An “equivalent stiffness” method for suspension roof analysis

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An "Equivalent Stiffness" Method for Suspension Roof Analysis

Une méthode de "rigidité équivalente" pour l'analyse de toits suspendus

Eine Methode der "äquivalenten Steifigkeit" zur Analyse von Hängedächern

DONALD P. GREENBERG
Dr.-Ing.
Cornell University
USA

1. INTRODUCTION

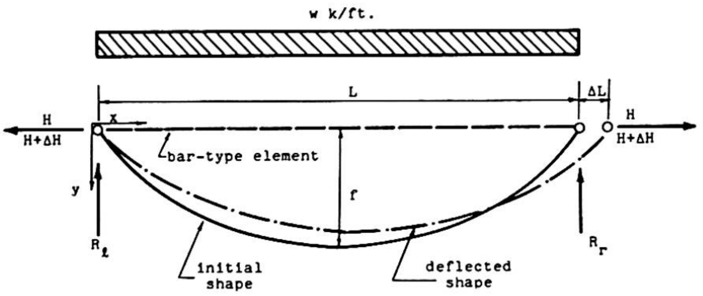
A method is derived to obtain the "equivalent stiffness" of a single cable subject to an initial uniform load. This method may be generalized to include any type of vertical loading including triangular, partial or point loading. The equivalent stiffness is defined as the force required to cause a relative unit displacement of the end points of the cable. This displacement is in the direction of the chord connecting the end points.

Once the value of the equivalent stiffness is found, an initially parabolic cable in a cable network may be replaced by an imaginary straight bar-type element of equivalent stiffness. A schematic diagram of this bar-type element is shown in Figure 1. The area of the bar is assumed equal to that of the cable, while its length is assumed equal to that of the chord connecting the end points of the cable. Thus, the bar-type element may be considered to be composed of a fictitious material with an "equivalent modulus of elasticity" such that the resistance provided by the cable and the bar-type element in the chordal direction are equal. The magnitude of this equivalent modulus depends primarily upon the sag-span ratio, the existing stress level, and the true modulus of elasticity of the cable material. The concept of an equivalent modulus was first investigated by Ernst(1) with regard to the lateral stiffness provided by the main cables of suspension bridges to their supporting towers.

An idealized model of a cable roof system composed of parabolic cables can be created from these imaginary bar-type members. The model may then be analyzed for stresses and deflections for any new loading condition. This procedure may greatly simplify the analysis of certain types of cable roof systems as well as improve the accuracy of the predicted results when compared to present methods of analysis.

2. ADVANTAGES OF IDEALIZED BAR-TYPE MODELS

In present methods of analysis the cable network is generally represented by a system of straight line cable segments connecting the nodal points (Figure 2). Stiffness equations are generated at each nodal point. The number and location of the nodal points depend primarily upon: a) the area of the roof, b) the spacing of the cable mesh, and c) the curvature of the roof surface.

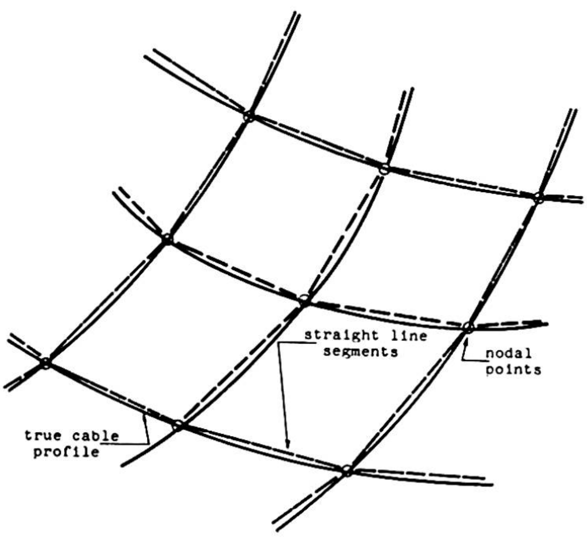


BAR-TYPE ELEMENT REPRESENTATION OF A UNIFORMLY LOADED CABLE

Figure 1

Basically, these methods are finite difference approaches, where the continuous trajectories of the cables are represented by a series of discrete points. To solve the stiffness equations, either direct solution methods, such as matrix inversion or Gaussian elimination, or iterative solution methods are utilized.(4)

