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SUSPENSION BRIDGES WITH A CONTINUOUS STIFFENING TRUSS.

PONTS SUSPENDUS AVEC POUTRE EN TREILLIS DE RAIDISSEMENT CONTINUE.

HÄNGEBRÜCKEN MIT DURCHLAUFENDEM VERSTEIFUNGS-FACHWERK.

S. TIMOSHENKO and S. WAY.

1. Introduction.

The theory of suspension bridges which takes into consideration the deflection of the stiffening truss has been applied hitherto only in the case of stiffening trusses with hinges at the supports[1]). In the present paper a method of analysis for stresses and deflections in suspension bridges with a continuous stiffening truss is given. General equations are applied to a numerical example and it is shown how the conditions of continuity of the stiffening truss at the supports affects the cable tension. Comparison of cable tension for continuous and for hinged stiffening trusses for various kinds of loading is also given in the paper.

2. Nomenclature.

We assume we are dealing with a three span symmetrical suspension bridge as shown in Fig. 1:

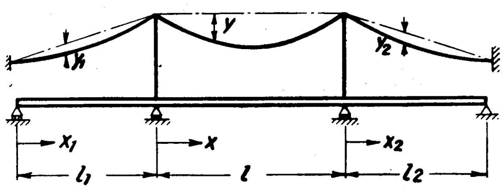

Fig. 1.

The two outer spans are of the same length. The distances x_1, x and x_2 are in each case measured from the left end of the span. We let:

E, I = YOUNG's modulus and the moment of inertia for the stiffening truss section.
p = The live load intensity at any point.
w, w_1 = Dead load intensity at any point in main or side spans respectively.
H_w = Horizontal component of cable tension due to the dead load.

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[1]) Dr. D. B. STEINMAN in the paper "General Deflection Theory for Suspension Bridges" presented before the Chicago Meeting of A. S. C. E., June, 1933, discussed the case of a continuous stiffening truss. In that paper, however, the conditions of continuity at the supports of the stiffening truss were overlooked, thus the theory presented at that time cannot be considered as satisfactory.

We shall assume the cable slides over the tops of the towers, so that H_w is the same in all spans.

When the live load p is applied, the cable and truss will deflect equal amounts, if we neglect stretching ot the hangars. Also we will have an additional cable tension, H, the same in all spans. We let η_1, η, η_2 = Live load deflections of cable and stiffening truss.

$$\beta = \frac{H}{H_w}$$

3. Basic Equations.

The equations of the deflection of the cable with zero live load are:

$$H_w \frac{d^2 y}{dx^2} = -w; \qquad H_w \frac{d^2 y_1}{dx_1^2} = -w_1; \qquad H_w \frac{d^2 y_2}{dx_2^2} = -w_1 \tag{1}$$

By integration we find

$$y = \frac{4fx(l-x)}{l^2}; \qquad y_1 = \frac{4f_1 x_1(l_1 - x_1)}{l_1^2}; \qquad y_2 = \frac{4f_2 x_2(l_1 - x_2)}{l_1^2} \tag{2}$$

where, f, f_1 and f_2 are deflections at the middle points:

$$f = \frac{wl^2}{8H_w}; \qquad f_1 = f_2 = \frac{w_1 l_1^2}{8H_w} \tag{2a}$$

Considering for the time being the center span and assuming some live load p acting, the equation (1) for the cable becomes:

$$(H_w + H)\frac{d^2}{dx^2}(y + \eta) = -(w + q) \tag{3}$$

where q is the additional hangar pull transmitted to the cable due to the live load. From (1) and (2) we find:

$$q = \beta w - H_w(1 + \beta)\frac{d^2\eta}{dx^2} \tag{4}$$

The load transmitted to the stiffening truss will be

$$p - q = p - \beta w + H_w(1 + \beta)\frac{d^2\eta}{dx^2}$$

The equation of vertical equilibrium for any element of the truss then gives:

$$EI\frac{d^4\eta}{dx^4} = p - \beta w + H_w(1 + \beta)\frac{d^2\eta}{dx^2} \tag{5}$$

The load $p-\beta w$ may be regarded as the negative second derivative of the bending moment M_0 due to this load and the moments at the supports. Then (5) becomes:

$$EI\frac{d^4\eta}{dx^4} = -\frac{d^2 M_0}{dx^2} + H_w(1 + \beta)\frac{d^2\eta}{dx^2}$$

and by integration we obtain

$$EI\frac{d^2\eta}{dx^2} = -M_0 + H_w(1 + \beta)\eta \tag{6}$$

Similarly, for spans 1 and 2 we obtain:

$$EI\frac{d^2\eta_1}{dx_1^2} = -M_0 + H_w(1+\beta)\,\eta_1 \tag{7}$$

$$EI\frac{d^2\eta_2}{dx_2^2} = -M_0 + H_w(1+\beta)\,\eta_2 \tag{8}$$

The equations (6), (7) and (8) hold for the stiffening truss of a continuous suspension bridge as well as for the truss of a bridge with hinged spans.

In addition to equations (6), (7) and (8) we have a fundamental equation [2]) for the determination of H. If we assume the hangar pull is uniformly distributed along the spans for purposes of finding H, we have:

$$\frac{HL_s}{AE} + \omega t L_t = \frac{8f_1}{l_1^2}\int_0^{l_1}\eta_1\,dx_1 + \frac{8f}{l^2}\int_0^{l}\eta\,dx + \frac{8f_1}{l_1^2}\int_0^{l_1}\eta_2\,dx_2 \tag{9}$$

where

$$\begin{aligned} L_s &= 2\int_0^{l_1}\left(\frac{ds_1}{dx_1}\right)^2 ds_1 + \int_0^{l}\left(\frac{ds}{dx}\right)^2 ds \\ L_t &= 2\int_0^{l_1}\left(\frac{ds_1}{dx_1}\right) ds_1 + \int_0^{l}\left(\frac{ds}{dx}\right) ds \end{aligned} \tag{10}$$

and s, s_1 are the arc lengths along the cable, t is the temperature rise, ω the coefficient of thermal expansion and A the cross sectional area of the cable.

The problem of a suspension bridge presents two tasks, first to find the deflection η, η_1, η_2 in terms of H, and second to find H by a cut and try method using equation (9).

4. Relation of the Suspension Bridge Problem to the Problem of a Continuous Beam with Axial Load.

Equation (5) will be recognized at once as the differential equation for a beam with lateral load $(p-\beta w)$ and axial load $H_w(1+\beta)$, as shown in Fig. 2:

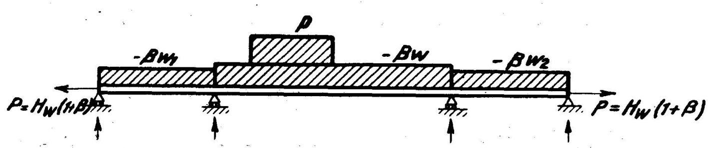

Fig. 2.

