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The Influence of Initial Stress on the Dynamic Behaviour of Elastic and Viscoelastic Plates

Influence de la tension initiale dans le comportement dynamique des plaques élastiques et visco-élastiques

Der Einfluß der Vorspannung auf das dynamische Verhalten elastischer und viskoelastischer Platten

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Introduction

The influence of initial stress on both the static and dynamic response of elastic bodies has been studied extensively by a variety of investigators. For example, the effect of initial axial stress on elastic torsion was dealt with by M. A. BIOT [1] and J. N. GOODIER [2] and the influence of initial stress on elastic waves was considered by M. A. BIOT [3]. In these studies, the pertinent equations of a three-dimensional solid under initial stress were derived by linearization of the nonlinear equations of elasticity.

What the influence of initial stress in bodies is concerned, one or two of whose dimensions are small as compared to the third, such as beams and plates, the governing equations were derived by considering directly a beam or plate element and formulating the equations of equilibrium or motion. The influence of initial stress on equilibrium of plates, uniformly distributed across the thickness, was established in this fashion by SAINT-VENANT [4, 5], long before the general three-dimensional theory of elasticity of a body under initial stress was formulated by M. A. BIOT [6].

The purpose of the present investigation is twofold. The first aim is to establish a connection between the equations of motion of a plate under initial stress and the general three-dimensional theory. The procedure adopted in achieving this aim is analogous to the one used recently by the author [7] in deriving a nonlinear plate theory, starting with assumption as to displacements and using a variational process. In the course of this process, certain simplifying assumptions have to be introduced in order to obtain the Saint-Venant equations, permitting thus to gain a better insight into that theory.

The second part of the study contains several applications of the derived equations of motion. The influence of initial stress on the natural frequency of free transverse vibrations of a rectangular plate is studied for different loadings, boundary conditions and aspect ratios. In addition, a case of simple visco-elastic behaviour is considered and related to an associated buckling problem.

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Symbols

x, y, z	initial coordinates of a particle
ξ, η, ζ	final coordinates of a particle
u, v, w	displacement components in the x, y, z direction, respectively
S_{ij}	components of initial stress
$\omega_x, \omega_y, \omega_z$	components of rotation, defined by eq. (2)
$t_{\mu\nu}$	components of increment of stress
$\tau_{\mu\nu}$	components of total stress
W_i	internal strain energy
$e_{\mu\nu}$	components of strain, defined by eq. (6)
$\bar{u}, \bar{v}, \bar{w}$	approximate plate displacements
h	plate thickness
u_0, v_0, w_0	plate displacements, defined by eq. (8)
$\bar{W}_i$	internal strain energy in plate
M_1, M_2, M_{12} $N_1^i, N_2^i, N_{12}^i, N_3^i$ M_1^i, M_2^i, M_{12}^i	plate stresses defined by eq. (13)
n, s	coordinates perpendicular to and along the plate boundary, respectively
f_x, f_y, f_z	components of boundary force per unit original area
W_e	work done by boundary forces
F_x, F_y, q m_x, m_y N_n^*, N_{ns}^* M_n^*, M_{ns}, Q_n	external plate forces and moments defined by eq. (20)
W_B	work done by body forces
X, Y, Z	components of body force
$\bar{X}, \bar{Y}$	components of body force defined by eq. (28)
T	kinetic energy
ρ	mass density
U	potential energy

L	kinetic potential
D	plate modulus
E	Young's modulus
ν	Poisson's ratio
η	viscosity coefficient
P_x	initial thrust
β^*, β_0	fraction of critical damping defined by eqs. (51), (53), respectively
ω	circular frequency
ω_0	circular frequency in the absence of initial thrust
$(P_x)_{cr}$	buckling load
a, b	plate length and width, respectively
$T(t)$	time function
m, n	number of half-waves in x, y direction, respectively
μ	fraction of critical loading defined by eq. (58)
P_e	Euler buckling load
k	buckling value
γ	specific weight
A	amplitude
ξ	body force parameter defined by eq. (66)

Three-Dimensional Theory of a Body under Initial Stress

A three-dimensional linear theory of elasticity for small deformations in a material under initial stress was established by M. A. Biot [6]. A particle of the material is identified by its initial coordinates x, y, z, the initial (equilibrium) state of the material being associated with initial components of stress

$$\begin{matrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{matrix} \tag{1}$$

If the material undergoes additional small deformations, so that the initia, coordinates x, y, z of a particle of the material become $\xi = x+u$, $\eta = y+v$l $\zeta = z+w$, the stresses will be changed. It is convenient to refer the components of stress, in the body deformed additionally, to axes 1, 2, 3 which are given by the local rotation of the material through an amount defined by

$$\begin{aligned} \omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \\ \omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \\ \omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \end{aligned} \tag{2}$$