Frequently these methods of analysis are restricted by excessive computational time and limited machine storage capacity. A large number of nodal points are required to sufficiently represent the true roof system. This results in a large number of simultaneous equations. For cable roof structures, the size of the resulting stiffness matrix often may exceed the direct storage capacity of many present day computers. In addition, even if the computer's storage capacity is sufficient, the machine time required to obtain a solution is often uneconomical. Iterative solution techniques, although reducing the storage problems and eliminating the need for matrix inversion, do not always converge due to the ill-conditioned aspects of the deflection equations.(2)



CABLE NETWORK REPRESENTED BY STRAIGHT LINE SEGMENTS CONNECTING NODAL POINTS

Figure 2

There are two distinct advantages to using the idealized bar-type models to represent parabolic cable segments. First, a smaller number of nodal points is required to represent the real structure, since each cable may be replaced by only one bar-type element.

This reduces the size of the total structure stiffness matrix which in turn has two decidedly beneficial results. The amount of computational machine time is shortened with its obvious accompanying economic advantages, and a smaller amount of information is required for input.

The second major advantage is that by using these bar-type models, each cable is represented as a continuous element, and not a series of straight line segments connected at discrete points. Thus, the true system is more accurately represented without an increase in the number of nodal points.

3. DERIVATION OF EQUATIONS FOR UNIFORM LOAD CASE

Assumptions

1. Small slopes compared to unity
2. Constant area per cable
3. Vertically applied uniform loading
4. Elastic material behavior
5. Initial profile of cable is parabolic
6. End points at equal elevation
7. Small changes in tension compared to the initial tension
8. Small displacements in the chordal direction compared to the initial length.

The sag and the arc length are expressed respectively by:

$$f = \frac{wL^2}{8H} \tag{1}$$

$$S = L\left(1 + \frac{8}{3}\frac{f^2}{L^2} + \ldots.\right) \tag{2}$$

where: f = sag
w = load per unit length
L = span length
H = horizontal component of tension
S = arc length of the cable

From differentiation, Equations (1) and (2) become:

$$df = \frac{wL}{4H} \cdot dL - \frac{wL^2}{8H^2}\,dH$$

$$df = \frac{2f}{L}\,dL - \frac{f}{H}\,dH \tag{3}$$

and,

$$ds = \left(1 - \frac{8}{3}\frac{f^2}{L^2}\right) dL + \left(\frac{16}{3}\frac{f}{L}\right) df \tag{4}$$

By substitution of Equation (3) into Equation (4),

$$ds = \left(1 + \frac{24}{3}\frac{f^2}{L^2}\right) dL - \left(\frac{16}{3}\frac{f^2}{LH}\right) dH \tag{5}$$

The elastic elongation for a change in horizontal tension[3], dH, is:

$$\Delta S = \int_0^S \frac{dH}{AE} \cdot \frac{ds}{dx}\,ds \tag{6}$$

where: A = area of the cable
E = modulus of elasticity

If only vertical loads are applied, the horizontal component of tension in the cable does not vary with the length, and thus the value of the change in horizontal tension, dH, is also constant. The integral of Equation (6) becomes:

$$\Delta S = \frac{dH}{AE} \int_0^S \frac{ds}{dx}\, ds = \frac{dH}{AE}(2S - L) \tag{7}$$

Substituting the value for the arc length from Equation (2);

$$\Delta S = \frac{dH}{AE}\left[L + \frac{16}{3}\frac{f^2}{L}\right] \tag{8}$$

For small changes in arc length, $\Delta S \rightarrow ds$, and therefore Equations (5) and (8) must be equal. Thus:

$$\frac{dH}{AE}\left[L + \frac{16}{3}\frac{f^2}{L}\right] = \left(1 + \frac{24}{3}\frac{f^2}{L^2}\right) dL - \frac{16}{3}\frac{f^2}{LH}\, dH \tag{9}$$

Rearranging Equation (9),

$$\frac{dH}{dL} = \frac{\left(1 + \frac{24}{3}\frac{f^2}{L^2}\right)}{\frac{16}{3}\frac{f^2}{LH} + \left(\frac{L + \frac{16}{3}\frac{f^2}{L}}{AE}\right)} \tag{10}$$