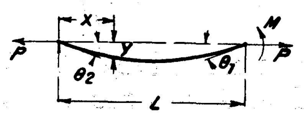

Fig. 3a.

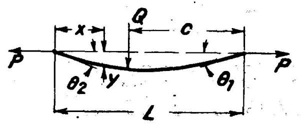

Fig. 3b.

The method of solution of such problems as that shown in Fig. 2 is well known. For the solution of a problem of this type it is convenient to have at hand certain formulae for the simpler structures shown in figures 3a and 3b.

[2]) See Johnson, Bryan and Turneaure, Modern Framed Structures, p. 283, Vol. II.

For the bar shown in Fig. 3a we have the following equations for the angles Θ_1, Θ_2 and deflection η.

$$\left.\begin{aligned} \Theta_1 = \frac{ML}{EI}\left(\frac{1}{KL\tanh KL} - \frac{1}{K^2L^2}\right); \quad \Theta_2 = \frac{ML}{EI}\left(\frac{1}{K^2L^2} - \frac{1}{KL\sinh KL}\right) \\ \eta = \frac{1}{EI}\left(-\frac{M}{K^2}\frac{\sinh Kx}{\sinh KL} + \frac{Mx}{K^2L}\right) \end{aligned}\right\} \quad (11)$$

in which $K = \sqrt{\dfrac{P}{EI}}$.

For the bar in Fig. 3b we have

$$\Theta_1 = -\frac{Q}{P}\frac{\sinh K(L-c)}{\sinh KL} + \frac{Q(L-c)}{PL}; \quad \Theta_2 = -\frac{Q}{P}\frac{\sinh Kc}{\sinh KL} + \frac{Qc}{PL} \quad (12)$$

Deflections at points on the left of Q are:

$$\eta = -\frac{Q\sinh Kc}{PK\sinh KL}\sinh Kx + \frac{Qcx}{PL} \quad (13a)$$

and on the right of Q:

$$\eta = -\frac{Q\sinh K(L-c)}{PK\sinh KL}\sinh K(L-x) + \frac{Q(L-c)(L-x)}{PL} \quad (13b)$$

The deflection due to a distributed load of intensity q may be found by using superpositon, i. e. substituting qdc for Q in the above formulae and integrating with respect to c in the proper limits.

5. Application to the Case of Partial Loading in the Main Span.

We shall assume the live load extends from $x = 0$ to $x = m$ in the main span as shown in Fig. 4.

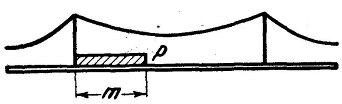

Fig. 4.

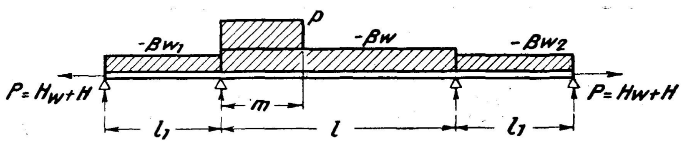

Fig. 5.

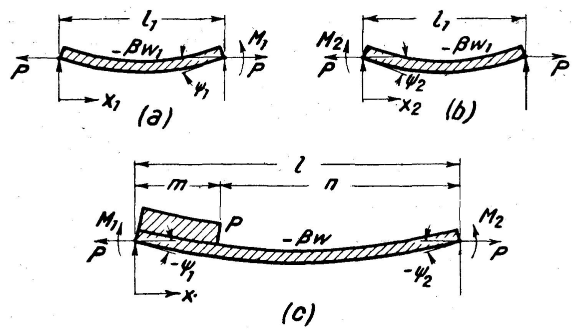

Fig. 6.

The stiffening truss may be regarded as a beam with axial and lateral loads as shown in Fig. 5.

The free body diagrams for the three spans are shown in Fig. 6.

For the bar in Fig. 6 (a), by using equations (11), (12), (13 a) and (13 b), we obtain:

$$\psi_1 = \frac{M_1 l_1}{EI}\left[\frac{1}{Kl_1 \tanh Kl_1} - \frac{1}{K^2 l_1^2}\right] + \int_0^{l_1}\left[\frac{-(-\beta w_1)\sinh Kc}{P \sinh Kl_1} + \frac{(-\beta w_1)c}{Pl_1}\right]dc =$$

$$\frac{M_1 l_1}{EI}\left[\frac{1}{Kl_1 \tanh Kl_1} - \frac{1}{K^2 l_1^2}\right] + \frac{\beta w_1(\cosh Kl_1 - 1)}{PK \sinh Kl_1} - \frac{\beta w_1 l_1}{2P} \tag{14}$$

$$\eta_1 = \frac{1}{EI}\left[\frac{-M_1}{K^2}\frac{\sinh Kx_1}{\sinh Kl_1} + \frac{M_1 x_1}{K^2 l_1}\right] + \frac{(-\beta w_1)}{PK^2}\left[\frac{(1-\cosh Kl_1)}{\sinh Kl_1}\sinh Kx_1 + \right.$$

$$\left. + \cosh Kx_1\right] - \frac{\beta w_1}{P}\left[\frac{l_1 x_1}{2} - \frac{x_1^2}{2} - \frac{1}{K^2}\right] \tag{15}$$

For the bar in Fig. 6 (b):

$$\psi_2 = \frac{M_2 l_1}{EI}\left[\frac{1}{Kl_1 \tanh Kl_1} - \frac{1}{K^2 l_1^2}\right] + \frac{\beta w_1(\cosh Kl_1 - 1)}{PK \sinh Kl_1} - \frac{\beta w_1 l_1}{2P} \tag{16}$$

$$\eta_2 = \frac{1}{EI}\left[\frac{-M_2}{K^2}\frac{\sinh K(l_1 - x_2)}{\sinh Kl_1} + \frac{M_2(l_1 - x_2)}{K^2 l_1}\right] + \frac{(-\beta w_1)}{PK^2} \times$$

$$\times \left[\frac{(1-\cosh Kl_1)}{\sinh Kl_1}\sinh Kx_2 + \cosh Kx_2\right] - \frac{\beta w_1}{P}\left[\frac{l_1 x_2}{2} - \frac{x_2^2}{2} - \frac{1}{K^2}\right] \tag{17}$$

For the bar in Fig. 6 (c) the contributions of the load p will be:

$$-(\psi_1)_p = \int_n^l\left(\frac{-p}{P}\frac{\sinh Kc}{\sinh Kl} + \frac{pc}{Pl}\right)dc = \frac{p}{P}\left[\frac{-(\cosh Kl - \cosh Kn)}{K \sinh Kl} + \frac{l^2 - n^2}{2l}\right] \tag{18}$$

$$-(\psi_2)_p = \int_n^l\left(\frac{-p \sinh K(l-c)}{P \sinh Kl} + \frac{p(l-c)}{Pl}\right)dc = \frac{p}{P}\left[\frac{(1-\cosh Km)}{K \sinh Kl} + \frac{m^2}{2l}\right] \tag{19}$$

