Theorems for a simplified second order limit analysis of elastic-plastic frames

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Theorems for a Simplified Second Order Limit Analysis of Elastic-Plastic Frames

Méthode réduite de seconde ordre pour la détermination de la charge limite des portiques élasto-plastiques

Hilfssätze für eine vereinfachte Traglastberechnung zweiter Ordnung elastisch-plastischer Rahmentragwerke

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Introduction

The use of simple (or "first order") limit analysis - assuming rigid-plastic material - is restricted to a limited class of frames as the computed failure load P_{FI} ("first order failure load") may give unsafe estimate in presence of axial forces.

Several attempts were made to include the effect of change in geometrie and thus to establish a "second order" limit analysis (resulting in a "second order failure load P_F), ranging from the most simple Rankine-formula to different computer methods [1], [2], [3], [4], [5].

Introducing simplifications this paper is to offer some theorems, which can be used as techniques for preliminary limit design of a class of simple frames, requiring generally an additional check by a more exact method. Because of lack of space the prove of theorems couldn't be reproduced and only reference can be given either to works treating the problem more generally [6] or to the authors previous reports [7], [8].

Attention is paid to the fact, that while the first order failure load depends basically on the value of full-plastic moment M_p of the cross sections only, the second order failure load is influenced by the flexural rigidity "EI" of the constituting members as well. Thus a preliminary design procedure has to include criteria for the required value of both flexural rigidity and full-plastic moment in case of a prescribed failure load P_F.

Assumptions

The model of a frame in the elastic-plastic range is taken as composed of perfectly elastic, initially straight members (of number s) and plastic hinges supposed to develop at certain cross sections only; their greatest possible number be m. The full-plastic moment of the cross sections is assumed to be constant independently of the axial force N acting in the corresponding member. Concentrated loads are allowed to act at joints only, increasing proportionally to a single load factor P (Fig.1.).

We restrict us to cases where in the equations expressing requirements of equilibrium and continuity the shortening of members due both to flexural deformation and direct axial compression can be neglected (excluding thus triangulated frames).

This way the analysis of a perfectly elastic frame can be carried out by solving two simultaneous matrix equations [1] of the form

$$P.\underline{q} = \underline{\underline{S}}.\ \underline{\delta} \qquad (1)$$

$$\underline{N} = P.\underline{q_1} + \underline{\underline{S_1}}\ .\ \underline{\delta} \qquad (2)$$

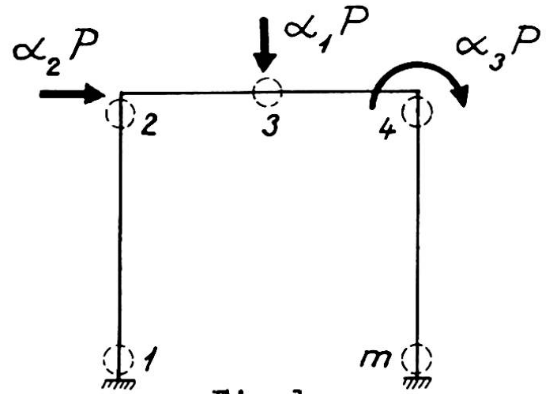

Fig.1.

where vector $\underline{q}$ and $\underline{q_1}$ depends on the distribution of external loads only (quantities α in Fig.1.); $\underline{\underline{S}}$ is the stiffness matrix (its elements being functions of the axial forces in the members), and vector $\underline{\delta}$ represents the "free" displacements of the joints. Second equation expresses, that vector $\underline{N}$ representing the axial forces in the members depends on external loads and displacements of joints as well.

Basic simplification will be introduced by omitting equation (2) and replacing it by

$$\underline{N} = P.\ \underline{\beta} \qquad (3)$$

i.e. assuming axial forces to increase proportionally to the load factor. $\underline{\beta}$ can be taken from the solution of a first order elastic analysis or rather of a rigid-plastic limit analysis. This assumption allows the use of superposition as well and thus bending moments M in an elastic-plastic frame can be expressed $M = M_e + M(\mathcal{X})$; the first term being bending moment of the perfectly elastic frame, the second term the moment originated be hinge-rotations $\mathcal{X}$ in the plastic hinges. Dividing the plastic hinges into "active" ($M = M_p$) and "inactive" ($|M| < |M_p|$) groups (of number i and $m-i$ respectively), the vector $\underline{M}$ representing the bending moments at the cross sections of the active plastic hinges can be written as

$$\underline{M} = \underline{M_e} + \underline{\underline{B}}\,\underline{\mathcal{X}} + \underline{\underline{A}}\,\underline{\mathcal{X}} = \underline{M_p} \qquad (4)$$

