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The Post-buckled Strength of Thin Walled Columns

La résistance de colonnes composées de plaques minces après flambement

Knicken dünnwandiger Stützen im überkritischen Bereich

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1. Introduction

In his paper on Thin Walled Structures, Professor Winter discusses the problem of the thin plate buckled in edge compression. He draws attention to the need of a theory that will enable a rigorous analysis of the advanced post-buckled state to be made, and hence an assessment of the ultimate load. It is the purpose of the present contribution to summarise just such a theory recently developed by the author[1] to determine the compressive strengths of thin walled rectangular columns, for which this single plate problem may be treated as a special case.

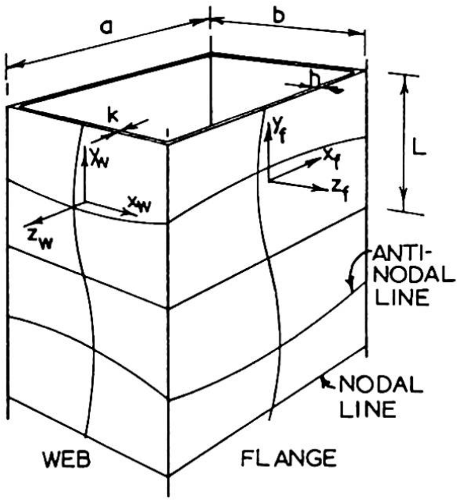

FIG I. LOCALLY BUCKLED COLUMN

The object of the analysis is to examine the response of the column shown in figure 1 to applied compressive strain. Local buckling is considered to occur entirely elastically, after which the strain is measured by the decreasing distance between the nodal and antinodal lines of the local buckles which remain straight horizontally. The post-buckled elastic response is obtained by a variational principle using precalculated deflected forms that enable an accurate solution to be quickly obtained. Plasticity is then introduced into the calculations when a plastic criterion is satisfied by the stress system at points in the column walls. The material is considered to be elastic-perfectly-plastic and governed by von-Mises' Criterion and the Prandtl-Reuss equations. The onset of plasticity causes an apparent loss of sectional stiffness in regions of the plate elements and as this effect spreads with the increasing deformations the total reactive force of the column is found to pass through a maximum and load shedding begins.

The complication of overall buckling interaction in the longer column is dealt with by finding the apparent bending stiffness of the section at stages in the post-buckled range, thus enabling the Euler buckling load to be computed. It is shown that overall failure of this type has no effect on the ultimate load of the square column provided the slenderness ratio is less than about half the 'critical slenderness ratio' (for which the Euler buckling stress is equal to the local buckling stress of the section).

2. Elastic Theory

The elastic post-buckled behaviour of the system of plates constituting the column is analysed by the Rayleigh-Ritz method[2]. Thus arbitrary expressions for the displacements u, v and ω in the x, y and z directions are chosen in terms of independent displacement parameters such that they satisfy all the kinematic boundary conditions. Values of the parameters corresponding to the approximate equilibrium solution are then obtained by minimising the expression for the elastic strain energy. In the present case, where the plate boundaries at the corners of the column suffer no appreciable direct stresses, the in-plane displacements u and v cannot be accurately represented by say simple truncated Fourier series, so that in order to limit the required number of parameters it becomes necessary to obtain more sophisticated expressions for u and v in advance. This is done in the following way. Von Kármán's large deflection equations[3]

$$\nabla^4\omega = \frac{t}{D}\left\{\frac{\partial^2 F}{\partial x^2}\cdot\frac{\partial^2\omega}{\partial y^2} + \frac{\partial^2 F}{\partial y^2}\cdot\frac{\partial^2\omega}{\partial x^2} - 2.\frac{\partial^2 F}{\partial x\partial y}\cdot\frac{\partial^2\omega}{\partial x\partial y}\right\} \qquad (1)$$

$$\nabla^4 F = E\left\{\left(\frac{\partial^2\omega}{\partial x\partial y}\right)^2 + \frac{\partial^2\omega}{\partial x^2}\cdot\frac{\partial^2\omega}{\partial y^2}\right\} \qquad (2)$$

are solved simultaneously for each element of the column section by a relaxation technique. The boundary conditions are,
1. along the corners of the column,