Now consider the extension of a straight bar-type element of length, L, cross-sectional area, A, and subjected to a change in tension, ΔH. This extension is expressed by the following:

$$\Delta L = \frac{\Delta H}{AE_e} \cdot L \tag{11}$$

where E_e = equivalent modulus of elasticity

Rearranging:

$$E_e = \frac{\Delta H}{\Delta L} \cdot \frac{L}{A} \tag{12}$$

As ΔL approaches zero, the ratio of ΔH/ΔL approaches the derivative dH/dL. Thus, by substitution of Equation (10) into Equation (12), the equivalent modulus of elasticity of an imaginary bar which will exhibit the same lateral stiffness as the true cable is obtained. Thus,

$$E_e = \frac{dH}{dL} \cdot \frac{L}{A} = \frac{\left(1 + \frac{24}{3}\frac{f^2}{L^2}\right)}{\left(\frac{16}{3}\frac{f^2}{L^2} \cdot \frac{1}{H/A}\right) + \left(\frac{1 + \frac{16}{3}\frac{f^2}{L^2}}{E}\right)} \tag{13}$$

Equation (13) is the expression derived for the equivalent modulus. Alternatively, the expression for the equivalent stiffness of an idealized bar-type element can be written as:

$$k_e = \frac{E_e \cdot A}{L} = \left\{ \frac{(1 + \frac{24}{3}\frac{f^2}{L^2})}{(\frac{16}{3}\frac{f^2}{L^2} \cdot \frac{1}{H/A}) + (\frac{1 + \frac{16}{3}\frac{f^2}{L^2}}{E})} \right\} \cdot \frac{A}{L}$$

where k_e = equivalent stiffness of the bar-type element

4. EFFECT OF PARAMETERS

A. Sag/span ratio

The most important parameter in calculating the equivalent modulus of elasticity is the sag/span ratio. A plot of the variation of the equivalent modulus of elasticity versus sag/span ratios is shown in Figure 3. As the sag increases for a given span, the lateral resistance offered by the cable decreases. As the sag/span ratio decreases, the cables become flatter, and the equivalent modulus of elasticity approaches the real modulus. This is easily explained mathematically since all the terms of f^2/L^2 in Equation (13) approach zero, and the equation reduces to:

$$E_e \rightarrow \frac{1}{\frac{1}{E}} \rightarrow E$$

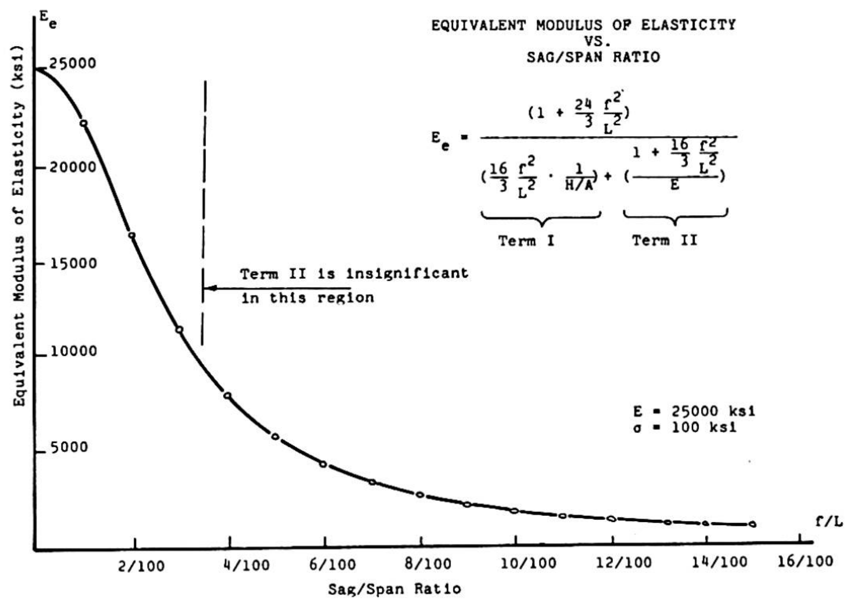


Figure 3

Two examples for calculating the equivalent modulus of elasticity are given below:

Example A

Small sag/span ratio

E = 25000 ksi

f/L = 1/100

$$E_e = \frac{1 + (8.0\text{x}10^{-4})}{1.33\text{x}10^{-5} + 4.00\text{x}10^{-5}}$$

$$E_e = \frac{1.0008}{5.33\text{x}10^{-5}} = 18{,}750 \text{ ksi}$$

Example B

Large sag/span ratio

E = 25000 ksi

f/L = 10/100

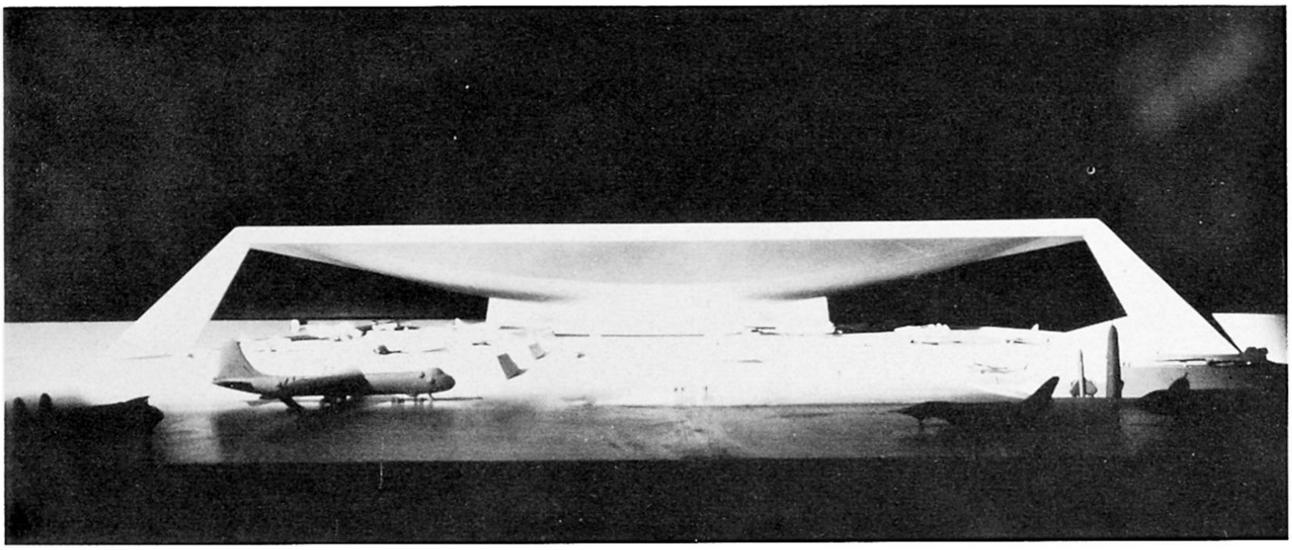
$$E_e = \frac{1 + (8.0\text{x}10^{-2})}{1.43\text{x}10^{-3} + 4.21\text{x}10^{-5}}$$

$$E_e = \frac{1.08}{1.4721\text{x}10^{-3}} = 731 \text{ ksi}$$

From these two examples it can be seen that the first term in the denominator plays the dominant role for all but very small sag/span ratios.

5. EXAMPLE OF PROPOSED AIR FORCE MUSEUM*

The advantages of the use of the bar-type element representation may best be illustrated by the method used to determine the forces and displacements of the proposed Air Force Museum in Dayton, Ohio.** A photograph of the architect's model is shown in Figure 4.



MODEL OF PROPOSED AIR FORCE MUSEUM
Figure 4

*The project was designed by the architectural firm of Roche, Dinkeloo and Associates of Hamden, Connecticut. Severud Associates of New York City served as the consulting engineers.

**The analysis of the stresses and deflections of the suspension roof was the responsibility of the author. The task of obtaining the idealized model and writing the computer program was done jointly with Associate Professors Richard White and Peter Gergely of Cornell University.