where i-vector $\underline{M_e}$ represents the moments of an elastic frame, i-vector $\mathcal{X}$ the rotations in active, m-i vector $\mathcal{X}_r$ the rotations in the inactive plastic hinges; the elements b_{pq} and a_{pq} of $i \times m$- and $i \times i$ matrix $\underline{\underline{B}}$ and $\underline{\underline{A}}$ give the moment at cross-section p of an elastic frame in presence of axial forces $\underline{N} = P.\underline{\beta}$ originated by an angular discontinuity $\mathcal{X}_q = 1$ in the cross section q. Positiv direction of M and $\mathcal{X}$ is given in Fig.2. Sign-convention is used to have $\underline{\mathcal{X}} \geq 0$ and $\underline{M} \geq 0$ in equation (4).

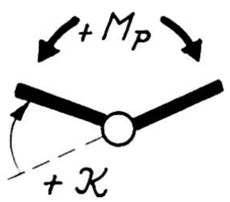

Fig.2.

We shall refer to as "active loading process" if at increasing load no local unloading in plastic hinges takes place. In this case no inactive plastic hinges exist and

$$\underline{M} = \underline{M}_e + \underline{\underline{A}}\,\underline{\mathcal{X}} = \underline{M}_p \qquad (5)$$

Stability considerations

Simplification introduced by equation (3) allows to formulate the condition of stability of the state of equilibrium defined by equation (4) as follows:

$$\sum_{b}\left(\int_0^{\ell} EIy''^2\,dx - P\int_0^{\ell}\beta_s\,y'^2\,dx\right) = 0 \qquad (6)$$

for any function y describing geometrically possible transverse displacements of the points of the members having angular discontinuities $\mathcal{X} \geq 0$ at the cross sections of active plastic hinges only. To facilitate stability investigation, the state of equilibrium defined by equation (4) (having i plastic hinges) should be accompanied by an elastic subsystem "i" (Fig.3.), loaded by axial forces $\underline{N} = P.\underline{\beta}$ only and containing real hinges of number i at the location of plastic hinges. The lowest critical load-factor causing buckling of this subsystem is denoted by $P_{cr,i}$ and is referred to as "deteriorated critical load" in the literature [1]. The buckling-mode of this subsystem at $P = P_{cr,i}$ be described by eigenfunction $\tilde{y}_i$ (containing angular discontinuities $\underline{\tilde{\mathcal{X}}}_i$ at the real hinges).

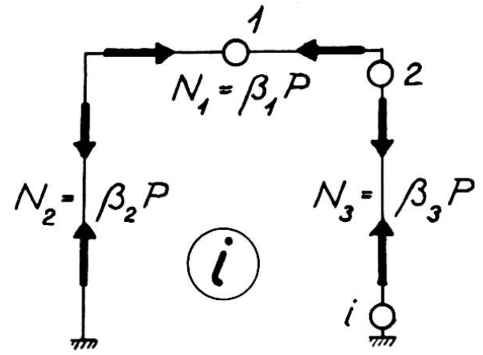

Fig.3.

Using above notations following statements can be done:

Theorem (I)

Supposed that at load factor P the elastic-plastic frame contains i active plastic hinges:

If $P < P_{cr,i}$ the state of equilibrium is stable, If $P_{cr,i} < P < P_{cr,i-1}$ ($P_{cr,i-1}$ being the lowest critical load of an elastic subsystem containing i-1 hinges only) two cases has to be dealt with:

If $\underline{\tilde{\mathcal{X}}}_i \geq 0$ the state of equilibrium is unstable,

If $\underline{\tilde{\mathcal{X}}}_i$ has negative component the state of equilibrium is stable.

Quadratic programming approach

Starting from the state defined by equations (4) and changing load factor to P + dP, the incremental forces and deformations can be described:

$$d\underline{M} = d(\underline{M}_e + \underline{\underline{B}}\,\underline{\mathcal{X}}_r) + d\underline{\underline{A}}\,\underline{\mathcal{X}} + \underline{\underline{A}}\,d\mathcal{X}\,. \tag{7.a}$$

Additionally the nature of plastic hinges requires:

$$d\underline{M} \leq 0; \quad d\mathcal{X} \geq 0 \quad \text{and} \quad d\underline{\mathcal{X}}.\,d\underline{M} = 0, \tag{7.b}$$

as either incremental rotation or decrease of full-plastic moment in the same hinge must be zero.

