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Autor(en): **Davidson, J.F.**

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The Vibration of a Slightly Curved Bar Carrying an End Mass

Oscillations d'une barre légèrement courbée et chargée par une masse à son extrémité

Die Schwingungen eines leicht gebogenen, an seinem Ende mit einer Masse behafteten Stabes

J. F. DAVIDSON, Cambridge, England

Notation

A	cross-sectional area of bar
a, b, δ, ϵ	arbitrary constants
B	flexural rigidity of bar
c	initial lateral displacement at middle of bar after application of load P_0
d	initial lateral displacement without load
E	Young's Modulus
f	lateral displacement of middle of bar due to vibration
h	distance from centroid of section to outer fibre
j	cantilever stiffness $= 3\,B/l^3$
K	stiffness of spring attached to mass M
k	radius of gyration of bar
l	length of bar
M	mass attached to end of bar
m	mass of bar
N	$2\,l^2 m/\pi^4 c^2$ mass in equivalent system
P	axial load in bar due to vibration
P_0	fixed axial load in bar
P_E	Euler load $\pi^2 B/l^2$
q	maximum stress in bar due to initial bending
R_α, R_β	amplitude ratios $(w/f)/(w/f)_{static}$ corresponding to frequencies α and β
t	time

u	total lateral displacement of bar at any point
u_0	initial lateral displacement of bar at any point
w	displacement of mass M due to vibration
x	distance along bar measured from fixed end
y	longitudinal contraction of bar due to curvature
α	anti phase frequency
β	in phase frequency
λ	$2ml^2\Omega^2/\pi^4c^2$ spring stiffness in analogous system
μ	$1/(AE/Kl+1)$
ρ	density of bar material
Ω	$\omega_L(1-P_0/P_E)^{1/2}$
ω_L	first circular frequency of bar $\pi^2(B/\rho A)^{1/2}/l^2$
ω_s	circular frequency of spring mass combination $= (\mu AE/Ml)^{1/2}$

1. Introduction

The effect of axial constraint on the lateral vibration of thin bars has been examined by various writers. Cowley and Levy (1918), Bateman (1929), Puwein (1939), Warren (1939), Nowacki (1949) showed that a constant axial force reduces the lateral frequency, which becomes zero at the Euler load. Woinowsky-Krieger (1950), Burgreen (1951) and Eringen (1952) considered an initially straight bar whose pinned ends were rigidly fixed in position. The longitudinal contractions normally associated with lateral vibration were thereby restricted, and the frequency made to depend upon amplitude in the same way as for a simple pendulum. Thus for infinitesimal amplitudes, the motion was simple harmonic, and the frequency unaltered by the axial restriction.

When one pinned end is attached to a mass constrained to move towards the fixed pin, the bar being initially straight, an effect similar to that of Woinowsky-Krieger is obtained, the longitudinal restriction being provided by the inertia of the mass. Vibrations of the mass, so that the bar acts merely as a spring without lateral movement, will also be possible, so that the system has two fundamental frequencies. When the bar is initially curved, the mass being constrained in the same way, infinitesimal lateral movements of the bar are accompanied by movements of the mass. In this way the lateral frequency of the curved bar is altered when its amplitude is very small, and the two fundamental frequencies of the system become coupled. The relation between these fundamental frequencies and the frequencies of the coupled system is obtained in the present paper. Two simple systems, made up of springs and masses, and having the same vibration characteristics as the coupled system, have been evolved to assist in visualising its behaviour.

2. Theory of the Curved Bar Carrying an End Mass

The system under consideration is shown in figure 1. R and Q are the pinned ends of the bar, R being fixed and Q free to move along the fixed axis Rx. Due to fixed applied loads, the bar, whose length is l, has initial lateral displacement, at distance x from R, of $u_0 = c \sin \pi x/l$. In the first place we notice that the ratio

$$\frac{\text{Time for stress wave to travel from } R \text{ to } Q}{\text{First natural period of bar}} = \frac{l(\rho/E)}{(\rho/E)\,2\,l^2/\pi k} = \frac{\pi k}{2l}$$

is small if the bar is thin, E being Young's Modulus, and ρ the density of the bar, whose radius of gyration is k. Thus if P is the additional axial load in the bar due to vibration, its variation with x may be neglected. Then if the total lateral displacement at any time is

