The significance of shake down loading

Autor(en): **Heyman, Jacques**

Objekttyp: **Article**

Zeitschrift: **IABSE congress report = Rapport du congrès AIPC = IVBH Kongressbericht**

Band (Jahr): **9 (1972)**

PDF erstellt am: **21.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-9537

The Significance of Shake Down Loading

La signification du "Shake Down" des charges

Die Bedeutung des "Shake Down" der Belastungen

JACQUES HEYMAN
Professor of Engineering
University of Cambridge
England, GB

The incremental collapse of frames, calculated according to accepted ideas of simple plastic theory, has been illuminated recently by a new formulation [1, 2]. Suppose that a unit load acting at section j of a frame produces an elastic bending moment μ_{ij} at section i of the frame; μ_{ij} is computed in the usual way on the assumption that the frame is initially stress free. Then the actual load W_j acting at j will give rise to an elastic bending moment of value

$$\mathcal{M}_i = \mu_{ij} W_j \quad . \tag{1}$$

If the value of the load W_j is not fixed, but can take on any value within the range

$$W_j^{min} \leqslant W_j \leqslant W_j^{max} \quad , \tag{2}$$

then the value of the elastic moment $\mathcal{M}_i$ will also fluctuate. To determine its largest value, $\mathcal{M}_i^{max}$ say, the value of W_j will be taken to be as large or as small as possible according as the unit moment μ_{ij} is positive or negative, denoted by $+\overset{+}{\mu}_{ij}$ and $-\overset{-}{\mu}_{ij}$, where $\overset{+}{\mu}_{ij}$ and $\overset{-}{\mu}_{ij}$ are themselves positive numbers. Thus

$$\mathcal{M}_i^{max} = \sum_j (\overset{+}{\mu}_{ij} W_j^{max} - \overset{-}{\mu}_{ij} W_j^{min}) \quad , \tag{3}$$

and, similarly,

$$\mathcal{M}_i^{min} = \sum_j (-\overset{-}{\mu}_{ij} W_j^{max} + \overset{+}{\mu}_{ij} W_j^{min}) \quad . \tag{4}$$

Now the basic equation for determining the full plastic moments $(M_p^s)_i$ of a frame so that is just on the point of incremental collapse by the formation of a mechanism with hinge rotations ϕ_i is

$$\sum (M_p^s)_i |\phi_i| = \sum (\mathcal{M}_i^{max} \overset{+}{\phi}_i - \mathcal{M}_i^{min} \overset{-}{\phi}_i) \quad . \tag{5}$$

The numbers ϕ_i^+ and ϕ_i^- are themselves positive, and the same sign convention for the hinge rotations $+\phi_i^+$ and $-\phi_i^-$ has been used as that for the unit moments μ_{ij}. Each term in the sum on the left-hand side of equation (5) is essentially positive, since it represents work dissipated at a rotating plastic hinge. Thus, introducing equations (3) and (4) into (5),

$$\sum_i (M_p^s)_i |\phi_i| = \sum_i \{\sum_j (\mu_{ij}^+ W_j^{max} - \mu_{ij}^- W_j^{min})\}\phi_i^+ - \sum_i \{\sum_j (-\mu_{ij}^- W_j^{max} + \mu_{ij}^+ W_j^{min})\}\phi_i^- \quad . \quad (6)$$

It is convenient to introduce a corresponding static design $(M_p^o)_i$, calculated for the same mechanism ϕ_i, but with the loads W_j all having their fixed maximum values W_j^{max}. This static design is thus given by

$$\sum_i (M_p^o)_i |\phi_i| = \sum_i \{\sum_j (\mu_{ij}^+ W_j^{max} - \mu_{ij}^- W_j^{max})\}\phi_i^+ - \sum_i \{\sum_j (-\mu_{ij}^- W_j^{max} + \mu_{ij}^+ W_j^{max})\}\phi_i^- \quad , \quad (7)$$

and it will be seen that this is almost identical with equation (6); the only difference is that in (7) all bending moments are due to W_j^{max}. Subtracting the two equations,

$$\sum_i \{(M_p^s)_i - (M_p^o)_i\}|\phi_i| = \sum_i \left[\phi_i^+\{\sum_j \mu_{ij}^-(W_j^{max} - W_j^{min})\} + \phi_i^-\{\sum_j \mu_{ij}^+(W_j^{max} - W_j^{min})\}\right] . (8)$$

Three important conclusions may be drawn immediately from a study of equation (8).