In the regions m and n the deflections due to the p load are:

$$(\eta_m)_p = \int_n^{l-x}\left[\frac{-p \sinh Kc}{PK \sinh Kl}\sinh Kx + \frac{pcx}{Pl}\right]dc + \int_{l-x}^{l}\left[\frac{-p \sinh K(l-c)}{PK \sinh Kl} \times \right.$$

$$\left. \times \sinh K(l-x) + \frac{p(l-c)(l-x)}{Pl}\right]dc =$$

$$= \frac{p}{P}\left[\frac{-(\cosh K(l-x) - \cosh Kn)}{K^2 \sinh Kl}\sinh Kx + \frac{[(l-x)^2 - n^2]x}{2l}\right] +$$

$$+ \frac{p}{P}\left[\frac{(1-\cosh Kx)}{K^2 \sinh Kl}\sinh K(l-x) + \frac{x^2(l-x)}{2l}\right] \tag{20}$$

$$(\eta_n)_p = \int_n^l\left[\frac{-p \sinh K(l-c)}{PK \sinh Kl}\sinh K(l-x) + \frac{p(l-c)(l-x)}{Pl}\right]dc =$$

$$= \frac{p}{P}\left[\frac{(1-\cosh Km)}{K^2 \sinh Kl}\sinh K(l-x) + \frac{m^2(l-x)}{2l}\right] \tag{21}$$

If, to these angles and deflections, we add the angles and deflections produced by the uniform load $-\beta w$ we obtain angles and deflections for the bar in Fig. 6 (c) in the following form:

$$-\psi_1 = \frac{p}{P}\left[\frac{-(\cosh Kl - \cosh Kn)}{K\sinh Kl} + \frac{l^2 - n^2}{2l}\right] + \frac{M_2 l}{EI}\left[\frac{1}{K^2 l^2} - \frac{1}{Kl\sinh Kl}\right] + $$
$$+ \frac{M_1 l}{EI}\left[\frac{1}{Kl\tanh Kl} - \frac{1}{K^2 l^2}\right] + \frac{\beta w(\cosh Kl - 1)}{PK\sinh Kl} - \frac{\beta w l}{2P} \tag{22}$$

$$-\psi_2 = \frac{p}{P}\left[\frac{(1 - \cosh Km)}{K\sinh Kl} + \frac{m^2}{2l}\right] + \frac{M_1 l}{EI}\left[\frac{1}{K^2 l^2} - \frac{1}{Kl\sinh Kl}\right] + $$
$$+ \frac{M_2 l}{EI}\left[\frac{1}{Kl\tanh Kl} - \frac{1}{K^2 l^2}\right] + \frac{\beta w(\cosh Kl - 1)}{PK\sinh Kl} - \frac{\beta w l}{2P} \tag{23}$$

$$\eta_m = \frac{p}{P}\left[\frac{-(\cosh K(l-x) - \cosh Kn)}{K^2\sinh Kl}\sinh Kx + \frac{[(l-x)^2 + n^2]x}{2l}\right] + $$
$$+ \frac{p}{P}\left[\frac{(1-\cosh Kx)}{K^2\sinh Kl}\sinh K(l-x) + \frac{x^2(l-x)}{2l}\right] + \frac{M_2}{EI}\left[\frac{-1}{K^2}\frac{\sinh Kx}{K^2\sinh Kl} + \frac{x}{K^2 l}\right] + $$
$$+ \frac{M_1}{EI}\left[\frac{-1}{K^2}\frac{\sinh K(l-x)}{\sinh Kl} + \frac{(l-x)}{K^2 l}\right] - \frac{\beta w}{PK^2}\left[\frac{(1-\cosh Kl)\sinh Kx}{\sinh Kl} + \right.$$
$$\left. + \cosh Kx\right] - \frac{\beta w}{P}\left[\frac{lx}{2} - \frac{x^2}{2} - \frac{1}{K^2}\right] \tag{24}$$

$$\eta_n = \frac{p}{P}\left[\frac{(1-\cosh Km)}{K^2\sinh Kl}\sinh K(l-x) + \frac{m^2(l-x)}{2l}\right] + \frac{M_2}{EI}\left[\frac{-1}{K^2}\frac{\sinh Kx}{\sinh Kl} + \frac{x}{K^2 l}\right] + $$
$$+ \frac{M_1}{EI}\left[\frac{-1}{K^2}\frac{\sinh K(l-x)}{\sinh Kl} + \frac{(l-x)}{K^2 l}\right] - \frac{\beta w}{PK^2}\left[\frac{(1-\cosh Kl)\sinh Kx}{\sinh Kl} + \right.$$
$$\left. + \cosh Kx\right] - \frac{\beta w}{P}\left[\frac{lx}{2} - \frac{x^2}{2} - \frac{1}{K^2}\right] \tag{25}$$

The equations for finding M_1 and M_2 come from equating the values of ψ_1 given in (14) and (22) and the values of ψ_2 given in (16) and (23). Before we do this we make the substitutions:

$$\varrho = \frac{pl}{H_w}; \quad P = H_w(1+\beta); \quad w = \frac{8fH_w}{l^2}; \quad w_1 = \frac{8f_1 H_w}{l_1^2}$$

We then have from (14) and (22):

$$\frac{\varrho}{1+\beta}\left[\frac{-(\cosh Kl - \cosh \frac{n}{l} Kl)}{Kl\sinh Kl} + \tfrac{1}{2}\left(1 - \left(\frac{n}{l}\right)^2\right)\right] + \left(\frac{M_2 l}{EI}\right)\left[\frac{1}{K^2 l^2} - \frac{1}{Kl\sinh Kl}\right] + $$
$$+ \left(\frac{M_1 l}{EI}\right)\left[\frac{1}{Kl\tanh Kl} - \frac{1}{K^2 l^2}\right] + \frac{8\beta}{1+\beta}\left(\frac{f}{l}\right)\frac{(\cosh Kl - 1)}{Kl\sinh Kl} - \frac{4\beta}{1+\beta}\left(\frac{f}{l}\right) + $$
$$+ \left(\frac{M_1 l}{EI}\right)\left[\frac{1}{Kl\tanh\left(\frac{l_1}{l}\right)Kl} - \left(\frac{l}{l_1}\right)\frac{1}{K^2 l^2}\right] + \frac{8\beta}{1+\beta}\left(\frac{f_1}{l}\right)\frac{\left(\cosh\frac{l_1}{l}Kl - 1\right)}{\left(\frac{l_1}{l}\right)Kl\sinh\left(\frac{l_1}{l}\right)Kl} - $$
$$- \frac{4\beta}{1+\beta}\left(\frac{f_1}{l_1}\right) = 0 \tag{26}$$