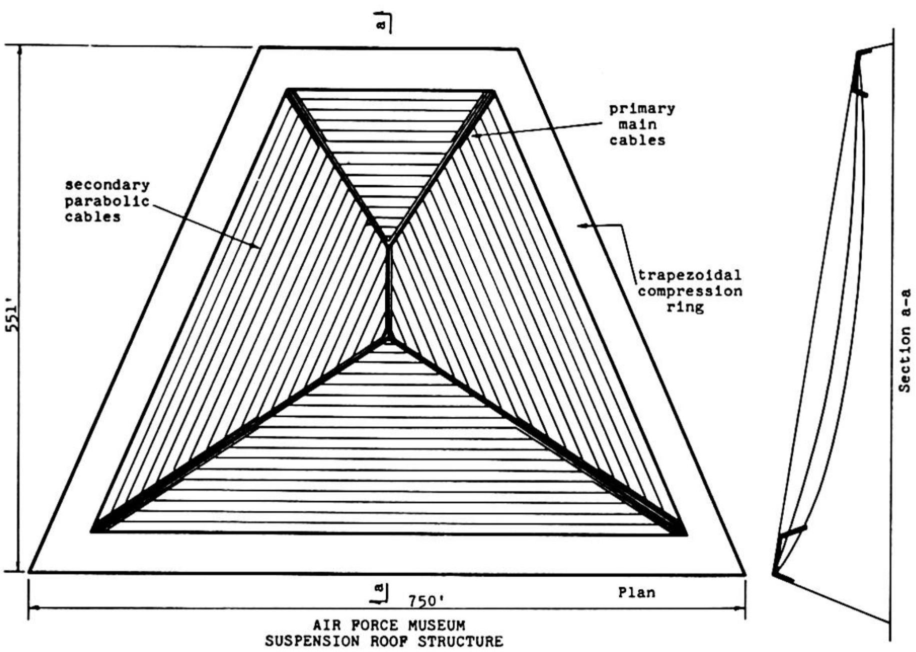


Figure 5

The roof is a suspension roof covering a large trapezoidal area. The structural system, which consists of primary main straight cables supporting sets of parallel secondary parabolic cables is shown in plan and section view in Figure 5. The vertical supports are provided only at the four corners of the trapezoid, and a concrete trapezoidal compression ring around the perimeter absorbs the thrusts from the main cables.

The use of a standard representation technique for the parabolic cables in the roof would be unwieldy; this would require such a large number of nodal points to represent the roof to sufficient accuracy that the direct core storage capacity of the available computer would be exceeded.*** In addition, the machine time required to solve the necessary set of simultaneous equations would have been uneconomical.

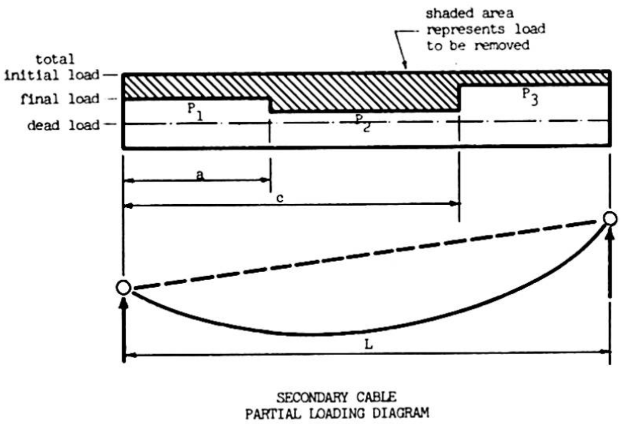


Figure 6

All secondary cables, which were initially of parabolic profile, were replaced by the idealized bar-type elements. The properties of these bar-type elements were determined by equations similar to those previously derived, except that they were generalized to include a set of three partial vertical loads per cable.

***The available computer was a CDC 1604.

By specifying the lengths and magnitudes of the line loadings to be added or removed from the initial loading (Figure 6), each secondary cable could then be subjected to a variety of loadings. Thus, the idealized model of the roof enabled analysis of the system for any set of asymmetric or point loading required by code. The resulting idealized model, consisting of only 74 nodal points, is schematically shown in Figure 7. The springs shown in this figure represent the idealized bar-type elements.