$$\omega_f = \omega_w = 0; \quad \frac{\partial\omega_f}{\partial x} = \frac{\partial\omega_w}{\partial x}; \quad h^3\left(\frac{\partial^2\omega_f}{\partial x^2}\right) = k^3\left(\frac{\partial^2\omega_w}{\partial x^2}\right) \qquad (3,4,5,6)$$

$$\frac{\partial^2 F_f}{\partial y^2} = \frac{\partial^2 F_w}{\partial y^2} = 0; \quad v_f = v_w\ ; \quad h\left(\frac{\partial^2 F_f}{\partial x\partial y}\right) = k\left(\frac{\partial^2 F_w}{\partial x\partial y}\right) \qquad (7,8,9,10)$$

representing zero out of plane displacements and direct stresses, continuity of slopes and vertical displacements, and equilibrium of bending moments and shear forces, the subscripts f and w referring to flange and web respectively,
2. along the nodal lines of local buckles,

$$v = -\frac{SL}{2}\ ; \quad \omega = \frac{\partial^2\omega}{\partial y^2} = \frac{\partial^2 F}{\partial x\partial y} = 0 \qquad (11,12,13,14)$$

where v in (11) is the vertical displacement relative to the antinodal lines corresponding to the applied strain S and equations (12,13,14) represent the symmetry requirements of zero out of plane displacements, bending moments and shear forces.

An assumed solution for (2) corresponding to a state of uniform compression

$$F = -\frac{\sigma_{cr} \cdot x^2}{2} \qquad (15)$$

is substituted into (1) and the solution ω of this equation is substituted back into (2) enabling an improved value for the stress function F to be obtained. The in-plane displacements u and v are related to ω and F by simple differential equations[4] and these are then integrated to give the following expressions[1]:

$$\begin{aligned} u = &-\frac{2\gamma(1+\nu)}{E}\left\{H_1\sinh(2\gamma x)+H_2\left[x.\cosh(2\gamma x)-\frac{(1-\nu)\sinh(2\gamma x)}{2\gamma(1+\nu)}\right]\right\}\cos(2\gamma y) \\ &-\frac{1}{E}\left\{4\gamma^2\int\psi(x)dx + \nu\frac{d(\psi(x))}{dx} + \int\frac{E}{4}\left(\frac{dw}{dx}\right)^2 dx\right\}\cos(2\gamma y) \\ &-\frac{\nu}{E}\left\{2C_1 x + \frac{d(\chi(x))}{dx} + \frac{1}{\nu}\int\frac{E}{4}\left(\frac{dw}{dx}\right)^2 dx\right\} + f(y) \end{aligned} \qquad (16)$$

$$\begin{aligned} v = &\frac{2\gamma(1+\nu)}{E}\left\{H_1\cosh(2\gamma x)+H_2\left[x.\sinh(2\gamma x)+\frac{\cosh(2\gamma x)}{\gamma(1+\nu)}\right]\right\}\sin(2\gamma y) \\ &+\frac{1}{2E\gamma}\left\{\nu 4\gamma^2\psi(x) + \frac{d^2(\psi(x))}{dx^2} + \frac{E}{4}\gamma^2 w^2\right\}\sin(2\gamma y) \\ &+\frac{1}{E}\left\{2C_1 + \frac{d^2(\chi(x))}{dx^2} + \frac{E}{4}\gamma^2 w^2\right\} y + g(x) \end{aligned} \qquad (17)$$

$$\omega = w(x)\cos(\gamma y) \qquad (18)$$

where

$$\psi(x) = \frac{\delta^2 E\gamma^2}{2}\left\{\frac{A^2\alpha^2}{16\gamma^4} - \frac{B^2\beta^2}{16\gamma^4} + R_0\frac{AB\cosh((\alpha+j\beta)x)}{(\alpha-j\beta)^2}\right\} \qquad (19)$$

$$w(x) = \delta(A\cosh(\alpha x) + B\cos(\beta x)) \qquad (20)$$

$$\begin{matrix}\alpha^2 \\ \beta^2\end{matrix} = \left[\pm\gamma^2 + \left(\frac{\sigma_{cr} t}{D}\gamma^2\right)^{1/2}\right] \qquad (21,22)$$

The constants and functions of integration H_1, H_2, C_1, f(y) and g(x) together with the function $\chi(x)$ are eliminated from the equations by satisfying the boundary conditions[1].