and from (16) and (23):

$$\frac{\varrho}{1+\beta}\left[\frac{\left(1-\cosh\frac{m}{l}Kl\right)}{Kl\sinh Kl}+\frac{1}{2}\left(\frac{m}{l}\right)^2\right]+\left(\frac{M_1 l}{EI}\right)\left[\frac{1}{K^2l^2}-\frac{1}{Kl\sinh Kl}\right]+$$
$$+\left(\frac{M_2 l}{EI}\right)\left[\frac{1}{Kl\tanh Kl}-\frac{1}{K^2l^2}\right]+\frac{8\beta}{1+\beta}\left(\frac{f}{l}\right)\frac{(\cosh Kl-1)}{Kl\sinh Kl}-\frac{4\beta}{1+\beta}\left(\frac{f}{l}\right)+$$
$$+\left(\frac{M_2 l}{EI}\right)\left[\frac{1}{Kl\tanh\frac{l_1}{l}Kl}-\frac{l}{l_1}\frac{1}{K^2l^2}\right]+\frac{8\beta}{1+\beta}\left(\frac{f_1}{l_1}\right)\frac{\left(\cosh\frac{l_1}{l}Kl-1\right)}{\left(\frac{l_1}{l}\right)Kl\sinh\left(\frac{l_1}{l}\right)Kl}-$$
$$-\frac{4\beta}{1+\beta}\left(\frac{f_1}{l_1}\right)=0 \tag{27}$$

For any particular bridge ϱ will be known and $\frac{f}{l}$ and $\frac{f_1}{l_1}$ will be known. Kl is related to β by

$$Kl=\sqrt{\frac{l^2H_w}{EI}(1+\beta)}$$

and the quantity $\frac{l^2H_w}{EI}$ will be known. Equations (26) and (27) will therefore give the quantities $\left(\frac{M_1 l}{EI}\right)$ and $\left(\frac{M_2 l}{EI}\right)$ in terms of β, $\frac{n}{l}$ and $\frac{m}{l}$. For any given load, therefore, the moments will be given as functions of β.

The cable tension will be found by placing expressions (15), (17), (24) and (25) in equation (9), which may be written in the following form:

$$\frac{K^2IHL_sl^2}{8fA}+\frac{H_w\omega tL_tl^2}{8f}+\frac{H\omega tL_tl^2}{8f}=$$
$$=(H_w+H)\left[\frac{f_1}{l_1^2}\frac{l^2}{f}\int_0^{l_1}\eta_1\,dx_1+\int_0^{l}\eta\,dx+\frac{f_1 l^2}{l_1^2 f}\int_0^{l_1}\eta_2\,dx_2\right] \tag{28}$$

By referring to (15), (17), (24) and (25) we see that the coefficient of β in the right hand member of (28) is

$$2\left(\frac{f_1 l^2}{l_1^2 f}\right)\int_0^{l_1}\left\{-\frac{w_1}{K^2}\left[\frac{1-\cosh Kl_1}{\sinh Kl_1}\sinh Kx_1+\cosh Kx_1\right]-w_1\times\right.$$
$$\left.\times\left[\frac{l_1x_1}{2}-\frac{x_1^2}{2}-\frac{1}{K^2}\right]\right\}dx_1-\int_0^{l}\left\{\frac{w}{K^2}\left[\frac{(1-\cosh Kl)\sinh Kx}{\sinh Kl}+\cosh Kx\right]+\right.$$
$$\left.+w\left[\frac{lx}{2}-\frac{x^2}{2}-\frac{1}{K^2}\right]\right\}dx$$

The terms in the right hand member which do dot involve β explicitly are:

$$\left(\frac{f_1 l^2}{l_1^2 f}\right)\int_0^{l_1}\left[\frac{-M_1\sinh Kx_1}{\sinh Kl_1}+\frac{M_1x_1}{l_1}\right]dx_1+\left(\frac{f_1 l^2}{l_1^2 f}\right)\int_0^{l_1}\left[\frac{-M_2\sinh K(l_1-x_2)}{\sinh Kl_1}+\right.$$
$$\left.+\frac{M_2(l_1-x_2)}{l_1}\right]dx_2+\int_0^{m}\left\{p\left[\frac{-\sinh Kl+\sinh Kx\cosh Kn+\sinh K(l-x)}{K^2\sinh Kl}+\right.\right.$$
$$\left.\left.+\frac{lx}{2}-\frac{x^2}{2}-\frac{n^2x}{2l}\right]-\frac{M_2\sinh Kx}{\sinh Kl}+\frac{M_2x}{l}-\frac{M_1\sinh K(l-x)}{\sinh Kl}+\frac{M_1(l-x)}{l}\right\}dx+$$

$$+\int_m^l\left\{p\left[\frac{(1-\cosh Km)\sinh K(l-x)}{K^2\sinh Kl}+\frac{m^2(l-x)}{2l}\right]-\frac{M_2\sinh Kx}{\sinh Kl}+\frac{M_2 x}{l}-\right.$$
$$\left.-\frac{M_1\sinh K(l-x)}{\sinh Kl}+\frac{M_1(l-x)}{l}\right\}dx\,.$$

If we perform the integrations, equation (28) can finally be presented in the form

$$\beta=\frac{U}{V} \tag{29}$$

in which [3])

$$U=\left[-\left(\frac{M_2 l}{EI}\right)\frac{(\cosh Kl-1)}{Kl\sinh Kl}+\tfrac{1}{2}\left(\frac{M_2 l}{EI}\right)-\left(\frac{M_1 l}{EI}\right)\frac{(\cosh Kl-1)}{Kl\sinh Kl}+\tfrac{1}{2}\left(\frac{M_1 l}{EI}\right)\right]+$$
$$+\left(\frac{f_1 l^2}{l_1{}^2 f}\right)\left[-\left(\frac{M_2 l}{EI}\right)\frac{(\cosh Kl_1-1)}{Kl\sinh Kl_1}+\tfrac{1}{2}\left(\frac{l_1}{l}\right)\left(\frac{M_2 l}{EI}\right)-\left(\frac{M_1 l}{EI}\right)\frac{(\cosh Kl_1-1)}{Kl\sinh Kl_1}+\right.$$
$$\left.+\tfrac{1}{2}\frac{l_1}{l}\left(\frac{M_1 l}{EI}\right)\right]+\frac{pl^3}{EI}\left[-\frac{m}{l}\left(\frac{1}{K^2l^2}\right)-\frac{(1-\cosh Km)}{K^3l^3\sinh Kl}+\frac{(\cosh Kl-\cosh Kn)}{K^3l^3\sinh Kl}+\right.$$
$$\left.+\frac{m^2}{4l^2}-\frac{m^3}{6l^3}\right]-\frac{H_w l^3\omega t}{8EIf}\cdot\frac{L_t}{l}$$