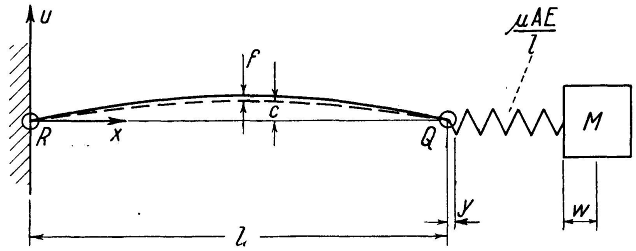

Fig. 1. Pin ended bar carrying end mass M via spring. Vibration about curved position

$$u = (f + c) \sin \frac{\pi x}{l}, \tag{2.1}$$

f being a function of time, the longitudinal contraction due to vibration will be

$$\frac{Pl}{AE} + \frac{1}{2}\int_0^l \left[\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u_0}{\partial x}\right)^2\right] dx = \frac{Pl}{AE} + \frac{\pi^2}{4l}(f^2 + 2cf). \tag{2.2}$$

If the mass M, which is constrained to move along the axis Rx, is attached to the pin Q by a spring of stiffness K, its total movement due to vibration is

$$w = P\left(\frac{1}{K} + \frac{l}{AE}\right) + \frac{\pi^2}{4l}(f^2 + 2cf). \tag{2.3}$$

A being the cross sectional area of the bar. The system may thus be visualised as a bar RQ, whose contraction is $y = \pi^2 (f^2 + 2cf)/4l$, carrying a spring of net stiffness $\mu A E/l$ where

$$\mu = \frac{1}{AE/Kl + 1}. \tag{2.4}$$

Also,

$$P = -Mw'' \tag{2.5}$$

dashes denoting differentiation with respect to time. Then eliminating w from (2.3) and (2.5), and neglecting f/c so that only small oscillations about the equilibrium position are considered, we get

$$P'' + \omega_s^2 P + \frac{\pi^2 \omega_s^2 M c f''}{2l} = 0, \tag{2.6}$$

where

$$\omega_s^2 = \mu A E/Ml. \tag{2.7}$$

The equation of lateral motion of the bar is

$$B\frac{\partial^4(u-u_0)}{\partial x^4}+(P+P_0)\frac{\partial^2 u}{\partial x^2}-P_0\frac{\partial^2 u_0}{\partial x^2}+\rho A\frac{\partial^2 u}{\partial t^2}=0, \tag{2.8}$$

t denoting time, B the flexural rigidity of the bar, and P_0 the fixed part of the axial load, due for example to gravity acting on the end mass. (2.8) may be simplified by the use of (2.1) so that

$$f''+\omega_L^2\left(1-\frac{P_0}{P_E}\right)f-\frac{P\pi^2(f+c)}{ml}=0, \tag{2.9}$$

where $\omega_L=\pi^2(B/\rho A)^{1/2}/l^2$ is the first circular frequency of the bar, $P_E=\pi^2 B/l^2$ is its Euler load, and m is its mass. Then neglecting f/c in (2.9) we get

$$f''+\Omega^2 f-\frac{\pi^2 c}{ml}P=0, \tag{2.10}$$

where

$$\Omega^2=\omega_L^2\left(1-\frac{P_0}{P_E}\right). \tag{2.11}$$

Eliminating P from between (2.6) and (2.10) gives

$$f^{\mathrm{IV}}+f''\left(\omega_s^2+\Omega^2+\omega_s^2\frac{\pi^4 c^2 M}{2l^2 m}\right)+f\omega_s^2\Omega^2=0, \tag{2.12}$$

of which the solution is

$$f=a\sin(\alpha t+\epsilon)+b\sin(\beta t+\delta),$$

where a, b, ϵ, δ are arbitrary constants depending upon the initial conditions and

$$\left.\begin{aligned}2\alpha&=\left[(\omega_s+\Omega)^2+\omega_s^2\frac{\pi^4c^2M}{2l^2m}\right]^{1/2}+\left[(\omega_s-\Omega)^2+\omega_s^2\frac{\pi^4c^2M}{2l^2m}\right]^{1/2},\\ \alpha\beta&=\omega_s\Omega.\end{aligned}\right\} \tag{2.13}$$

Then α and β are the circular frequencies of the coupled system, and $\alpha>\beta$. Thus α represents an "anti-phase" mode of vibration in which the strut bends towards the line RQ figure 1 simultaneously with the mass M approaching Q. Similarly β represents the "in-phase" mode. The ratio w/f which is (mass deflexion)/(strut deflexion) can be obtained from (2.3) and (2.10). Its ratio to the value of w/f caused by a small static load on the mass M is, for the α mode,

$$R_\alpha=(w/f)/(w/f)_{static}=\frac{\beta^2-\omega_s^2}{\alpha^2+\beta^2-\omega_s^2}, \tag{2.14}$$

and for the β mode,

$$R_\beta=(w/f)/(w/f)_{static}=\frac{\alpha^2-\omega_s^2}{\alpha^2+\beta^2-\omega_s^2}. \tag{2.15}$$

