 &= s\left[-c + \frac{m}{2}(1+c)\right]. \end{aligned}$$

The physical consequence of introducing ϕ' is apparent. It divides the sway angle ϕ into a part ϕ' directly dependent on the shear force and a part due to the end rotations of the member.

n is the appropriate stiffness factor to use when a member can sway freely ($\theta_B = F = 0$). It is arguable that greater simplicity and parallelism of results might have been obtained if another carry-over factor had been defined to give M_{BA} in terms of M_{AB} in this case instead of o which corresponds to $-sc$.

However n and o have the advantage of reducing to unity when stability effects are negligible. Either set of eqs. (I) or (II) may be used depending on the problem. The m, n, o functions were first used in reference [7] and first tabulated in reference [1]. They are also shown in fig. 2.

Use of Stability Functions

Consider the portal shown in fig. 3 where F is so small in comparison with P_1 and P_2 that stability effects in the beam can be neglected. Then

$$\begin{aligned} M_{BA} &= k_1[s_1\theta_B - s_1(1+c_1)\phi], \\ F_1 l &= 2A_1 k_1\phi - s_1(1+c_1)k_1\theta_B, \\ M_{BC} &= k_3[4\theta_B + 2\theta_C] \end{aligned}$$

and similarly for the other leg.

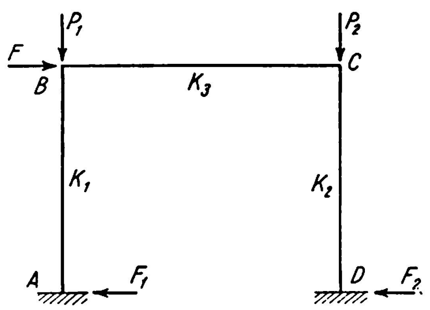

Fig. 3.

We have the two joint equations of equilibrium and the shear equation $F = F_1 + F_2$. Therefore

$$\begin{aligned} 0 &= (s_1 k_1 + 4k_3)\theta_B + 2k_3\theta_C - s_1(1+c_1)k_1\phi, \\ 0 &= 2k_3\theta_B + (s_2 k_2 + 4k_3)\theta_C - s_2(1+c_2)k_2\phi, \\ Fl &= -s_1(1+c_1)k_1\theta_B - s_2(1+c_2)k_2\theta_C + 2(A_1 k_1 + A_2 k_2)\phi. \end{aligned}$$

These equations enable us to solve directly for θ_B, θ_C and ϕ and thus the proplem is solved. In particular the critical load of the structure is given by the vanishing of the determinant of the equations. Numerous detailed calculations [8] have shown that the lateral deflections of rigid frames (also therefore sway critical loads) are not at all sensitive to the distribution of the loads on the columns provided that the same total load is taken. It is in fact sufficiently accurate to distribute the total load so that the columns have the same value of $\frac{P}{P_E}$ and thus the same value of their stability functions. The physical reason for this is that transferring load from say column A to column B increases the stiffness of A at the same time as it reduces the stiffness of B and to a first order causes no change in the overall lateral stiffness.

Calculations for symmetrical bents with equal column loads are more simply carried out by means of eqs. (II). Thus for the two storey portal shown in fig. 4 only the two joint equations of equilibrium are required

i.e.
$$0 = (n_2 k_2 + 6 k_4)\,\theta_C - o_2 k_2 \theta_B \qquad - m_2 \frac{F l}{4},$$
$$0 = -o_2 k_2 \theta_C \qquad + (n_1 k_1 + n_2 k_2 + 6 k_3)\,\theta_B - (m_1 + m_2)\frac{F l}{4}.$$

As before the sway critical load is obtained by putting $F = 0$ and thus the n, o functions are peculiarly suitable for critical load calculations.

References [7, 9, 10, 11] give examples of the use of m, n, o functions for the calculation of critical loads of tall building frames. They can be used for slope deflection or relaxation solutions. An example of the results is given in fig. 5 which is taken from reference [7]. It shows how the sway critical load

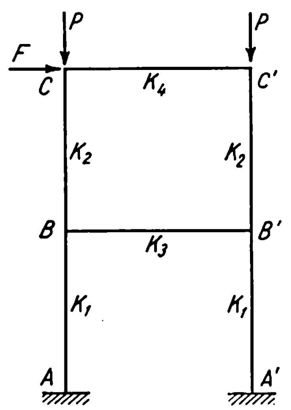

Fig. 4.