First, the whole of the right-hand side of equation (8) is positive or zero. The range of loading $(W_j^{max} - W_j^{min})$ is essentially positive or zero, while the products $\phi^+\mu^-$ and $\phi^-\mu^+$ are positive by definition. Thus the equation indicates that the values of full plastic moment M_p^s required to prevent incremental collapse in a given mechanism ϕ will exceed (or at best equal) the corresponding values M_p^o for static collapse. In other words, a frame subjected to fluctuating loads will always require more material than a frame subjected to steady peak values of those loads.

Secondly, only the range of loading $(W_j^{max} - W_j^{min})$ occurs in equation (8), and not the absolute values of the loads. Now the equation is a measure of the difference between the incremental collapse design M_p^s and the static collapse design M_p^o; thus this difference in design cannot be affected by any dead load (or any other load of fixed magnitude). That is, the dead loads will affect the actual value of M_p^o, but are not concerned in any increase to M_p^s to guard against incremental collapse.

Thirdly, it is only products $\phi^+\mu^-$ and $\phi^-\mu^+$ which appear in equation (8). The difference between M_p^s and M_p^o arises only from loads which produce unit negative elastic moments at sections where there are positive hinge rotations, or which produce unit positive elastic moments at sections where there are negative hinge rotations.

As a numerical example, the uniform fixed-ended beam of fig.1 carries loads W_1 and W_2, where the values of the loads can vary randomly and independently within the ranges

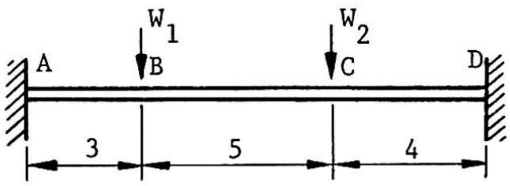
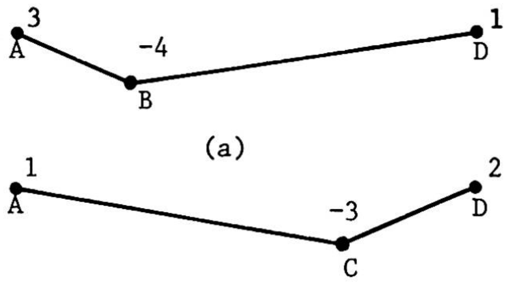


Fig.1



(b) Fig.2

$$\left.\begin{aligned} 0 \leqslant W_1 \leqslant 352 \quad , \\ 0 \leqslant W_2 \leqslant 270 \quad . \end{aligned}\right\} \tag{9}$$

If the loads W_1 and W_2 have their fixed maximum values, then the conventional methods of plastic design lead to the static value of full plastic moment:

$$M_p^o = 536 \quad . \tag{10}$$

Static collapse occurs by mechanism (b) of fig.2.

If, on the other hand, the loads are allowed to vary between the limits (9), then a shakedown analysis must be made. Conventional elastic theory leads to the following table of bending moments:

Table 1

Section	Moment due to $W_1 = 352$	$W_2 = 270$	$\mathcal{M}$	$\mathcal{M}^{max}$	$\mathcal{M}^{min}$
A	594	240	834	834	0
B	-297	30	-267	30	-297
C	-22	-320	-342	0	-342
D	198	480	678	678	0

The column labelled $\mathcal{M}$ in table 1 represents the static elastic solution when both loads act together with their full values, and it may be noted that the design given by equation (10) can be recovered from the fundamental equation

$$\sum (M_p^o)_i |\phi_i| = \Sigma \mathcal{M}_i \phi_i \quad , \tag{11}$$

(where, indeed, the elastic moments $\mathcal{M}_i$ can be replaced by any distribution of moments in equilibrium with the given external loading). Using the values $\mathcal{M}_i$ from table 1, equation (11) applied to mechanism (a) of fig.2 gives

$$8M_p^o = (834)(3) + (-267)(-4) + (678)(1) \quad ,$$

$$\text{or} \qquad M_p^o = 531 \quad ; \tag{12}$$

a similar calculation for mechanism (b) gives the more critical $M_p^o = 536$ of equation (10).