$$V=\frac{(K^2l^2)IL_s}{8fAl^2}\left(\frac{H_w l^2}{EI}\right)+\left(\frac{H_w l^2}{EI}\right)\frac{\omega tl}{8f}\left(\frac{L_t}{l}\right)+\frac{16f}{l}\left(\frac{H_w l^2}{EI}\right)\frac{(\cosh Kl-1)}{K^3l^3\sinh Kl}+$$
$$+\frac{2}{3}\frac{f}{l}\left(\frac{H_w l^2}{EI}\right)-\frac{8f}{l}\frac{1}{(K^2l^2)}\left(\frac{H_w l^2}{EI}\right)+\frac{32}{K^3l^3}\frac{cf_1}{l_1}\frac{l}{l_1}\left(\frac{H_w l^2}{EI}\right)\frac{(\cosh Kl_1-1)}{\sinh Kl_1}+$$
$$+\frac{4}{3}\frac{f_1 l_1 c}{l^2}\left(\frac{H_w l^2}{EI}\right)-\frac{16 f_1 c}{l_1(K^2l^2)}\left(\frac{H_w l^2}{EI}\right)$$

This equation, along with (26) and (27), form the basis for numerical calculations. We shall now consider a numerical example to illustrate the method.

6. Numerical Example. [4])

Let $\frac{H_w l^2}{EI}=41.29$; $H_w=3.667\cdot 10^6$ lbs.

$l=800$ ft.; $EI=56.84\cdot 10^9$ lbs. ft.²

$l_1=400$ ft.; $\frac{f_1 l^2}{l_1{}^2 f}=1$; $w=w_1$

$\frac{f}{l}=0.105$; $\frac{f_1}{l_1}=0.0525$; $\frac{l}{l_1}=2$

$$\varrho=\frac{pl}{H_w}=0.28361$$

[3]) With $M_1=M_2=0$, this formula checks with that of JOHNSON, BRYAN and TURNEAURE, Modern Framed Structures, p. 288.

[4]) The numerical data in this example are the same as in the paper presented by Dr. D. B. STEINMAN at the joint meeting of A. S. M. E. and A. S. C. E. at Chicago in June, 1933.

$$\frac{L_t}{l} = 2.4975 \quad ; \quad \frac{L_s}{l} = 2.59375\,; \quad \frac{I}{A\,l^2} = 0.34880 \cdot 10^{-4}$$

$$\omega = 65 \cdot 10^{-7}\,; \quad t = 60^0\ \mathrm{F} \quad ; \quad A = 87.8\ \mathrm{in.}^2$$

$$I = 1960\ \mathrm{in.}^2\ \mathrm{ft.}^2$$

The equations (26) and (27) become:

$$\frac{0.28361}{1+\beta}\left[\frac{-\left(\cosh Kl - \cosh \frac{n}{l}\,Kl\right)}{Kl \sinh Kl} + \tfrac{1}{2}\left(1-\left(\frac{n}{l}\right)^2\right)\right] + \left(\frac{M_2\,l}{EI}\right) \times$$
$$\times \left[\frac{1}{K^2\,l^2} - \frac{1}{Kl\sinh Kl}\right] + \left(\frac{M_1\,l}{EI}\right)\left[\frac{1}{Kl\tanh Kl} - \frac{1}{K^2\,l^2}\right] + \frac{0.8400\,\beta}{1+\beta} \times$$
$$\times \frac{(\cosh Kl - 1)}{Kl \sinh Kl} - \frac{0.6300\,\beta}{1+\beta} + \frac{0.8400\,\beta}{1+\beta}\,\frac{\left(\cosh \frac{Kl}{2} - 1\right)}{Kl \sinh \frac{Kl}{2}} + \left(\frac{M_1\,l}{EI}\right) \times$$
$$\times \left[\frac{1}{Kl \tanh \frac{Kl}{2}} - \frac{2}{K^2\,l^2}\right] = 0$$

$$\frac{0.28361}{1+\beta}\left[\frac{1-\cosh \frac{m}{l}Kl}{Kl\sinh Kl} + \tfrac{1}{2}\left(\frac{m}{l}\right)^2\right] + \left(\frac{M_1\,l}{EI}\right)\left[\frac{1}{K^2\,l^2} - \frac{1}{Kl\sinh Kl}\right] +$$
$$+ \left(\frac{M_2\,l}{EI}\right)\left[\frac{1}{Kl\tanh Kl} - \frac{1}{K^2\,l^2}\right] + \frac{0.8400\,\beta}{1+\beta}\,\frac{(\cosh Kl - 1)}{Kl\sinh Kl} - \frac{0.6300\,\eta}{1+\beta} +$$
$$+ \frac{0.8400\,\beta}{1+\beta}\,\frac{\left(\cosh\frac{Kl}{2} - 1\right)}{Kl\sinh\frac{Kl}{2}} + \left(\frac{M_2\,l}{EI}\right)\left[\frac{1}{Kl\tanh\frac{Kl}{2}} - \frac{2}{K^2\,l^2}\right] = 0$$

A numerical calculation must now be made to determine for the above equations the values of $\frac{M_1\,l}{EI}$ and $\frac{M_2\,l}{EI}$ for various values of n and β. Values of $\frac{M_1\,l}{EI}$ are found to be:

Table I.

Values of $\frac{M_1\,l}{EI}$

	$\beta = -0.1$	$\beta = 0$	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.3$
$n = l$	— 0.1489	0	+ 0.1401	+ 0.2723	+ 0.3979
$n = 0.9\,l$	— 0.1740	— 0.0249	+ 0.1155	+ 0.2479	+ 0.3739
$n = 0.8\,l$	— 0.2290	— 0.0784	+ 0.0629	+ 0.1969	+ 0.3238
$n = 0.7\,l$	— 0.2936	— 0.1410	+ 0.0025	+ 0.1379	+ 0.2666
$n = 0.6\,l$	— 0.3574	— 0.2031	— 0.0561	+ 0.0812	+ 0.2118
$n = 0.5\,l$	— 0.4139	— 0.2561	— 6.1081	+ 0.0316	+ 0.1638
$n = 0.4\,l$	— 0.4606	— 0 3006	— 0.1508	— 0.0092	+ 0.1243
$n = 0.3\,l$	— 0.4959	— 0.3342	— 0.1829	— 0.0402	+ 0.0947
$n = 0.2\,l$	— 0.5197	— 0.3566	— 0.2043	— 0.0607	+ 0.0749
$n = 0.1\,l$	— 0.5322	— 0.3686	— 0.2160	— 0.0718	+ 0.0644
$n = 0$		— 0.3721	— 0.2190	— 0.0750	+ 0.0615

Values of $\frac{M_2 l}{EI}$ are found to be:

Table II.