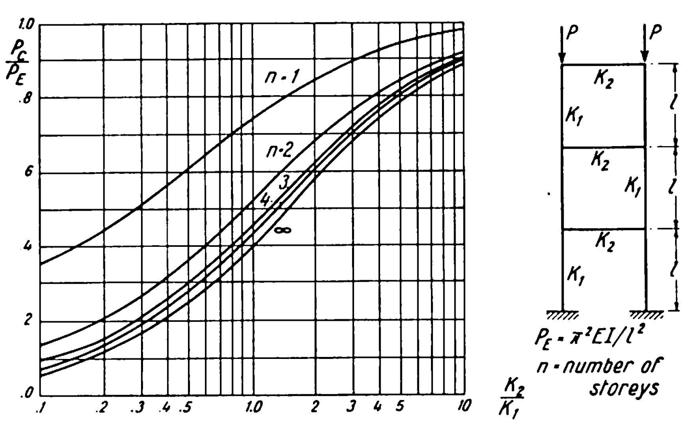

Fig. 5.

of a symmetrical single bay portal depends on the number of storeys. Some useful relations between s, c, m, n, o etc. are given in reference [8].

Members with Plastic Hinges

Suppose that there is a plastic hinge at B so that $|M_{BA}| = M_P$ and is specified. Eqs. (I) become

$$\begin{aligned} M_{AB} - c\,M_{BA} &= s\,(1-c^2)\,k\,(\theta_A - \phi), \\ M_{AB} + M_{BA} + F\,l &= -\pi^2 \rho\, k\, \phi. \end{aligned} \qquad \text{(I a)}$$

Eqs. (II) become

$$\begin{aligned} M_{AB} + \frac{o}{n} M_{BA} &= -\frac{\pi^2 \rho}{n} k\,\theta_A - \frac{F\,l}{n}, \\ M_{AB} + M_{BA} + F\,l &= -\pi^2 \rho\, k\, \phi. \end{aligned} \qquad \text{(IIa)}$$

As an example of the use of eqs. (II) and (II a) we will follow through the development of plastic hinges for the simple portal shown in fig. 6 where the

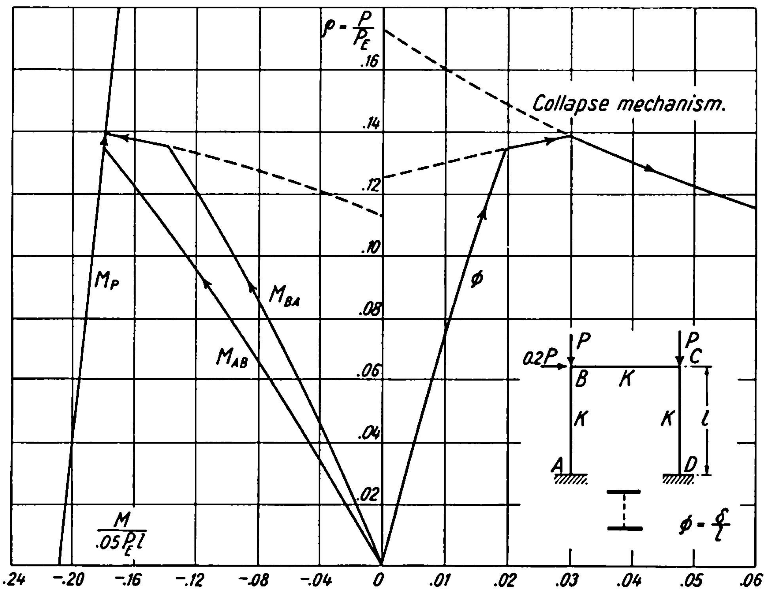

Fig. 6.