The last two columns of table 1 give values of $\mathcal{M}^{max}$ and $\mathcal{M}^{min}$ as the loads vary between their limits (9). Using equation (5) with mechanism (a),

$$8M_p^s = (834)(3) + (-297)(-4) + (678)(1) \quad ,$$

$$\text{or} \qquad M_p^s = 546 \quad ; \tag{13}$$

similarly, for mechanism (b),

$$6M_p^s = (834)(1) + (-342)(-3) + (678)(2)$$

$$\text{or} \qquad M_p^s = 536 \quad . \tag{14}$$

Clearly equation (13), mechanism (a), is more critical; the design full plastic moment must be increased from the value 536 of equation (10) for the static case to the value 546 of equation (13) in order to prevent incremental collapse.

Now the formulation of equation (8) shows how this increase arises. Since only products $\phi^+\mu^-$ or $\phi^+\mu^-$ can enter into the calculations, the first step is to examine the signs of the elastic bending moments and of the corresponding hinge rotations. The two mechanisms of fig.2 have positive (hogging) hinge rotations at the ends A and D of the beam, and negative (sagging) rotations at the internal points B and C. From table 1 it is seen that the signs of the elastic bending moments due to the load W_1 are precisely the same as those of the hinge rotations at the four critical sections; the conclusion is that the load W_1 cannot contribute at all to any increase in the value of M_p, (from M_p^o to M_p^s).

Similarly, the signs of the bending moments due to W_2 are the same as the signs of the hinge rotations for mechanism (b); thus mechanism (b) must give the same design value for M_p whether the loads are static or fluctuating, and this is confirmed by the identity of equations (10) and (14).

The only opposition in sign of bending moment and of hinge rotation occurs for section B with mechanism (a) and the load W_2; in this simple example, it is this single contribution which increases the value of M_p from the 531 of equation (12) to the 546 of equation (13).

This discussion indicates that equation (8) can be simplified for the purpose of calculation of shakedown limits. The right-hand side may be written

$$\sum_i\{\phi_i^+(\sum_j \mu_{ij}^- \bar{W}_j) + \phi_i^-(\sum_j \mu_{ij}^+ \bar{W}_j)\} \equiv \sum_i(\phi_i^+\bar{\mathcal{M}}_i^- + \phi_i^-\bar{\mathcal{M}}_i^+) \equiv \sum_i|\phi_i\mathcal{M}_i^*| \quad , \qquad (15)$$

where $\bar{W}_j$ represents the range of loading $(W_j^{max} - W_j^{min})$, leading to a change of elastic bending moment $\bar{\mathcal{M}}_i$, denoted plus or minus according as the change is an increase or a decrease from the datum. As before, only the products of a <u>positive</u> change of moment $\bar{\mathcal{M}}_i^+$ with a <u>negative</u> hinge rotation ϕ_i^-, and vice versa, are taken, and this is indicated by the final short notation $|\phi_i\mathcal{M}_i^*|$ of (15). Thus, finally, if the static collapse equation is written

$$\text{Static:} \quad \sum(M_p^o)_i|\phi_i| = \sum(M_F)_i\phi_i \quad , \qquad (16)$$

where $(M_F)_i$ represents any convenient set of bending moments in equilibrium with the <u>maximum</u> values of the loads, then the incremental collapse equation for the same mechanism but with fluctuating values of the loads may be written

$$\text{Incremental:} \quad \sum(M_p^s)_i|\phi_i| = \sum(M_F)_i\phi_i + \sum|\mathcal{M}_i^*\phi_i| \quad . \qquad (17)$$

The numerical example of fig.1 may be reworked by means of a rearrangement of table 1:

<u>Table 2</u>

Section	Maximum positive and negative change in bending moment		(a) 		(b) 	
	$\bar{\mathcal{M}}^+$	$\bar{\mathcal{M}}^-$	ϕ	$\mathcal{M}_\phi^*$	ϕ	$\mathcal{M}_\phi^*$
A	834	0	3	0	1	0
B	30	-297	-4	120		
C	0	-342			-3	0
D	678	0	1	0	2	0
	Static collapse:		$8M_p^o = 4248$		$6M_p^o = 3216$	
	Incremental collapse:		$8M_p^s = 4368$		$6M_p^s = 3216$	

As a second example, the collapse of the fixed-base portal frame will be investigated, both under static and under fluctuating loads. Figure 3 shows the frame, of uniform section, acted upon by loads V and H; the values of V and H are supposed to vary randomly and independently within the ranges

$$\left.\begin{array}{l} 0 \leqslant V \leqslant V_o \;, \\ 0 \leqslant H \leqslant H_o \;. \end{array}\right\} \qquad (18)$$

Also shown in fig.3 are sketch elastic solutions for H = 1 and V = 1, together with the three possible modes of collapse.