The post-buckled problem is solved using combinations of terms in the equations (16, 17, 18) in association with the displacement parameters in such a way as to retain the necessary kinematic boundary conditions (3, 4, 5, 9). Applied in this manner the Rayleigh-Ritz technique yields curves of average axial stress versus strain composed of two straight lines intersecting at the critical local buckling stress of the column in question. The divergence from the known exact solution for the square column[5], where the elastic post-buckled stiffness gradually decreases with strain, is of the order of two percent in the range of stresses considered here. For rectangular columns the ratio of post to prebuckled compressive stiffnesses is found to vary with the ratio b/a as shown in figure 2, passing through a maximum at about b/a = .6. In the following discussion of the plastic theory it will appear that this variation in the elastic post buckled stiffnesses is reflected in a corresponding variation in the ultimate strengths.

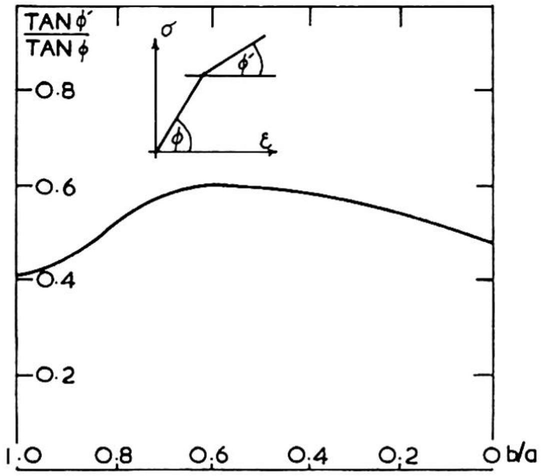

FIG 2 STIFFNESS RATIOS.

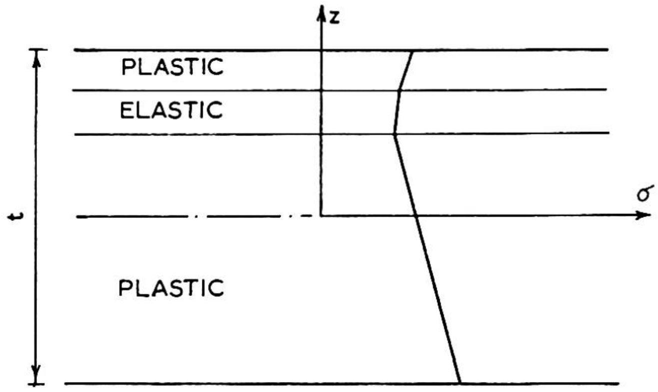

FIG 3 ASSUMED STRESS DISTRIBUTION IN THE ELASTO-PLASTIC SECTION.

3. Plastic Theory

The buckled plate element constitutes a problem in plane stress for which von Mises' Criterion takes the form

$$(\sigma_x)^2 + (\sigma_y)^2 - \sigma_x.\sigma_y + 3(\tau_{xy})^2 = (\sigma_Y)^2. \qquad (23)$$

Assuming that plane sections remain plane this can be expressed in terms of the elastic strains as follows[6]:

$$a_1\left(\frac{z}{t}\right)^2 + a_2\left(\frac{z}{t}\right) + a_3 = (\sigma_Y)^2\frac{(1-\nu^2)^2}{E^2} \qquad (24)$$

where $a_1(\varepsilon_b^2)$, $a_2(\varepsilon_b, \varepsilon_m)$ and $a_3(\varepsilon_m^2)$ are second order functions of membrane strains (ε_m) and bending strains (ε_b) at a point. Thus the values of (z/t) bounding the elastic and plastic regions in the cross-section are the solutions of the quadratic (24) and it can be demonstrated that for the general elasto-plastic section shown in figure 3 a region of elasticity is always sandwiched between two plastic regions[6]. In a plastic material it is possible to relate only <u>increments</u> of stress to <u>increments</u> of strain in terms of the current stress state represented by the stress deviators 's'. Prager and Hodge[7] have derived the incremental stress strain relations for the general 3-dimensional case using the Prandtl-Reuss equations and the requirement that the stress changes must satisfy the differential form of the yield criterion. In 2-dimensions their expressions reduce to

$$\Delta\sigma_x = (E/(1-\nu^2))(\Delta\varepsilon_x + \nu\Delta\varepsilon_y - \Phi(s_x + \nu s_y)) \qquad (25)$$