Furthermore, the new components of stress will be taken per unit initial area before deformation and as functions of the coordinates x, y, z. Designating by $t_{\mu\nu}$ the components of increments of stress due to additional deformation, the total stress $\tau_{\mu\nu}$ for the same state is

$$\begin{array}{lll} \tau_{11} = S_{xx} + t_{11} & \tau_{12} = S_{xy} + t_{12} & \tau_{13} = S_{xz} + t_{13} \\ \tau_{21} = S_{yx} + t_{21} & \tau_{22} = S_{yy} + t_{22} & \tau_{23} = S_{yz} + t_{23} \\ \tau_{31} = S_{zx} + t_{31} & \tau_{32} = S_{zy} + t_{32} & \tau_{33} = S_{zz} + t_{33} \end{array} \tag{3}$$

It is shown in [6], that the internal strain energy in a body under initial stress is given by the expression

$$W_i = \tfrac{1}{2} \sum t_{\mu\nu} e_{\mu\nu} + \sum S_{\mu\nu} \epsilon_{\mu\nu} \tag{4}$$

and its variation by

$$\delta W_i = \sum t_{\mu\nu} \delta e_{\mu\nu} + \sum S_{\mu\nu} \delta \epsilon_{\mu\nu}. \tag{5}$$

The notation S_{11}, S_{12}, etc. is used for S_{xx}, S_{xy}, etc. $e_{\mu\nu}$ are the components of strain of the linear theory of elasticity,

$$\begin{array}{ll} e_{xx} = \dfrac{\partial u}{\partial x}, & e_{yz} = \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right), \\ e_{yy} = \dfrac{\partial v}{\partial y}, & e_{zx} = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right), \\ e_{zz} = \dfrac{\partial w}{\partial z}, & e_{xy} = \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial y}\right). \end{array} \tag{6}$$

$\epsilon_{\mu\nu}$ are the components of strain of a second order theory, taken in the form [7].

$$\begin{aligned} \epsilon_{11} &= e_{xx} + \tfrac{1}{2}(\omega_z^2 + \omega_y^2) \\ \epsilon_{22} &= e_{yy} + \tfrac{1}{2}(\omega_x^2 + \omega_z^2) \\ \epsilon_{33} &= e_{zz} + \tfrac{1}{2}(\omega_y^2 + \omega_x^2) \\ \epsilon_{23} &= e_{yz} - \tfrac{1}{2}\omega_y \omega_z \\ \epsilon_{31} &= e_{zx} - \tfrac{1}{2}\omega_z \omega_x \\ \epsilon_{12} &= e_{xy} - \tfrac{1}{2}\omega_x \omega_y \end{aligned} \tag{7}$$

Derivation of Equations of Motion of an Initially Stressed Plate

Deformations

A plate of constant thickness h is referred to an x, y, z Cartesian coordinate system, the x, y-plane being the middle plane of the plate, the z-axis being hence normal to that plane. The approximate displacements $\bar{u}$, $\bar{v}$, $\bar{w}$ are taken as those characterizing the plate theory with moderately large amplitudes [7]

$$\begin{aligned}
\bar{u} &= u_0(x,y) - z\frac{\partial w_0}{\partial x},\\
\bar{v} &= v_0(x,y) - z\frac{\partial w_0}{\partial y},\\
\bar{w} &= w_0(x,y).
\end{aligned} \tag{8}$$

The displacements of the classical plate theory are obtained by letting

$$u_0 = v_0 = 0.$$

The components of strain of the classical plate theory are obtained, by substituting the forms (8), with $u_0 = v_0 = 0$, into expressions (6)

$$\begin{aligned}
\bar{e}_{xx} &= -z\frac{\partial^2 w_0}{\partial x^2},\\
\bar{e}_{yy} &= -z\frac{\partial^2 w_0}{\partial y^2},\\
\bar{e}_{zz} &= 0,\\
\bar{e}_{yz} &= 0,\\
\bar{e}_{zx} &= 0,\\
\bar{e}_{xy} &= -z\frac{\partial^2 w_0}{\partial x\,\partial y}.
\end{aligned} \tag{9}$$

The components of strain of a large deflection plate theory are obtained by substituting the forms (8) into expressions (7)

$$\begin{aligned}
\bar{\epsilon}_{11} &= \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2,\\
\bar{\epsilon}_{22} &= \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2,\\
\bar{\epsilon}_{33} &= \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2,\\
\bar{\epsilon}_{23} &= 0,\\
\bar{\epsilon}_{31} &= 0,\\
\bar{\epsilon}_{12} &= \frac{1}{2}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} - 2z\frac{\partial^2 w_0}{\partial x\,\partial y} + \frac{\partial w_0}{\partial y}\frac{\partial w_0}{\partial x}\right).
\end{aligned} \tag{10}$$

Plate deformations may produce only large rotations ω_x, ω_y but not ω_z. Thus, in calculating the components of strain from eq. (7), second order terms with ω_z were neglected.