Values of $\frac{M_2 l}{EI}$

	$\beta = -0.1$	$\beta = 0$	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.3$
$n = l$	-0.1489	0	$+0.1401$	$+0.2723$	$+0.3979$
$n = 0.9\,l$	-0.1524	-0.0033	$+0.1368$	$+0.2692$	$+0.3950$
$n = 0.8\,l$	-0.1651	-0.0154	$+0.1253$	$+0.2582$	$+0.3844$
$n = 0.7\,l$	-0.1887	-0.0379	$+0.1039$	$+0.2375$	$+0.3646$
$n = 0.6\,l$	-0.2231	-0.0714	$+0.0717$	$+0.2068$	$+0.3350$
$n = 0.5\,l$	-0.2707	-0.1160	$+0.0291$	$+0.1660$	$+0.2957$
$n = 0.4\,l$	-0.3274	-0.1689	-0.0228	$+0.1161$	$+0.2476$
$n = 0.3\,l$	-0.3909	-0.2313	-0.0816	$+0.0594$	$+0.1927$
$n = 0.2\,l$	-0.4556	-0.2937	-0.1421	$+0.0006$	$+0.1355$
$n = 0.1\,l$	-0.5106	-0.3473	$-0{,}1946$	-0.0506	$+0.0853$
$n = 0$		-0.3721	-0.2190	-0.0750	$+0.0615$

When we substitute the given numerical values, equation (29) becomes:

$$\beta = \frac{11.710\left[-\frac{m}{l}\frac{1}{K^2 l^2} - \frac{(1-\cosh Km)}{K^3 l^3 \sinh Kl} + \frac{(\cosh Kl - \cosh Kn)}{K^3 l^3 \sinh Kl} + \frac{m^2}{4\,l^2} - \frac{m^3}{6\,l^3}\right] - 0.047878 + \left(\frac{M_2 l}{EI} + \frac{M_1 l}{EI}\right)\left[-\frac{(\cosh Kl - 1)}{\sinh Kl} - \frac{\left(\cosh\frac{Kl}{2} - 1\right)}{\sinh\frac{Kl}{2}} + \frac{3}{4}Kl\right]\frac{1}{Kl}}{44.468\cdot 10^{-4} K^2 l^2 + 3.6607 - \frac{69.367}{K^2 l^2} + \frac{69.367}{K^3 l^3}\cdot\left[\frac{(\cosh Kl - 1)}{\sinh Kl} + 2\frac{(\cosh Kl_1 - 1)}{\sinh Kl_1}\right]}$$

For various assumed values of β substituted in the right side of the above equation, we find values of β as follows:

Table III.

	$\beta = -0.1$	$\beta = 0$	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.3$
$n = l$	-0.06316	-0.01646	0.02745	0.06887	0.10818
$n = 0.9\,l$	-0.06070	-0.01400	0.02992	0.07137	0.11071
$n = 0.8\,l$	-0.05147	$-0\ 00459$	0.03944	0.08107	0.12049
$n = 0.7\,l$	-0.03499	0.01211	0.05639	0.09813	0.13776
$n = 0.6\,l$	-0.01273	0.03426	0.07906	0.12104	0.16084
$n = 0.5\,l$	0.01219	0.05998	0.10477	0.14704	0.18700
$n = 0.4\,l$	0.03730	0.08572	0.13047	0.17301	0.21310
$n = 0.3\,l$	0.05941	0.10781	0.15311	0.19585	0.23615
$n = 0.2\,l$	0.07586	0.12456	0.17006	0.21359	0.25344
$n = 0.1\,l$	0.08511	0.13393	0.17954	0.22263	0.26319
$n = 0$		0.13641	0.18208	0.22509	0.26579

By plotting β — calculated against β — assumed for each value of n, we obtain a set of curves of the type shown in Fig. 7. The points where these curves cross the 45^0 line determine the values of β for each load.

The moments as given in Table I have been plotted against β for various values of m, in Fig. 8. Using the values of β found in Fig. 7, the quantity $\frac{M_1 l}{EI}$ can be found for each load from Fig. 8. $\frac{M_2 l}{EI}$ can be found for each load from Fig. 9. Values of $\frac{M_1 l}{EI}$ and $\frac{M_2 l}{EI}$ and β for various values of n are given in Table IV.

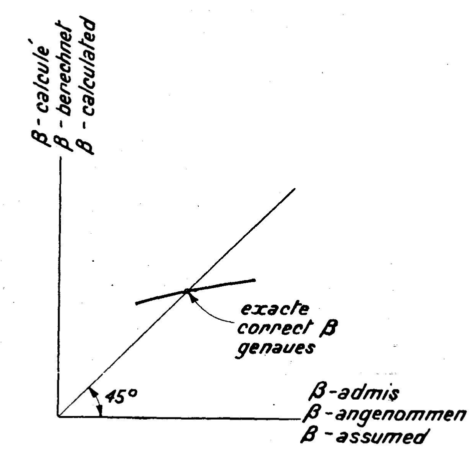
β - calculé
β - berechnet
β - calculated
exacte
correct β
genaues
45°
β-admis
β-angenommen
β-assumed

Fig. 7.

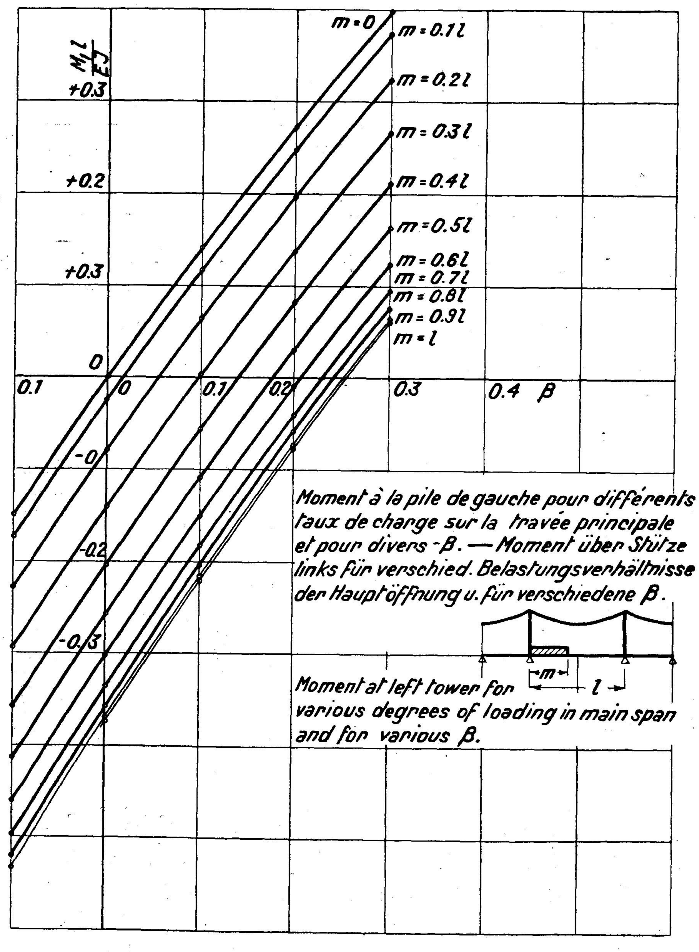
$\frac{M_1 l}{EJ}$
+0.3
+0.2
+0.3
0
0.1
0
0.1
0.2
0.3
0.4 β
-0.1
-0.2
-0.3
m = 0
m = 0.1l
m = 0.2l
m = 0.3l
m = 0.4l
m = 0.5l
m = 0.6l
m = 0.7l
m = 0.8l
m = 0.9l
m = l
Moment à la pile de gauche pour différents taux de charge sur la travée principale et pour divers -β. — Moment über Stütze links für verschied. Belastungsverhältnisse der Hauptöffnung u. für verschiedene β.
m
l
Moment at left tower for various degrees of loading in main span and for various β.