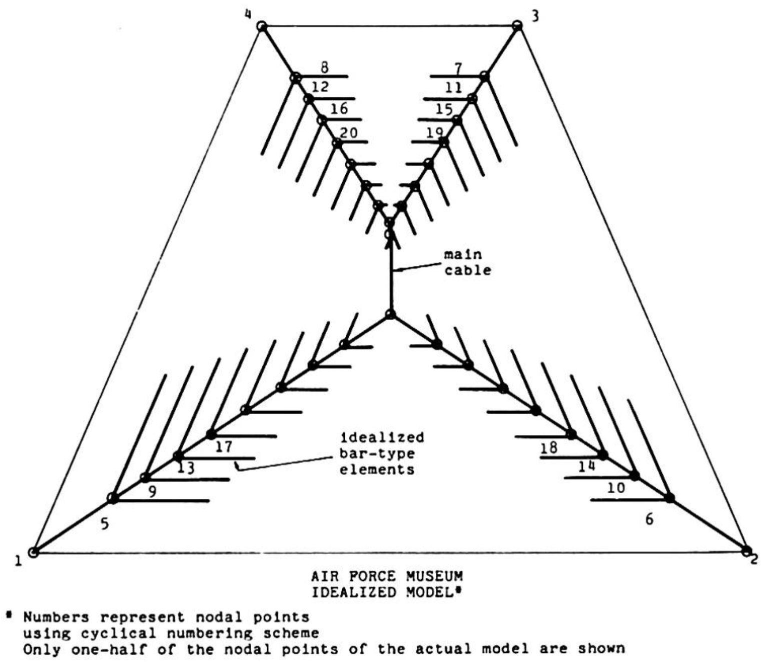


Figure 7

All vertical loads were applied to the secondary cables incrementally to conform with the limitations of the derivation. Secondary cable reactions were then applied to the main cable network, which included the stiffness of the secondary cables. The deformation equations of the total suspension roof were formed using Siev's(5) method of analysis which guarantees a convergent solution. This formulation assumes elastic material behavior and includes the effects of changes in geometry due to large deformations. Using the schematic representation shown in Figure 7, the nodal points were cyclically numbered in such a manner, that combined with the use of diagonal subscripting, the storage requirements of the structural stiffness matrix were minimized. A Gaussian elimination process, adapted for diagonal subscripting, was used to solve the resulting set of simultaneous equations. A description of the computer program is shown following. The system was successfully analyzed for both uniform loading, and partial loading cases.

Computer Program for Analysis of Air Force Museum Using "Equivalent Stiffness" Method

1. Begin Program.

2. Read initial equilibrium conditions (tensions, loads, geometry), material properties, types and increments of loading.

3. Compute initial lengths of primary cables.

4. Compute unstressed lengths of primary cables to use as a base for calculating future tensions.

5. Compute lengths of imaginary bar-type elements representing secondary cables.

6. Compute incremental loads from the secondary cables which will be applied to the total structure and find new secondary cable profiles and tensions.

7. Compute equivalent modulus of elasticity of bar-type elements.

8. Generate total structural stiffness matrix based on current conditions of geometry, loads, and tensions, including the stiffness contribution of the bar-type elements. Store as a diagonally-subscripted band matrix.

9. Apply load increments from (6) to the total structure.

10. Solve for incremental deflections in each direction at each nodal point using the stiffness equations and a Gaussian elimination technique adapted for diagonal-subscripting.

11. Compute new geometry and tensions from the linear solutions of (10).

12. Sum equilibrium at each joint to determine unbalance in each direction due to the "linearization" of the deflection equations.

13. Using current geometry and tensions from (11), reload the structure using the unbalanced residuals from (12) and return to (8).

14. Repeat until unbalanced loads become negligible. Solution is then converged for one increment of load.

15. If final load condition has not yet been reached, add another load increment by returning to (6).

16. Repeat until final load condition has been attained.

17. Print final geometry, stresses, and total deflections, including secondary cable profiles.

18. End Program.

6. BIBLIOGRAPHY

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7. SUMMARY

A method is presented to derive the "equivalent stiffness" of a uniformly loaded, parabolic cable which depends primarily on the sag/span ratio of the cable. To simplify the analysis of certain suspension structures, parabolic cables may be replaced by imaginary bar-type elements of equivalent stiffness. This replacement reduces the number of nodal points required to accurately represent these specific structures, and thus has the advantage of reducing both the computer solution time and the input data.