As pointed out elsewhere in the literature [9],[10] problems of this kind can be solved by a quadratic programming approach, as (7.a) and (7.b) can be written in form of

$$\underline{u} = \underline{a} - \underline{\underline{A}}\,\underline{x} \tag{8.a}$$

$$u \geq 0; \quad \underline{x} \geq 0 \quad \text{and} \quad \underline{x}\,.\,\underline{u} = 0 \tag{8.b}$$

By introducing the scalar function

$$z(\underline{x}) = \underline{a}\,\underline{x} - \frac{1}{2}\underline{x}\,\underline{\underline{A}}\,\underline{x}$$

the solution of problem (8) can be defined as a non-negative vector $\underline{x} = \underline{x}_1 \geq 0$, in case of which the value of function $z=z(x_1)$ doesn't exceed the values $z(\underline{x}_1 + d\underline{x})$ in its vicinity, provided $\underline{x}_1 + d\underline{x} \geq 0$. By virtue of known mathematical theorems [11] a solution always exist if $\underline{x}\,\underline{\underline{A}}\,\underline{x} < 0$ for $\underline{x} \geq 0$, $\underline{x} \neq 0$. As condition $\underline{x}\,\underline{\underline{A}}\,\underline{x} = d\underline{\mathcal{X}}\,\underline{\underline{A}}\,d\underline{\mathcal{X}} < 0$ for $d\mathcal{X} \geq 0$ is fullfilled in a stable state of equilibrium; following statement can be done:

Theorem (II)

Starting from a stable state of equilibrium at load factor P, equilibrium will exist at P + dP as well. Thus failure load (peak load) can be reached only in an unstable state of equilibrium.

If dealing with active loading process only and supposing that plastic hinges can develop in cross sections of number i only with given direction of rotation (chosen to be positiv), the moments and hinge-rotations at a load factor can be determined by transforming equation (5):

$$\underline{M}_p - \underline{M} = \underline{M}_p - \underline{M}_e - \underline{\underline{A}}\,\underline{\mathcal{X}} \tag{9.a}$$

and considering additionally that according to the nature of plastic hinges:

$$\underline{M}_p - \underline{M} \geq 0; \quad \underline{\mathcal{X}} \geq 0 \quad \text{and} \quad (\underline{M}_p - \underline{M}).\underline{\mathcal{X}} = 0. \tag{9.b}$$

Problem (9) can be regarded as integrated form of (8), made equivalent by assuming that local unloading in plastic hinges is excluded. Problem (9) can be rewritten in form of (8) again and investigating the properties of $i \times i$ matrix $\underline{\underline{A}}$ (its components being function of P), we can state:

Theorem (III)

If $P < P_{cr,i}$ there exists one and only one solution.

Theorem (IV)

If $P_{cr,i} < P < P_{cr,i-1}$, the existence and number of solutions depend on the sign of the components of i-vector

$$\underline{\mathcal{K}}_o = \underline{\underline{A}}^{-1} (\underline{M}_p - \underline{M}_e)$$

If all components of $\underline{\mathcal{K}}_o$ are positive ($\underline{\mathcal{K}}_o > 0$), two different solutions exist, one describing a stable, the other an unstable state of equilibrium.

If all components of $\underline{\mathcal{K}}_o$ are non-negative, but at least one of them equals zero ($\mathcal{K}_{oj} = 0$), a single solution exists, describing an unstable state of equilibrium.

If not all components of $\underline{\mathcal{K}}_o$ are non-negative, no solution exists.

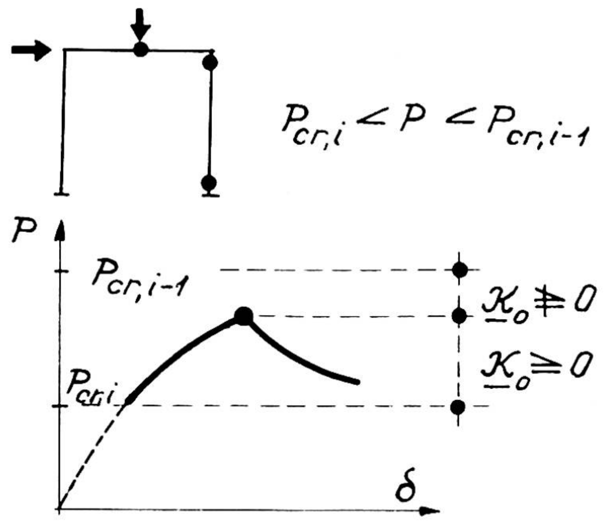

Fig.4.