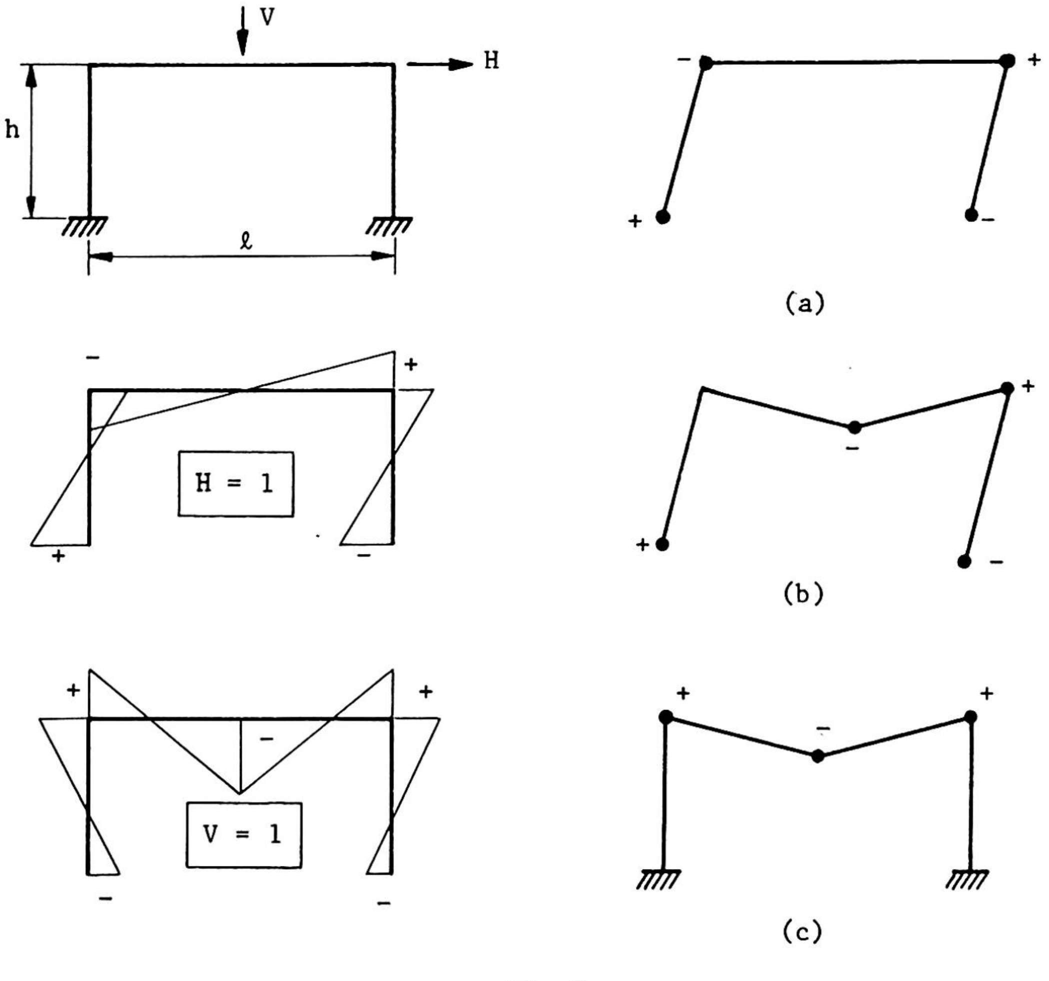


Fig.3

Comparing the elastic solution for H = 1 with mode (a), it will be seen that the signs of the bending moments at the hinge positions are the same in all cases as the signs of the hinge rotations. The conclusion is that there will be no "$\mathcal{M}^*\phi$" terms arising from the side load H for mode (a), and that therefore the terms in H will be identical for the static collapse and the incremental collapse equation. On the other hand, the elastic solution for

$V = 1$ indicates an opposition in sign for the hinges in the left-hand column for mode (a), so that there will be an "$\mathcal{M}^*\phi$" contribution from the load V.