$$\Delta\sigma_y = (E/(1-\nu^2))(\Delta\varepsilon_y + \nu\Delta\varepsilon_x - \Phi(s_y + \nu s_x)) \qquad (26)$$

$$\Delta\tau_{xy} = G\ (\Delta\gamma_{xy} - 2\Phi\tau_{xy}) \qquad (27)$$

where

$$\Phi = \frac{s_x(\Delta\varepsilon_x + \nu\Delta\varepsilon_y) + s_y(\Delta\varepsilon_y + \nu\Delta\varepsilon_x) + (1-\nu)\Delta\gamma_{xy}\tau_{xy}}{s_x(s_x + \nu s_y) + s_y(s_y + \nu s_x) + 2(1-\nu)\tau^2_{xy}} \qquad (28)$$

Making the assumption that the plastic stress increments vary linearly with z between calculated surface stresses and the known stresses at the elasto-plastic boundary as in figure 3, it is possible to integrate the response to strain increments over the cross section including both the elastic and plastic regions to obtain an apparent value for the sectional stiffness at a particular point. An appropriate value for the total strain energy of the system under an imposed set of displacements may then be obtained by numerically integrating the corresponding discrete expressions for strain energy per unit area over a representative area of the column.

The variational method for satisfying the equilibrium will be applicable provided no strain reversals occur[8] and in the present case of the column subject to monotonically increasing strain this requirement is satisfied[1]. Since the plastic behaviour is non linear and depends on the current stress state the calculation progresses into the plastic regime via a step by step Runge-Kutta type procedure. The post-buckled stiffness begins to fall off rapidly until the ultimate load is attained.

Results are given in figures 4 and 5, figure 4 showing the behaviour of the square column with various values of the ratio σ_Y/σ_{cr} of the order to be expected in civil structures, and figure 5 the behaviour of rectangular columns with constant σ_Y/σ_{cr} and differing b/a ratios.