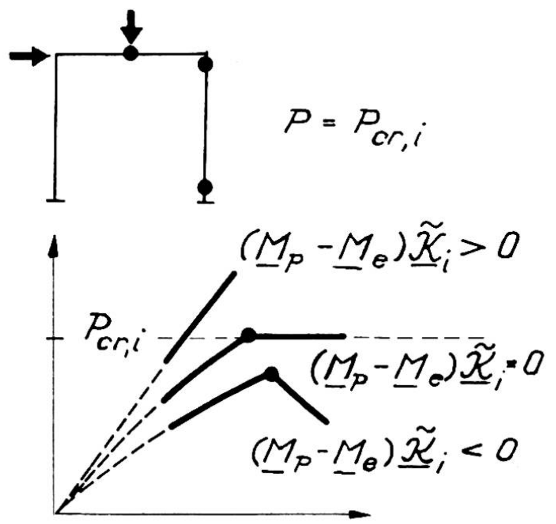

Fig.5.

Theorem (V)

As special case let be $P = P_{cr,i}$. The existence and number of solutions depends on the sign of the scalar product $(\underline{M}_p - \underline{M}_e)\,\underline{\tilde{\mathcal{K}}}_i$.

If $(\underline{M}_p - \underline{M}_e)\,\underline{\tilde{\mathcal{K}}}_i > 0$ one solution exists, describing a stable state of equilibrium.

If $(\underline{M}_p - \underline{M}_e)\tilde{\underline{\varkappa}}_i < 0$ no solution exists.

If $(\underline{M}_p - \underline{M}_e)\tilde{\underline{\varkappa}}_i = 0$ an infinite number of solution exists, describing indifferent states of equilibrium.

Theorem (IV) and (V) can be illustrated by Fig.4. and Fig.5.

Application I. Limit design of "stiff frames"

In a limit design problem the value of P_F be given and the flexural rigidity EI of the members and full plastic moment M_p of cross sections are to be determined. The design problem can be solved on various ways as both EI and M_p contribute to the value of the failure load. So additional restrictions can be given.

We require additionally, that failure should take place if the number of plastic hinges has reached the number n of plastic hinges contained in a yield-mechanism (rigid-plastic collapse-mechanism), thus $i = n$. Frames designed this way will referred to as "stiff frames". Supposing active loading process, theorems (I) - (IV) can be applied.

The subsystem n corresponding to a yield-mechanism is unstable in presence of any forces and thus $P_{cr,n} = 0$ can be taken and the corresponding buckling-mode (eigenfunction) $\tilde{y}_n$ coincides with the displacements of a rigid-plastic yield-mechanism, having angular discontinuities $\tilde{\underline{\varkappa}}_n$ most easily to determine. According to theorem (I) the chosen yield-mechanism prescribes not only the location, but possible rotational direction of plastic hinges as well, as $\tilde{\underline{\varkappa}}_n \geq 0$ has to be taken. The additional requirement given above states, that

$$0 < P_F < P_{cr,n-1}$$

$P_{cr,n-1}$ being the deteriorated critical load of a subsystem produced by removing any of the hinges in the yield-mechanism. As $P_{cr,n-1}$ can be written symbolically

$$P_{cr,n-1} = \frac{c_{n-1}\, EI}{L^2}$$

c_{n-1} being a constant, EI and L representing flexural-rigidity and geometrical data, the criterion for flexural rigidity can be given in the form

$$EI > \frac{L^2}{c_{n-1}} P_F \qquad (10)$$

The required values of full plastic moments $\underline{M}_p$ should be determined according to theorem (IV)

$$\underline{\varkappa}_o = \underline{\underline{A}}^{-1}(\underline{M}_p - \underline{M}_e) \geq 0 \quad \text{for } P \leq P_F \qquad (11)$$

This later requirement can be illustrated practically by Fig.6., as according to virtual-work considerations $\underline{\varkappa}_o$ represents hinge-rotations of the frame under the action of external loads and full-plastic moments at the hinges of number n (axial forces supposed to be $\underline{N} = P.\beta$). Thus this method is equivalent to that referred to as "last hinge method" in the literature [1].