Two analogous systems having the same properties as the strut-mass combination are shown in figure 2. The equations of small oscillations about equilibrium are in each case (2.6) and (2.10), so that the analogies hold good

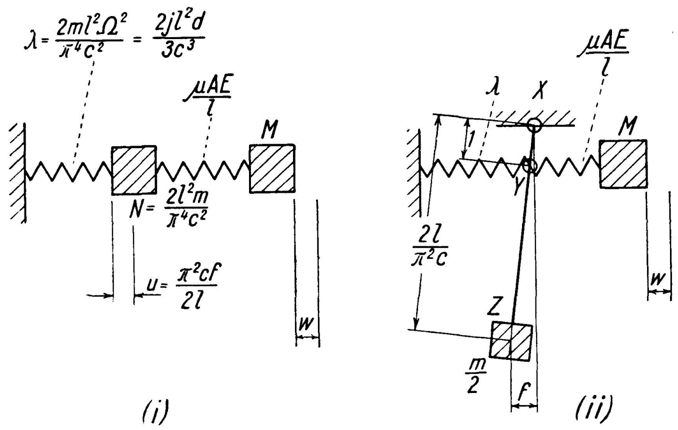

Fig. 2. Systems with the same characteristics as the strut-spring-mass combination of figure 1

only when f/c is small. In (I) figure 2, the strut is replaced by a mass $N = 2\,l^2 m/\pi^4 c^2$, carried by a spring of stiffness

$$\lambda = 2\,m\,l^2 \Omega^2/\pi^4 c^2 = 2\,j\,l^2 d/3\,c^3. \tag{2.16}$$

$j = 3\,B/l^3$ is the stiffness of the strut when used as a cantilever, and d is its initial central deflexion without axial load. Thus d is the "initial imperfection", and when the constant axial load P_0 is applied, the central deflexion becomes c, and if the strut is always in the form of a half sine wave,

$$\frac{c}{d} = \frac{1}{1 - P_0/P_E} = \frac{\omega_L{}^2}{\Omega^2} \tag{2.17}$$

from (2.11). Thus if the initial imperfection d is small, and P_0 is nearly P_E, $l^2 d/c^3$ will be of order 1, and from (2.16) λ is of order j, and therefore represents a comparatively soft spring.

In analogy (II) figure 2, X is a fixed hinge, and $X\,Y\,Z$ a rigid link whose end Z has the same movement as the centre of the strut and carries half its mass. This illustrates the large mechanical advantage, $2\,l/\pi^2 c$, which the strut mass has over the mass M.

Numerical Results in a Typical Case

The maximum stress q in the bar due to the initial bending is given by

$$q = E\,h\,c\,\frac{\pi^2}{l^2}, \tag{2.18}$$

h being the distance between the centroid and the outer fibre. For a high strength aluminium alloy q may be 10 tons/sq. in. and $E = 4550$ tons/sq. in., so that if $l/k = 150$ and $k/h = 0.578$, as for a rectangular bar, we get, from (2.18), $c/l = 0.0193$. A reasonable value of M may be obtained by taking $M/P_E = 0.25$ so that

$$\frac{M}{m} = 0.25\,\frac{\pi^2 E\,k^2}{\rho\,l^3},$$

and putting $l = 100$ in. and $\rho = 0.096$ lbs/cu.in., we get $M/m = 117$, so that $\pi^4 c^2 M/2l^2 m = 2.12$. Using this value in (2.13) we obtain α/Ω, α/ω_s, β/Ω and β/ω_s in terms of Ω/ω_s and these values are shown plotted in figure 3, together with the amplitude ratios R_α and R_β from (2.14) and (2.15). Thus when $\Omega/\omega_s \to \infty$ the coupling is small and the two component frequencies are obtained. But for values of Ω/ω_s from 0 to 2.0 the coupled and component frequencies are widely different, so that the only condition for no coupling is $\Omega/\omega_s \to \infty$.

The conditions necessary to obtain each value of Ω/ω_s are determined by combining (2.7) and (2.11) to give

$$\frac{\Omega}{\omega_s} = \frac{\pi^2 k}{l}\left[\frac{M}{\mu m}\left(1 - \frac{P_0}{P_E}\right)\right]^{1/2}.$$

In the present case we use the given values of M/m, l/k, and obtain

$$\frac{\Omega}{\omega_s} = 0.711\left[\frac{1}{\mu}\left(1 - \frac{P_0}{P_E}\right)\right]^{1/2}, \qquad (2.19)$$

so that any value of Ω/ω_s may be obtained by using appropriate values of μ and P_0/P_E. From (2.4) we note that $\mu \leqq 1$ and if $P_0 = 0$, the minimum value of Ω/ω_s from (2.19) is 0.711. At this minimum point, the bar carries the mass without an intermediate spring, and figure 3 shows that the coupled and component frequencies differ by a factor of about 1.8.