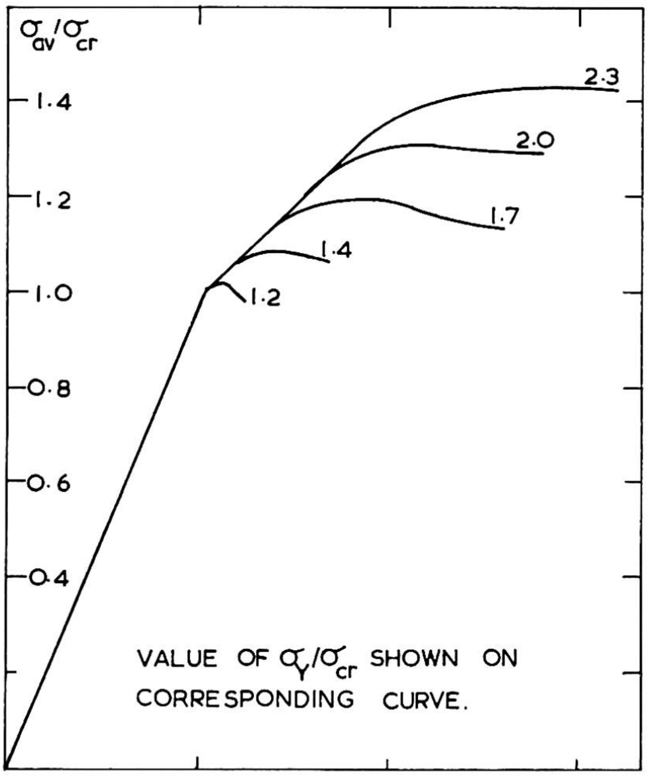

FIG 4 AVERAGE STRESS VERSUS STRAIN - SQUARE SECTIONS

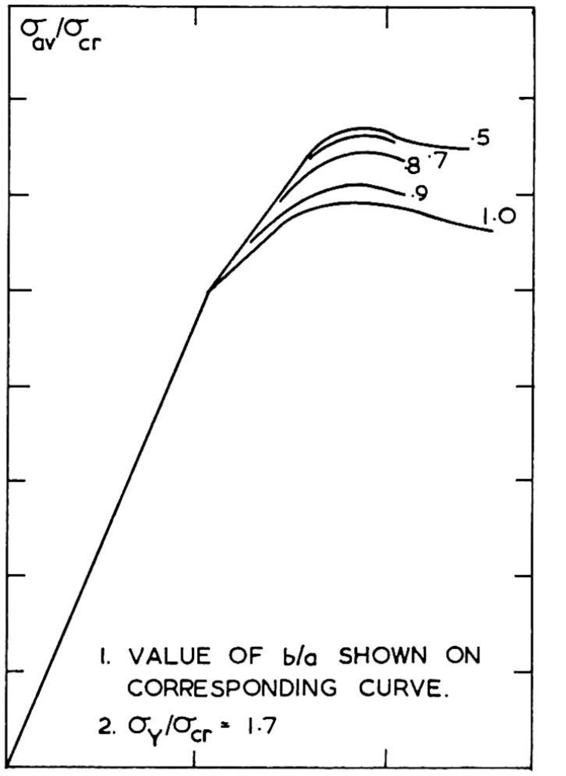

FIG 5 RECTANGULAR SECTIONS

An interesting feature of the results is that the corners of the columns go completely plastic in a predominantly compressive mode at about the same time as the ultimate load is attained, which lends support to the 'equivalent width' design approximation discussed by Professor Winter. The theoretical results for the square column agree well with the results of some simply supported plate tests recently reported by Dwight and Ractliffe[9].

4. Effect on Ultimate Strengths of Overall Buckling

Professor Winter further mentions the interaction between ultimate strengths and overall buckling in longer columns. This effect, which has hitherto been disregarded in this paper, may be dealt with theoretically by considering the apparent internal bending stiffness 'K' of the section at stages in its post-buckled range. K is proportional to the reactive moment produced by the application of an infinitesimal bending strain S' to the column, and since the nodal and antinodal lines of the existing local buckles must remain straight by symmetry, such a bending strain may be measured by a change in their inclination as shown in figure 6. Given K, the Euler buckling stress of a column of slenderness ratio (ℓ/r) follows from the formula

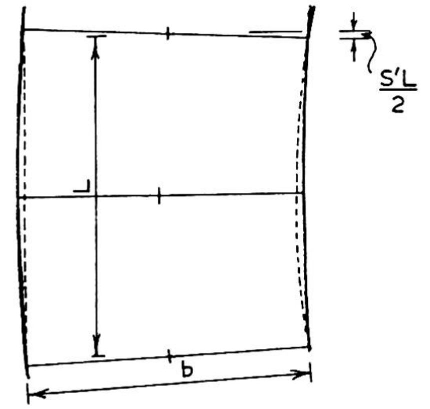

FIG 6. LOCALLY BUCKLED COLUMN UNDER BENDING STRAIN S'

$$\sigma_E = \frac{\pi^2 K}{(\ell/r)^2 I} \tag{29}$$

and can therefore be found by a graphical construction on the curve of K versus σ_{av} for the section in question[6]. σ_E may then be postulated as the failure stress of the column provided it is reached before the local buckling ultimate stress.

A variational principle is again used to obtain the infinitesimal equilibrium state corresponding to the applied bending strain. Approximate expressions for u', v' and ω' are obtained by solving the differential form of von Kármán's equations (the superscript denoting quantities of infinitesimal magnitude):

$$\nabla^4\omega' = \frac{t}{D}\left\{\frac{\partial^2 F}{\partial y^2}\cdot\frac{\partial^2\omega'}{\partial x^2} + \frac{\partial^2 F'}{\partial y^2}\cdot\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\cdot\frac{\partial^2\omega'}{\partial y^2} + \frac{\partial^2 F}{\partial x^2}\cdot\frac{\partial^2\omega}{\partial y^2} - 2.\frac{\partial^2 F}{\partial x\partial y}\cdot\frac{\partial^2\omega'}{\partial x\partial y} - 2.\frac{\partial^2 F'}{\partial x\partial y}\cdot\frac{\partial^2\omega}{\partial x\partial y}\right\} \tag{30}$$

$$\nabla^4 F' = E\left\{2.\frac{\partial^2\omega'}{\partial x\partial y}\cdot\frac{\partial^2\omega}{\partial x\partial y} - \frac{\partial^2\omega'}{\partial y^2}\cdot\frac{\partial^2\omega}{\partial x^2} - \frac{\partial^2\omega'}{\partial x^2}\cdot\frac{\partial^2\omega}{\partial y^2}\right\} \tag{31}$$

in which ω and F are known from the previous purely compressive calculation. The increment of strain energy U' is then minimised with respect to the chosen displacement parameters. The required value of K is directly related to U' by the formula

$$K = \frac{U'}{L}\left(\frac{b}{S'}\right)^2 \qquad (32)$$

The results of the interaction calculation for square columns are shown in figure 7 where it will be seen that overall buckling has no effect on ultimate strengths provided slenderness ratios are less than about $.5(\ell/r)_{cr}$.