Fig. 8.

Table IV.

n	0	0.1 l	0.2 l	0.3 l	0.4 l	0.5 l
β	0.2430	0.2386	0.2230	0.1930	0.1535	0.1082
$\frac{M_1 l}{EI}$	— 0.016	— 0.0188	— 0.0290	— 0.0500	— 0.0742	— 0.096
$10^{-3} \cdot M_1$	— 1140	— 1335	— 2060	— 3550	— 5270	— 6820
$10^{-3} \cdot H$	891	875	818	708	563	397
$\frac{M_2 l}{EI}$	— 0.016	+ 0.003	0.032	0.050	0.052	0.041
$10^{-3} \cdot M_2$	— 1140					

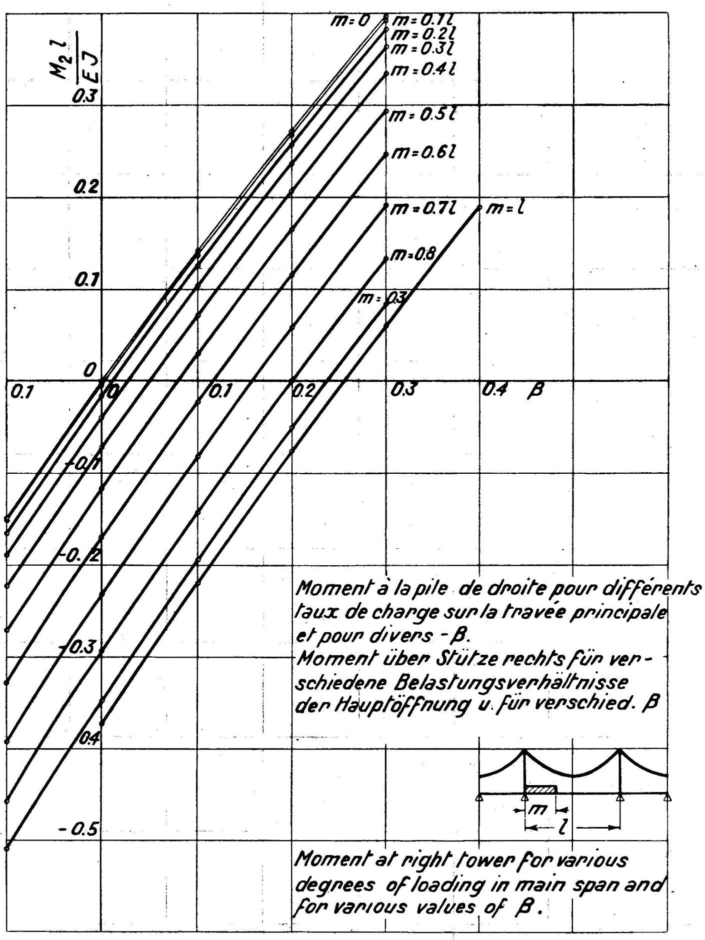

Fig. 9.

n	0.6 l	0.7 l	0.8 l	0.9 l	1.0 l
β	0.0625	0.0222	— 0.0085	— 0.0258	— 0.0305
$\frac{M_1 l}{EI}$	— 0.110	— 0.1085	— 0.0908	— 0.0628	— 0.0445
$10^{-3} \cdot M_1$	— 7810	— 7710	— 6450	— 4460	— 3160
$10^{-3} \cdot H$	229	81.4	— 31.1	— 94.6	— 111.9
$\frac{M_2 l}{EI}$	0.018	— 0.007	— 0.028	— 0.041	— 0.0445
$10^{-3} \cdot M_2$					— 3160

Curves for $\frac{M_1 l}{EI}$, $\frac{M_2 l}{EI}$ and β have been plotted in Fig. 10, using the loaded portion of the main span as abscissa. We see that the moment at the left tower is greatest when the load extends over 35 % of the main span.