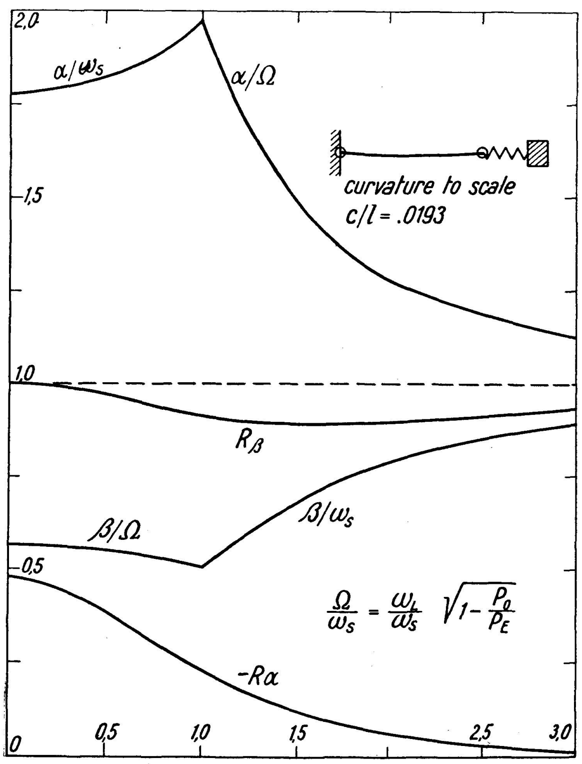

Fig. 3. Frequency ratios. α obtained with mass and bar anti-phase, β obtained with mass and bar in-phase. R_α and $R_\beta = (w/f)/(w/f)_{static}$

The character of the motion can be further understood from the amplitude ratios. Thus R_β remains almost 1, so that the displacements of the "in-phase" mode are similar to those with a static load. When $\Omega \to \infty$, $R_\alpha \to 0$ since the anti-phase motion is made up of a high frequency strut vibration, with the mass remaining almost fixed.

3. Conclusions

An analysis of the foregoing type might be applied to a built up structure such as a triangulated truss. Thus the vibration of such a structure consists of movements of the dead weight carried, together with compression and tension of the component members, and is analogous to the vibration of the spring and mass in the example given. If at the same time the component members are curved, due for example to end bending moments, their vibrations will become coupled with those of the whole structure. The coupling will be small only if the lateral frequency of the component member is much greater than that of the whole structure.

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Summary

A pin ended bar, slightly bent in the form of a half sine wave, is considered. One end is fixed and the other is free to move along the line joining the pins. A mass, constrained to move along the same line, is attached to the moving end by a spring. The oscillations of the bar are small compared with its curvature.

In these circumstances the system has two degrees of freedom, and the resulting pair of frequencies is obtained in terms of the natural frequency of the straight bar and of the spring mass combination. In a practical case, the frequences of the coupled system are shown to differ from the natural frequencies by a factor of 1.8.

Résumé

Nous avons étudié le cas d'une barre montée à articulation et incurvée en forme de demi-sinusoïde. L'une des extrémités est fixe et l'autre peut se déplacer le long de la corde. A cette extrémité mobile, est fixée, à l'aide d'un ressort, une masse qui doit se déplacer suivant la même ligne. Les déviations de la barre sont faibles par comparaison avec sa flèche.

Dans ces conditions, le système comporte deux degrés de liberté et les deux fréquences qui se manifestent peuvent être exprimées en fonction des fréquences propres de la barre droite et de la masse fixée élastiquement. Dans l'un des cas examinés, les fréquences du système couplé s'écartent des fréquences propres dans le rapport de 1,8.

Zusammenfassung

Wir betrachten einen gelenkig gelagerten, nach einer halben Sinuswelle gekrümmten Stab. Das eine Ende ist fest, das andere kann sich längs der Stabsehne bewegen. An dieses bewegliche Ende ist mittelst einer Feder eine Masse befestigt, die sich in derselben Wirkungslinie bewegen soll. Die Ausschläge des Stabes sind klein verglichen mit seiner Ausbiegung.

Unter diesen Umständen besitzt das System zwei Freiheitsgrade, und die beiden auftretenden Frequenzen lassen sich ausdrücken durch die Eigenfrequenzen des geraden Stabes und der abgefederten Masse. An einem der untersuchten Fälle weichen die Frequenzen des gekoppelten Systems von den Eigenfrequenzen um das 1,8-fache ab.