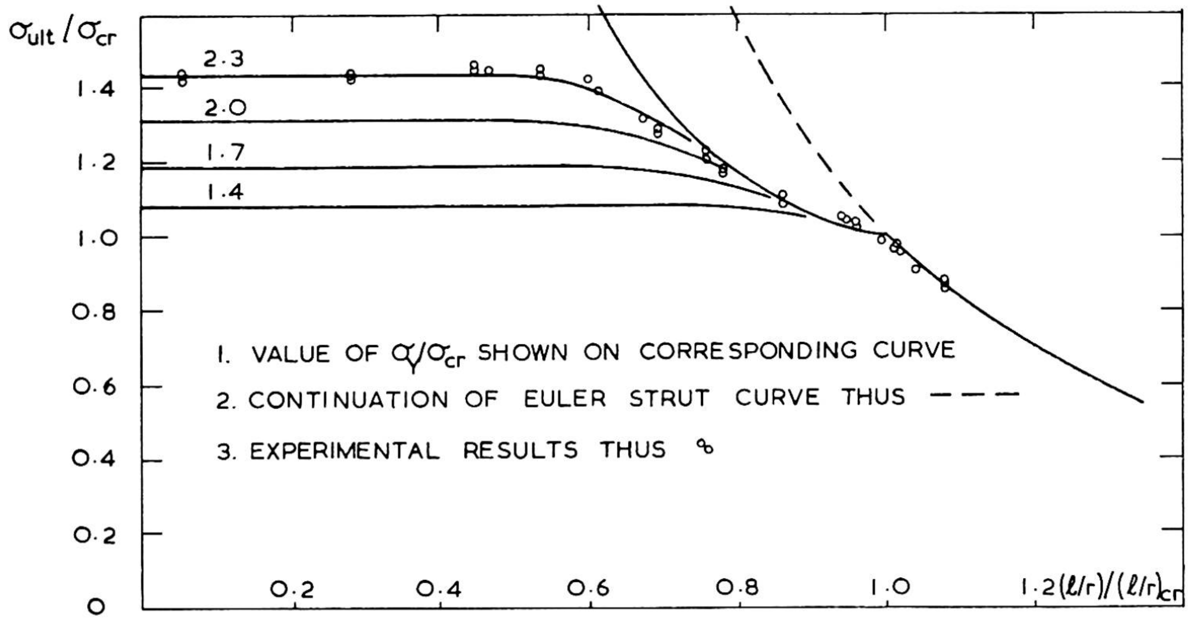

FIG. 7 ULTIMATE STRESSES OF SQUARE THIN WALLED COLUMNS

5. Experiments

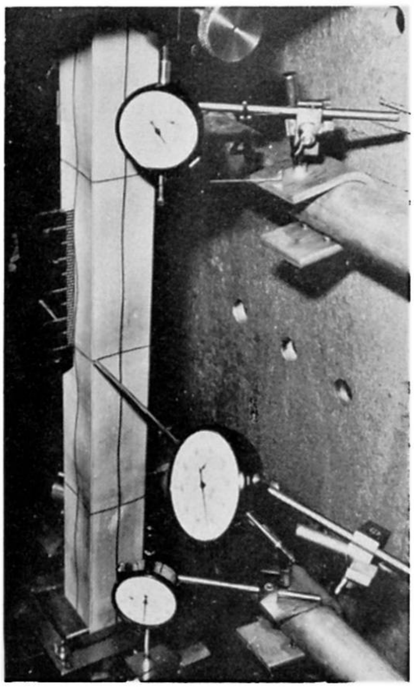

FIG 8 COLUMN TEST

In order to check the interaction theory, a series of experiments were performed on model columns cut from thin walled aluminium tubing[10]. The square section of nominal dimensions a(=b) = 2.00in, t = .040in, was drawn from the structural aluminium HT30/WP. Separate tests on the compressive characteristics of the material showed it to have a fairly marked yield as required by the theory.