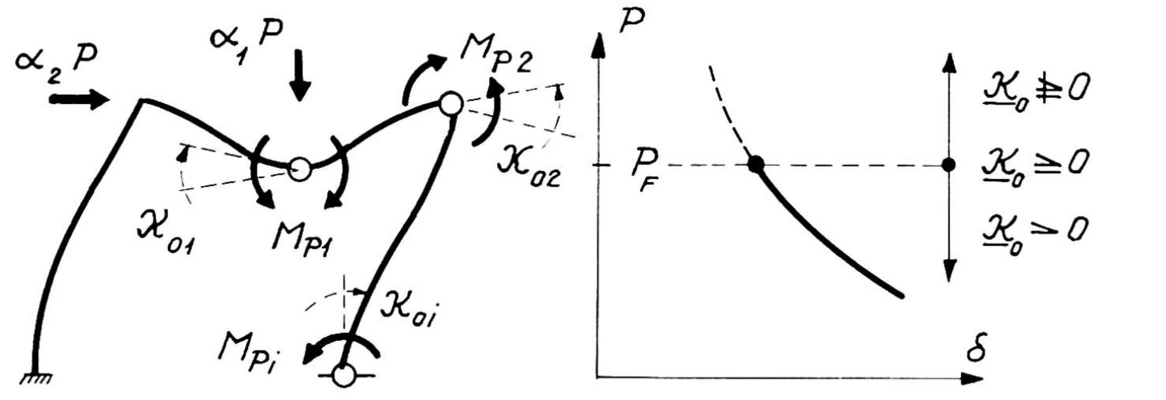

Fig. 6.

If $m > n$, more than one yield mechanism should be considered. To make appropriate choice, following theorem can be used.

Theorem (VI)

Supposing active loading process, at

$$P = P_F < P_{cr,n-1}$$

the moments will not exceed the value of full plastic moment at any of the cross sections m, if unequality (11) is fullfilled for all groups of hinges of number n corresponding to a possible yield-mechanism.

This theorem can be formulated as a minimum principle for failure loads P_F computed with respect the different possible yield-mechanisms (Fig. 7.) or as a maximum principle for a multiplier, if the ratio of full-plastic moments is previously prescribed.

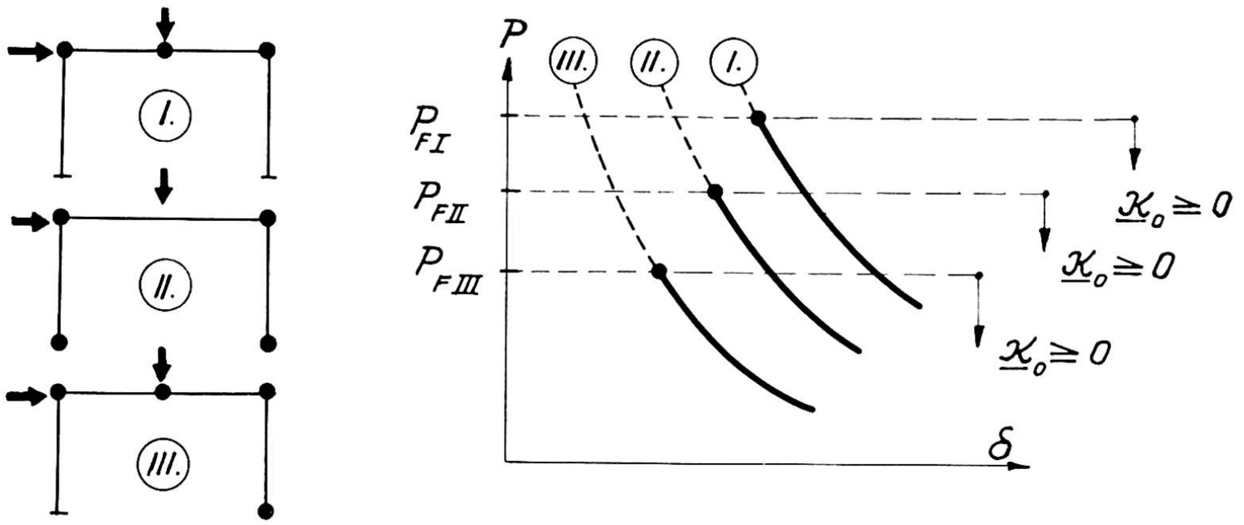

Fig. 7.

Application II. Limit design of "flexible frames"

If requirement (10) results in an unrealistic flexural rigidity, it has to be allowed to reach failure load in presence of a lower number of plastic hinges, than that transforming the structure into a yield-mechanism ($i < n$). Frames designed this way will referred to as "flexible frames". A possible way of limit design easy to carry out results from an additional restriction in form of

$$P_F = P_{cr,i} = \frac{c_i EI}{L^2} \,, \qquad EI = \frac{L^2}{c_i} P_F \tag{12}$$

which can be regarded as criterion for the flexural rigidity required.

Using theorem (V) the values of full-plastic moments in cross section i can be determined as - supposing active loading process - equilibrium can exist at $P = P_{cr,i}$ only if:

$$(\underline{M}_p - \underline{M}_e)\, \underline{\tilde{\mathcal{X}}}_i \geq 0$$

This condition can brought to a more convenient form by using virtual work consideration, resulting in

$$P_F \hat{\sum} \alpha_j \tilde{u}_j \leq \underline{M}_p \underline{\tilde{\mathcal{X}}}_i \tag{13}$$

where $\tilde{u}_j$ and $\underline{\tilde{\mathcal{X}}}_i$ represent the displacements and hinge-rotations due to the buckling-mode (eigenfunction $\tilde{y}_i$) of the plastic subsystem "i" at $P = P_F = P_{cr,i}$ (Fig.8.). This inequality resembles the virtual-work inequality used in a single-plastic limit analysis, but displacements and rotations of a rigid-plastic yield mechanism should be replaced by those belonging to the buckling-mode described by eigenfunction $\tilde{y}_i$.