The final solution requires, of course, expressions for the elastic bending moments in the frame; using these known values, it will be found that collapse by mode (a) leads to the following equations

$$\text{Mode (a)} \left\{ \begin{aligned} &\text{Static:} && H_o h = 4M_p^o \,, \\ &\text{Incremental:} && H_o h + \frac{V_o \ell}{8}\left(\frac{3\ell}{2\ell+h}\right) = 4M_p^s \,; \end{aligned} \right\} \quad (19)$$

Similarly, examination of the unit bending moment distributions in fig.3 shows at once that the side load H will make no extra contribution to mode (b) of incremental collapse, and the vertical load V will make no extra contribution to the incremental collapse equation for mode (c). The final equations are

$$\text{Mode (b)} \left\{ \begin{aligned} &\text{Static:} && H_o h + \frac{V_o \ell}{2} = 6M_p^o \,, \\ &\text{Incremental:} && H_o h + \frac{V_o \ell}{8}\left(\frac{9\ell+4h}{2\ell+h}\right) = 6M_p^s \,; \end{aligned} \right\} \quad (20)$$

$$\text{Mode (c)} \left\{ \begin{aligned} &\text{Static:} && \frac{V_o \ell}{2} = 4M_p^o \,, \\ &\text{Incremental:} && \frac{H_o h}{2}\left(\frac{3h}{\ell+6h}\right) + \frac{V_o \ell}{2} = 4M_p^s \,. \end{aligned} \right\} \quad (21)$$

Equations (19), (20) and (21) are plotted schematically in the interaction diagram of fig.4 (this diagram is drawn for $\ell = 2h$, but the general features of the diagram will be preserved for other ratios of ℓ/h). It will be noted

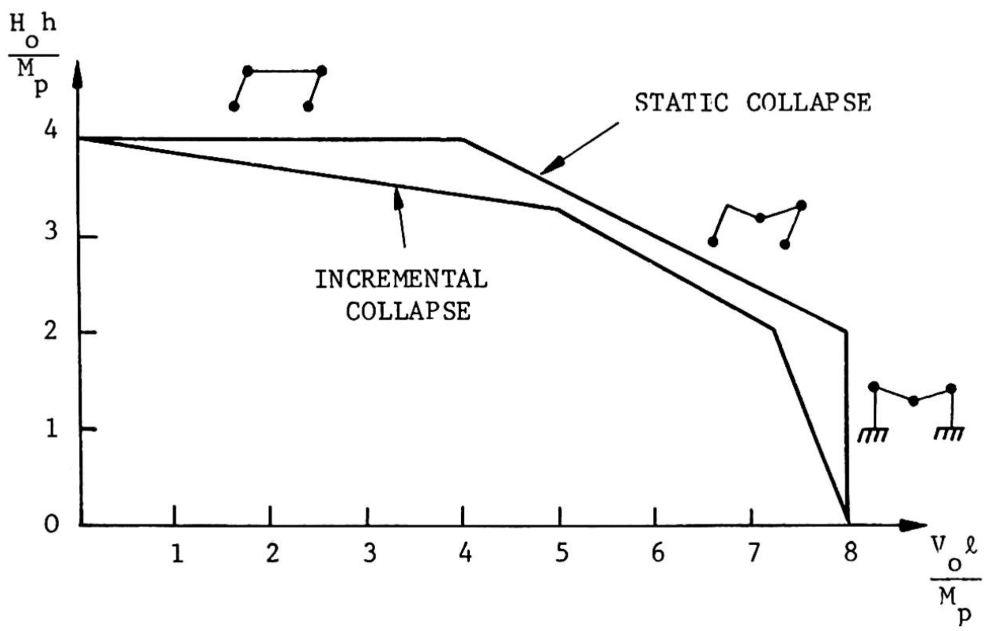


Fig.4

that the "yield surface" for incremental collapse lies entirely within (as it must) the corresponding yield surface for static collapse.

References

1. M. H. Ogle. Shakedown of steel frames. Ph.D. Thesis (Cambridge University), 1964.

2. J. Heyman. Plastic design of frames: Vol.2, Applications. Cambridge, 1971.

Summary

A new formulation of the basic equation of incremental collapse shows immediately which loads acting on a frame are of significance in shakedown design, and which loads are not. A simple numerical example illustrates the procedure, and an interaction diagram is given for the collapse of the fixed-base portal frame.