members are assumed to remain elastic until a plastic hinge forms. During the elastic range

$$\begin{aligned} M_{BA} &= k\,n\,\theta_B - 0.05\,m\,P\,l, \\ M_{BC} &= 6\,k\,\theta_B, \\ 0 = M_{BA} + M_{BC} &= k\,(n+6)\,\theta_B - 0.05\,m\,P\,l, \\ 0.1\,P\,l &= 2\,A\,k\,\phi'. \end{aligned}$$

Therefore
$$\frac{M_{BA}}{0.05 P_E l} = \frac{-6 m \rho}{n+6},$$

$$\frac{M_{AB}}{0.05 P_E l} = -m\rho\left[\frac{o}{n+6}+1\right],$$

$$\phi = 0.05\,\pi^2 \rho\left[\frac{1}{A}+\frac{m}{2(n+6)}\right].$$

These values are shown graphed in fig. 6. During this range the elastic critical load towards which the bending moments and deflections are increasing asymptotically is given by $n+6=0$ i. e. by $\rho=0.748$.

M_{AB} is greater than M_{BA} and therefore the first plastic hinges form at the bases of the columns.

After a hinge forms at A we have $M_{AB}= -M_P$ and so from eqs. (IIa)

$$M_{BA} = \frac{o\,M_P - \pi^2 \rho\, k\,\theta_B - 0.1\,Pl}{n},$$

$$0 = k(6n-\pi^2\rho)\,\theta_B + o\,M_P - 0.1\,Pl.$$

Therefore
$$\frac{M_{BA}}{0.05 P_E l} = \left[\frac{o\,M_P}{0.05 P_E l} - 2\rho\right]\frac{6}{6n-\pi^2\rho}.$$

Also
$$\pi^2 \rho\, k\,\phi = -M_{BA} + M_P - 0.1\,Pl.$$

Therefore
$$\phi = \frac{0.05}{\rho}\left[\frac{-M_{BA}+M_P}{0.05 P_E l}\right] - 0.100.$$

During this range the critical load towards which the bending moments and deflections are increasing asymptotically is given by $6n-\pi^2\rho=0$ i. e. by $\rho=0.185$.

To plot the moments and deflections during this second stage we require the relation between the plastic moment on the column and the axial load on it. For this example we will take $\frac{M_P}{0.05\,P_E\,l}=0{,}21\,(1-\rho)$.

The axial yield load of the column P_Y is given by $M_P=0$ i. e. by $\rho=\frac{P}{P_E}=1$ so that in fact we are now calculating the example for $P_Y=P_E$. For mild steel with a yield stress of 15.25 ton sq. in. and $E=13{,}400$ ton sq. in. this corresponds to a value of $\frac{l}{r}$ equal to 93. The numerical constant is that appropriate to an idealised parallel plate section. M_{BA} and ϕ for the second stage can now be calculated and are also shown in fig. 6. Valuable checks on the calculation are that the two graphs for M_{BA} and the two graphs for ϕ both intersect at the value of $\rho=0.135$ for which $M_{AB}=M_P$.

The value of ϕ for the collapse mechanism with hinges at the top and bottom of the columns is also shown. To determine the mechanism

$$0.1\,Pl + Pl\phi = -(M_{AB}+M_{BA}) = 2\,M_P,$$

$$\frac{M_P}{0.05\,P_E l} = 0.21\,(1-\rho).$$

Therefore
$$\phi = \frac{0.021}{\rho} - 0.121.$$

The final collapse load of the portal is given by the intersection of the second stage with the collapse mechanism i.e. by $\rho = 0.139$. If the second critical load had been lower than the load existing at the formation of the first pair of hinges then the subsequent deflection curve would have had a negative slope and would have represented an unstable condition. In this case the collapse load of the structure would occur at the formation of the first pair of hinges and not at the intersection of the second stage with the collapse mechanism.

The rigid plastic collapse load or limit load P_Y ignoring stability effects is obtained from the collapse mechanism by putting $\phi = 0$ i.e. by $\rho_Y = 0.173$. For this portal the interaction between stability and plasticity effects reduces the failure load to $\rho_F = 0.139$ i.e. to 80% of the rigid plastic collapse load and this for a ratio of limit load to original elastic critical load of

$$\frac{\rho_Y}{\rho_C} = \frac{0.173}{0.748} = 0.232.$$

Fig. 7 taken from ref. [8] shows a more complicated example. The deflections of a two storey frame are shown for three ratios of lateral load. Over three hundred cases of different types of rectangular frames have been solved in order that the general field of the interaction between the failure loads, the limit loads and the critical loads may be studied. The generalised results have also been compared with a series of tests on model frames.