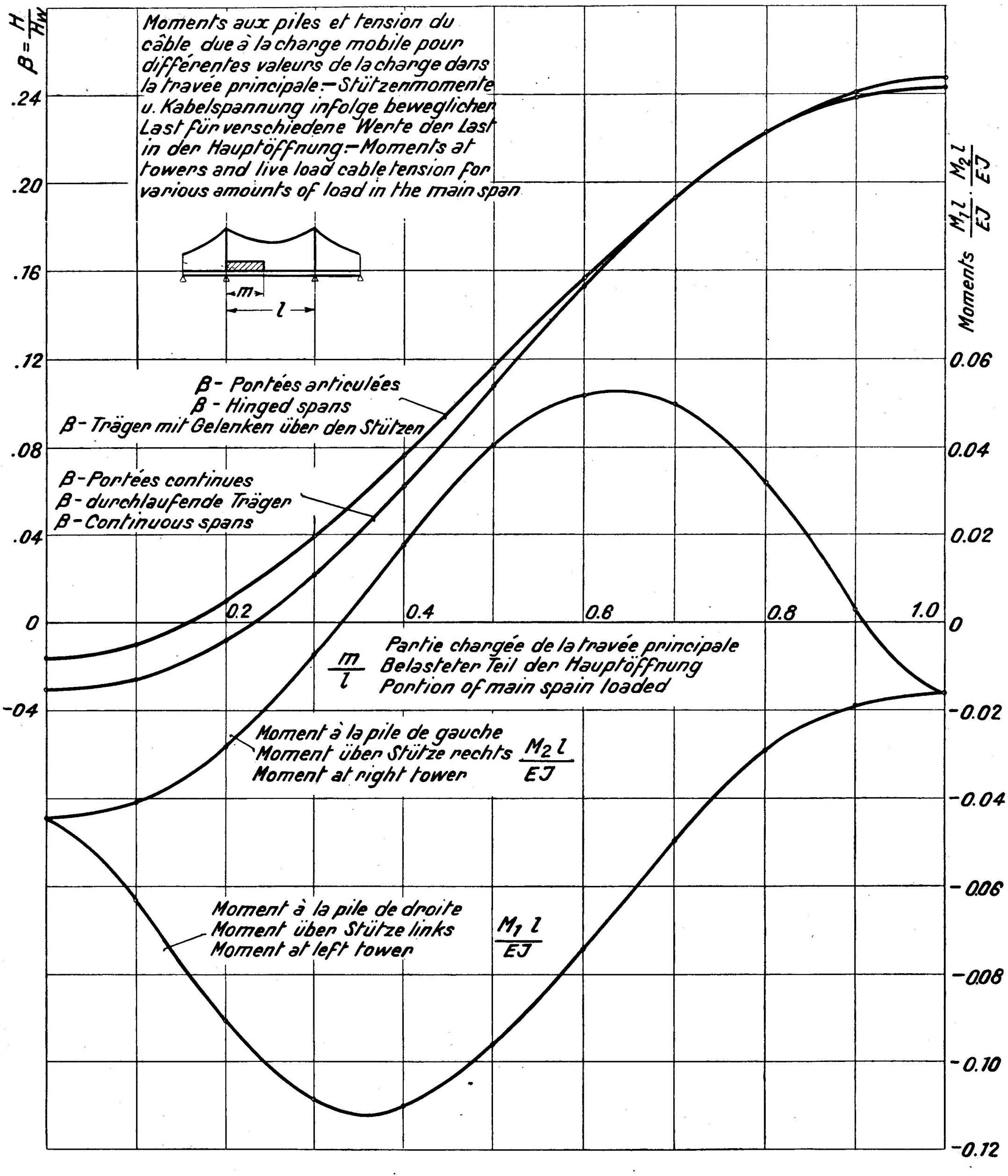

Fig. 10.

It is interesting to show the effect of continuity at the supports on the cable tension. In the case of hinges at the supports, the moments at the supports are zero; the terms having $\frac{M_1 l}{EI}$ and $\frac{M_2 l}{EI}$ as coefficients in equation (29) will drop out. From this special case of equation (29) we can calculate values of β for various assumed values of β with the following results:

Table V.

	$\beta = -0.1$	$\beta = 0$	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.3$
$n = l$	-0.01697	-0.01646	-0.01604		
$n = 0.9\,l$	-0.01008	-0.00962	-0.00924		
$n = 0.8\,l$	0.00964	0.00995	0.01021		
$n = 0.7\,l$	0.03980	0.03985	0.03987		
$n = 0.6\,l$	0.07731	0.07683	0.07664		
$n = 0.5\,l$	0.11836	0.11767	0.11703	0.11636	0.11567
$n = 0.4\,l$	0.15952		0.15742	0.15641	0.15538
$n = 0.3\,l$	0.19695		0.19419	0.19287	0.19154
$n = 0.2\,l$	0.22712		0.22387	0.22233	0.22079
$n = 0.1\,l$	0.24683		0.24329	0.24163	0.23995
$n = 0$	0.25371		0.25010	0.24839	0.24668

By making curves similar to those in Fig. 7, we find the values of β for various amounts of loading in the main span to be:

Table VI.

n	0	$0.1\,l$	$0.2\,l$	$0.3\,l$	$0.4\,l$	$0.5\,l$
	0.2475	0.2408	0.2219	0.1930	0.1568	0.1170
$10^{-3} \cdot H$	908	884	814	708	575	429
n	$0.6\,l$	$0.7\,l$	$0.8\,l$	$0.9\,l$	l	
	0.0767	0.0399	0.0100	-0.0097	-0.0165	
$10^{-3} \cdot H$	281	146	36.6	-35.4	-60.6	

The values of β at various loads for the case of a hinged span have been plotted in Fig. 10. At small loads we see H is negative. This is due to the fact that the temperature rise of 60°, which we have assumed, expands the cable, allowing part of the dead load to be taken by the stiffening truss.

Moments and cable tension due to live load are given in dimensionless form in fig. 10 in order that the results may be applied no matter what system of units is used.

Summary.

The method presented here for calculating the stresses and change of shape in suspension bridges with continuous stiffening girders is derived from the differential equation of the sag of the cable, the change in which caused by the live load must be equal to the deflection of the stiffening girder, assuming that the length of the suspension rods remains unchanged. This, in conjunction with the conditions for vertical equilibrium, gives the differential equation of the curve of deflection of the stiffening girder, which may be considered as differential equation of a continuous beam with at the same time axial tension. The solution of this is known; it allows the change in sag of the cable to be determined, after which the tension in the cable is given by a known basic equation. In applying the result to a numerical example, the influence of continuity on the tension of the cable is shown. The numerical tables that are provided should facilitate the practical adoption of the method of research here given.

Résumé.

La méthode proposée pour le calcul des contraintes et des déformations dans les ponts suspendus équipés avec poutres raidisseuses continues est

déduite de l'équation différentielle du câble suspendu, dont la déformation sous l'influence de la charge roulante, en supposant constante la longueur des tringles de suspension, doit être équivalente au propre fléchissement de l'élément raidisseur. On déduit de là, en faisant intervenir la condition d'équilibre vertical, l'équation différentielle de la courbe de flexion de l'élément raidisseur, qui peut être considérée comme représentant l'équation différentielle d'une poutre continue soumise simultanément à une traction axiale. La solution de ce problème est connue; elle permet la détermination des déformations du câble, dont on déduit, au moyen d'une équation fondamentale connue, la traction sur ce câble.

L'auteur met en évidence l'influence de la continuité sur la traction sur le câble, au moyen d'une application pratique. Les tableaux de calculs joints montrent les possibilités pratiques d'emploi de cette méthode d'investigation.

Zusammenfassung.

Die vorgelegte Methode zur Berechnung der Spannungen und Formänderungen in Hängebrücken mit durchlaufendem Versteifungsträger geht aus von der Differentialgleichung des Kabeldurchhangs, dessen Veränderung infolge Verkehrslast bei unveränderlich angenommener Länge der Hängestangen gleich der Durchbiegung des Versteifungsträgers sein muß. Daraus ergibt sich in Verbindung mit der vertikalen Gleichgewichtsbedingung die Differentialgleichung der Biegungslinie des Versteifungsträgers, die als Differentialgleichung eines durchlaufenden Balkens mit gleichzeitigem Axialzug aufgefaßt werden kann. Die Lösung dieser Aufgabe ist bekannt; sie erlaubt die Bestimmung der Durchhangsveränderungen des Kabels, worauf sich der Kabelzug aus einer bekannten Grundgleichung ergibt. Bei der Anwendung auf ein Zahlenbeispiel wird der Einfluß der Kontinuität auf den Kabelzug gezeigt. Die mitgeteilten Zahlentafeln sollen die praktische Anwendung der gegebenen Untersuchungsmethode erleichtern.