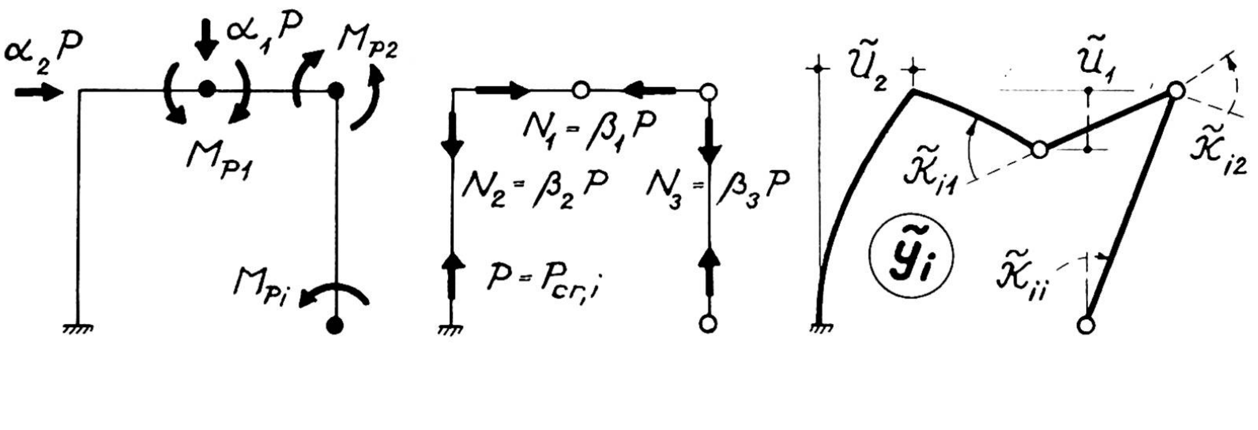

Fig.8.

Using this method an additional check must be carried out with respect to the moments at cross sections being not represented as possible locations of plastic hinges in the subsystem.

Application III. Bifurcation of equilibrium

Attention is to be paid to the fact, that the assumption of active loading process disregards the possibility of bifurcation of equilibrium which can take place in stable state of equilibrium as well [6], due to the "two-faced" nature of plastic hinges.

As illustrativ example a symmetric and symmetrically loaded frame should be regarded with only two possible locations 1. and 2. for plastic hinges, assuming that at a load factor

$$P_{cr,2} < P < P_{cr,1}$$

($P_{cr,1}$ and $P_{cr,2}$ denoting the deteriorated critical load of a subsystem containing one and two real hinges respectively) both plastic hinges are active (Fig.9.).

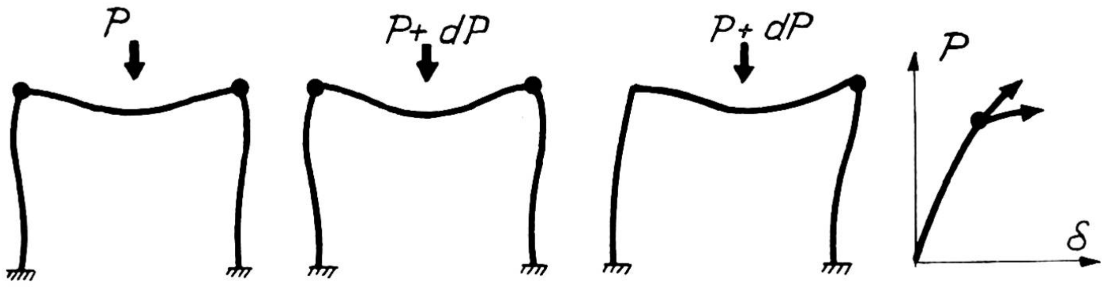

Fig.9.