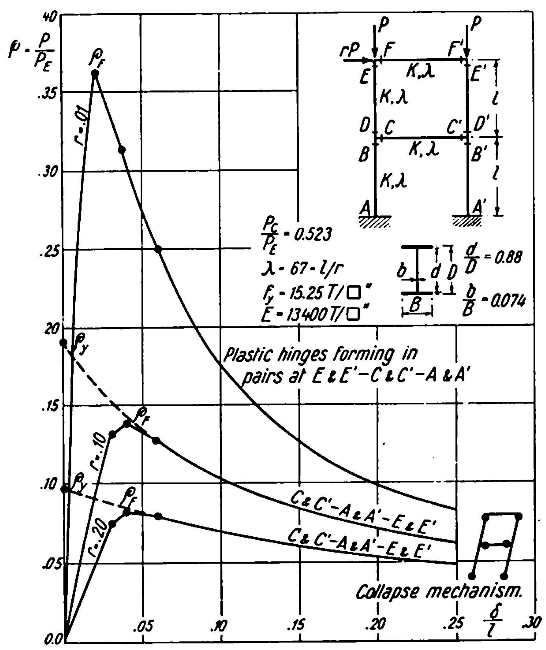

Fig. 7.

Acknowledgment

This paper summarises some of the analytical methods developed for a research project on the combined effects of plasticity and stability on the failure loads of structures, conducted by the Department of Structural Engineering, College of Science and Technology, Manchester, during the last nine years. The authors wish to thank their other colleagues and research students who have assisted in the general programme and thus made this paper possible.

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Summary

A new tabulation of stability functions for prismatic members was published in 1956[1]). The functions tabulated are

s the non-dimensional stiffness
c the carry over factor

and three new functions m, n, o which are convenient for dealing with cases of sway of members. The independent variable is taken as the ratio of the

[1]) Stability Functions for Structural Frameworks, R. K. LIVESLEY and D. B. CHANDLER, Manchester University Press, 1956.

axial load in a member to the Euler load of a member. It is not yet generally realised how convenient these functions are for the stability analysis of rigid jointed frames and the paper therefore gives examples of their use. In particular the evaluation of critical loads and the effect of plastic hinges in members are treated.

Résumé

Un nouveau tableau de fonctions de stabilité pour barres prismatiques a été publié en 1956. Dans ce tableau, figurent les fonctions suivantes:

Coefficient de rigidité s (sans dimension)
Facteur de transmission c

ainsi que trois nouvelles fonctions m, n et o, qui peuvent être employées dans les problèmes de déplacement latéral des barres. Le rapport entre l'effort axial de compression et la charge de flambage d'Euler de la barre a été adopté comme variable indépendante. Le caractère pratique de ces fonctions pour les études de stabilité des cadres rigides n'est encore que peu connu. L'auteur en donne donc quelques exemples d'application. Il traite en particulier de la détermination des charges critiques et de l'effet des articulations plastiques dans les barres.

Zusammenfassung

Im Jahre 1956 wurde eine neue Zusammenstellung von Stabilitätsfunktionen für prismatische Stäbe veröffentlicht. Folgende Funktionen sind tabelliert worden:

Die dimensionslose Steifigkeit s
der Übertragungsfaktor c

sowie drei neue Funktionen m, n und o, welche bei seitlichem Ausweichen in Stäben angewendet werden können. Als unabhängige Variable wird das Verhältnis der axialen Druckkraft zur Eulerschen Knicklast des Stabes angenommen. Es ist noch wenig bekannt, wie bequem diese Funktionen bei Stabilitätsuntersuchungen von steifen Rahmen sind. Die Arbeit gibt daher einige Beispiele für den Gebrauch dieser Funktionen. Besonders werden die Bestimmung der kritischen Belastungen und die Wirkung von plastischen Gelenken in Stäben behandelt.