Only initially straight columns have so far been examined analytically and in the tests reported here, the effect of initial overall curvature was specifically excluded by using adjustable end fittings that enabled the eccentricity of the applied load to be continuously varied. The 'correct' load position was found at the early stages of the test by observing the rate of increase of curvature with load and adjusting the fittings till this approached zero.

The compression machine was made as stiff as possible to enable the load shedding regions to be followed. Columns were tested in batches of five with slenderness ratios varying between 5 and 100 - covering the range of interest. The results are included in figure 7 and a typical specimen at its ultimate load is shown in figure 8.

6. Conclusions

The post-buckled behaviour of thin walled rectangular columns has been analysed, and by including the effect of plasticity ultimate loads have been predicted.

It has been shown that for columns of square section, overall buckling does not affect these strengths provided slenderness ratios are less than $.5(\ell/r)_{cr}$, a result which may be expressed in the form of the design criterion:

$$\frac{\ell}{r} < \sqrt{\frac{3(1-\nu^2)}{4}}\left(\frac{b}{t}\right) \qquad (33)$$

for overall stability.

7. References

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Symbols

The following symbols are used generally throughout the paper. Other symbols are defined when they occur.

D	-	flexural rigidity of an elastic plate, $Et^3/12(1-\nu^2)$
E	-	Young's modulus of elasticity
F	-	Airy's stress function
G	-	shear modulus of elasticity
I	-	second moment of area of section
L	-	half wavelength of local buckles
$\mathcal{R}$	-	real part of a complex expression
U	-	strain energy
j	-	$\sqrt{-1}$
ℓ	-	effective length of column
r	-	radius of gyration of column section
t	-	general thickness of plate
γ	-	π/L
γ_{xy}	-	shear strain
δ	-	arbitrary amplitude of local buckles
ν	-	Poisson's ratio
σ_{av}	-	average stress on column (load/sectional area)
σ_{cr}	-	critical local buckling stress
σ_{ult}	-	ultimate average stress on column
σ_Y	-	material yield stress

$$\nabla^4(\ ) = \frac{\partial^4(\ )}{\partial x^4} + 2.\frac{\partial^4(\ )}{\partial x^2 \partial y^2} + \frac{\partial^4(\ )}{\partial y^4}$$

SUMMARY

A method of analysing the post-buckled behaviour of a rectangular column made up of thin plate elements is presented. Plastic yielding of the material as the deformations become large is taken into account and this is shown to result in a progressive decrease in the post-buckled stiffness and eventual load shedding.

The effect of overall buckling on the column strength is also assessed and conclusions are drawn as to the allowable values of slenderness ratio for which this effect may be ignored.

RÉSUMÉ

On présente dans cette communication une méthode d'analyser comment se comporte après son flambage local une colonne rectangulaire composée de plaques minces. Il est tenu compte du comportement plastique de la matière quand la déformation s'agrandit, et il en ressort que la rigidité après flambage décroit progressivement et on aboutit à la perte du chargement.

On évalue aussi l'effet du flambage total sur la résistance de la colonne, et on tire des conclusions concernant les souplesses pour lesquelles on peut ignorer cet effet.

ZUSAMMENFASSUNG

Eine Methode wird vorgelegt, das Verhalten nach Knickung einer aus dünnen Platten bestehenden rechteckigen Säule zu ermitteln. Plastisches Fliessen des Materials bei grosswerdenden Formänderungen ergibt eine zunehmende Minderung der Steifheit nach Knickung und eventuell einen Abfall der Belastung.

Die Wirkung Eulerscher Knickung auf die Säulenstärke wird auch berechnet; daraus werden die Schlankheitswerte erschlossen, bei denen man diese Wirkung nicht zu beachten braucht.

Reply to Dr. P.S. Bulson's questions

Dr. Bulson raised a very interesting point regarding the interaction between column and local buckling. His object was to derive a simple rule to enable designers to obtain an approximate interaction curve. I agree that having obtained his reduced modulus E*, a good approximation to this curve can be produced by the construction he proposes. Unfortunately this value for E* cannot be found in simple closed form, although the amount of calculation necessary to obtain it is an order of magnitude less than that necessary for the full interaction curve. For square columns E* comes out to be 0.62E where E is the Young's modulus of the material in question. From an observation of the results, the plasticity transition appears to be sufficiently sudden for the rounding off of Dr. Bulson's construction to be unnecessary.