The possible direction of rotation in plastic hinges is given and taken to be positive. As the buckling-mode at $P = P_{cr,2}$ of the subsystem containing two real hinges is antimetric, according to theorem (II) $\underset{\sim}{\varkappa}_2 \ngtr 0$ and the equilibrium is stable. Analysing incremental deformations by equation

$$\text{at } P \qquad \underline{\underline{M}} = \underline{\underline{M}}_e + \underline{\underline{A}}\,\underline{\varkappa},$$

$$\text{at } P + \Delta P\text{:} \quad d\underline{\underline{M}} = d\underline{\underline{M}}_e + d\underline{\underline{A}}\,\underline{\varkappa} + \underline{\underline{A}}\,d\underline{\varkappa}$$

$$d\underline{\underline{M}} \leq 0, \qquad d\varkappa \geq 0, \qquad d\underline{\underline{M}} \cdot d\underline{\varkappa} = 0$$

Supposing now, that at $P + \Delta P$ a symmetric state of equilibrium with two active hinges exists,

$$d\underline{\underline{M}} = 0 \quad \text{and}$$

$$d\,\underline{\varkappa} = \underline{A}^{-1}\,(d\underline{M}_e + d\underline{\underline{A}}\,\underline{\varkappa}) > 0,$$

according to theorem (IV) a second solution exists. Thus at P, **although** having stable state of equilibrium two different loading paths are possible: a symmetric one and another including side-sway as well.

Appendix

As illustration, the prove of theorem (IV) is given as follows:

(a) If $P_{cr,i} < P < P_{cr,i-1}$ and $\underline{\tilde{\mathcal{X}}}_i$ angular discontinuities of eigenfunction $\tilde{y}_i$ are all positive

$$\underline{\mathcal{X}} = \underline{\underline{A}}^{-1}\,\underline{M} \geq 0, \quad \text{when } \underline{M} \geq 0 \tag{14}$$

This can be proved by describing the displacements of the structure originated by moments $\underline{M}$ acting at the hinges by function y and rotations $\underline{\mathcal{X}}$ and investigating the expression

$$\sum_s \left[EI\int_0^\ell \left(y'' + c_j\tilde{y}_i''\right)dx - P\beta_s\int_0^\ell \left(y' + c_j\tilde{y}_i'\right)^2 dx\right] > 0 \tag{15}$$

$$c_j = -\frac{\mathcal{X}_j}{\tilde{\mathcal{X}}_{ij}} \tag{16}$$

as $y + c_j\, y_i$ are geometrically possible displacements of sub-system "i-1" being stable at $P < P_{cr,i-1}$. Transforming and regarding the virtual work equations

$$\underline{M}\,\underline{\mathcal{X}} = -\sum\left(EI\int_0^\ell y''^2 dx - P\beta_s\int_0^\ell y'^2 dx\right),$$

$$\underline{M}\,\underline{\tilde{\mathcal{X}}}_i = -\sum_s\left(EI\int_0^\ell y''\tilde{y}_i'' dx - P\beta_s\int_0^\ell y'\tilde{y}_i' dx\right),$$

we receive:
$$-\underline{M}\,\underline{\mathcal{X}} - 2\,\frac{\mathcal{X}_j}{\tilde{\mathcal{X}}_{ij}}\,\underline{M}\,\underline{\tilde{\mathcal{X}}}_i > 0 \tag{17}$$

Multiplying by $\mathcal{X}_{ij}\ M_j > 0$ and adding up similar expressions for $j = 1,2\ldots i$:

$$(\underline{M}\,\underline{\mathcal{X}})\ (\underline{M}\,\underline{\tilde{\mathcal{X}}}_i) > 0 \tag{18}$$

which together with equation (17) proves statement (14).

The solution of problem (9) can be written in the form

$$\underline{\mathcal{X}} = \underline{\underline{A}}^{-1}(\underline{M}_p - \underline{M}_e) - \underline{\underline{A}}^{-1}(\underline{M}_p - \underline{M}) \geq 0, \tag{19.a}$$

$$\underline{M}_p - \underline{M} \geq 0 \quad \text{and } (\underline{M}_p - \underline{M})\,\underline{\mathcal{X}} = 0. \tag{19.b}$$

As the second term in (19.a) according to (14) is non-negative, $\underline{\mathcal{X}} \geq 0$ is impossible if

$$\underline{\underline{A}}^{-1}\,(\underline{M}_p - \underline{M}_e) \ngeq 0,$$

and no solution exists.

If
$$\underline{\mathcal{X}}_o = \underline{\underline{A}}^{-1}\,(\underline{M}_p - \underline{M}_e) \succ 0, \tag{20}$$

one solution is given by (20). A second solution can be found as well. With reference to those stated earlier in connection with problem (8), a local minimum of the function

$$z(\underline{\mathcal{X}}) = (\underline{M}_p - \underline{M}_e)\mathcal{X} - \frac{1}{2}\underline{\mathcal{X}}\,\underline{\underline{A}}\,\underline{\mathcal{X}} \tag{21}$$

in the subspace $\mathcal{X} \geq 0$ will define a solution. Chosing one component of $\underline{\mathcal{X}}$ to be $\mathcal{X}_j = 0$, and omitting all elements of $\underline{M}_p$, $\underline{M}_e$ and $\underline{\underline{A}}$ with indices j, we have an i - 1 - dimensional expression

$$\tilde{z}(\underline{\tilde{\mathcal{X}}}) = (\underline{\tilde{M}}_p - \underline{\tilde{M}}_e)\,\underline{\tilde{\mathcal{X}}} - \frac{1}{2}\underline{\tilde{\mathcal{X}}}\,\underline{\underline{\tilde{A}}}\,\underline{\tilde{\mathcal{X}}} \tag{22}$$

which has always a minimum (denoted by $z_{j,min}$) for $\underline{\tilde{\mathcal{X}}} \geq 0$ as because of $P < P_{cr,i-1}$ $\underline{\underline{\tilde{A}}}$ is negative definite. The value $z_{j,min}$ is therefore a minimum on the boundary-plane of subspace $\underline{\mathcal{X}} \geq 0$ defined by $\mathcal{X}_j = 0$. A series of such minimums can obtained by chosing j = 1,2...i. This among them having the lowest value be denoted by $\tilde{z}_{min}$ (at a location given by $\underline{\tilde{\mathcal{X}}}_o$).

Considering equations (20), (21) and (22) the values of function $\tilde{z}$ on the boundary of subspace $\underline{\mathcal{X}} \geq 0$ can be expressed as

$$\tilde{z} = \frac{1}{2}\underline{\mathcal{X}}_o\,\underline{\underline{A}}\,\underline{\mathcal{X}}_o - \frac{1}{2}\underline{r}\,\underline{\underline{A}}\,\underline{r} \tag{23}$$

where $\underline{r}$ denotes the vector between point $\underline{\mathcal{X}} = \underline{\mathcal{X}}_o$ and the point on the boundary. As because of $P_{cr,i} < P$ $\underline{\underline{A}}$ is indefinite, the boundary contains points, where

$$\tilde{z} < \underline{\mathcal{X}}_o\,\underline{\underline{A}}\,\underline{\mathcal{X}}_o$$

and so

$$\tilde{z}_{min} < \frac{1}{2}\underline{\mathcal{X}}_o\,\underline{\underline{A}}\,\underline{\mathcal{X}}_o \; ; \qquad \underline{r}_o\,\underline{\underline{A}}\,\underline{r}_o > 0, \tag{24}$$

where vector $\underline{r}_o$ connects points given by $\underline{\mathcal{X}}_o$ and $\underline{\tilde{\mathcal{X}}}_o$. Equation (23) and (24) prove, that z_{min} is a minimum on the boundary and a local minimum in the subspace $\underline{\mathcal{X}} \geq 0$, thus defining a second solution of problem (19), representing a stable state of equilibrium, being z_{min} on the boundary of the subspace $\underline{\mathcal{X}} \geq 0$. A third solution is impossible, as supposing its existence $\underline{\tilde{\mathcal{X}}}'_o$ and using notations of problem (8)

$$\underline{u} = \underline{a} - \underline{\underline{A}}\,\underline{\tilde{\mathcal{X}}}_o, \quad \underline{\tilde{\mathcal{X}}}_o \geq 0, \quad \underline{u} \geq 0, \quad \underline{u}\cdot\underline{\tilde{\mathcal{X}}}_o = 0$$

$$\underline{u}' = \underline{a} - \underline{\underline{A}}\,\underline{\tilde{\mathcal{X}}}'_o, \quad \underline{\tilde{\mathcal{X}}}'_o \geq 0, \quad \underline{u}' \geq 0, \quad \underline{u}'\cdot\underline{\tilde{\mathcal{X}}}'_o = 0$$

would require

$$(\underline{\tilde{\mathcal{X}}}_o - \underline{\tilde{\mathcal{X}}}'_o)\,\underline{\underline{A}}\,(\underline{\tilde{\mathcal{X}}}_o - \underline{\tilde{\mathcal{X}}}'_o) > 0,$$

which is impossible according theorem (I), as the vector $\underline{\tilde{\mathcal{X}}}_o - \underline{\tilde{\mathcal{X}}}'_o$ connecting two points on the boundary of subspace $\underline{\mathcal{X}} \geq 0$ can't be composed of non-negative components only.

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Summary

The second order limit analysis - including the effect of change in geometrie - of certain class of frames can be simplified by assuming (i) axial forces to increase proportionally to a load factor (ii) disregarding local unloading in plastic hinges and (iii) taking full-plastic moments to be independent of axial forces. Using these assumptions paper offers some theorems to be used in preliminary limit design of frames.