Strain Energy

Inserting the approximate expressions for the strains (9) and (10) into the expression for the strain energy (4), an approximate expression for the strain energy $\overline{W}_i$ is obtained

$$\overline{W}_i = \iiint\limits_{V_0} \left[\frac{1}{2} t_{11} z \frac{\partial^2 w_0}{\partial x^2} - \frac{1}{2} t_{22} z \frac{\partial^2 w_0}{\partial y^2} - t_{12} z \frac{\partial^2 w_0}{\partial x \, \partial y}\right] d x \, d y \, d z$$
$$+ \iiint\limits_{V_0} \left[S_{11} \left(\frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right)\right.$$
$$+ S_{22} \left(\frac{\partial v_0}{\partial y} - z \frac{\partial^2 w_0}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right) \tag{11}$$
$$+ S_{33} \left(\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right)$$
$$\left. + S_{12} \left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} - 2 z \frac{\partial^2 w_0}{\partial x \, \partial y} + \frac{\partial w_0}{\partial y} \frac{\partial w_0}{\partial x}\right)\right] d x \, d y \, d z .$$

Since the z-dependence of all the components of strain has been made explicit through the assumptions (8), an integration through the thickness h of the plate from $z = -\frac{h}{2}$ to $z = \frac{h}{2}$ can be performed immediately, with the result

$$\overline{W}_i = \iint\limits_{S_0} \left[-\frac{1}{2} M_1 \frac{\partial^2 w_0}{\partial x^2} - \frac{1}{2} M_2 \frac{\partial^2 w_0}{\partial y^2} - M_{12} \frac{\partial^2 w_0}{\partial x \, \partial y}\right] d x \, d y$$
$$+ \iint\limits_{S_0} \left[N_1^i \left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right) - M_1^i \frac{\partial^2 w_0}{\partial x^2}\right.$$
$$+ N_2^i \left(\frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right) - M_2^i \frac{\partial^2 w_0}{\partial y^2} \tag{12}$$
$$+ N_3^i \left(\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right)$$
$$\left. + N_{12}^i \left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} + \frac{\partial w_0}{\partial y} \frac{\partial w_0}{\partial x}\right) - 2 M_{12}^i \frac{\partial^2 w_0}{\partial x \, \partial y}\right] d x \, d y$$

the integration being extended over the initial surface S_0 of the plate. In this process the following plate stresses are defined automatically

$$M_1 = \int_{-h/2}^{h/2} t_{11} z \, d z; \quad M_2 = \int_{-h/2}^{h/2} t_{22} z \, d z; \quad M_{12} = \int_{-h/2}^{h/2} t_{12} z \, d z;$$
$$N_1^i = \int_{-h/2}^{h/2} S_{11} \, d z; \quad N_2^i = \int_{-h/2}^{h/2} S_{22} \, d z; \quad N_{12}^i = \int_{-h/2}^{h/2} S_{12} \, d z; \quad N_3^i = \int_{-h/2}^{h/2} S_{33} \, d z; \tag{13}$$
$$M_1^i = \int_{-h/2}^{h/2} S_{11} z \, d z; \quad M_2^i = \int_{-h/2}^{h/2} S_{22} z \, d z; \quad M_{12}^i = \int_{-h/2}^{h/2} S_{12} z \, d z .$$

The variation of strain energy is then given by the expression

$$\delta \overline{W}_i = \iint\limits_{S_0} \left[-M_1 \delta \frac{\partial^2 w_0}{\partial x^2} - M_2 \delta \frac{\partial^2 w_0}{\partial y^2} - 2\, M_{12}\, \delta \frac{\partial^2 w_0}{\partial x\, \partial y} \right] dx\, dy$$
$$+ \iint\limits_{S_0} \left[N_1{}^i \delta \frac{\partial u_0}{\partial x} + N_1{}^i \frac{\partial w_0}{\partial x} \delta \frac{\partial w_0}{\partial x} - M_1{}^i \delta \frac{\partial^2 w_0}{\partial x^2} \right.$$
$$+ N_2{}^i \delta \frac{\partial v_0}{\partial y} + N_2{}^i \frac{\partial w_0}{\partial y} \delta \frac{\partial w_0}{\partial y} - M_2{}^i \delta \frac{\partial^2 w_0}{\partial y^2}$$
$$+ N_3{}^i \frac{\partial w_0}{\partial x} \delta \frac{\partial w_0}{\partial x} + N_3{}^i \frac{\partial w_0}{\partial y} \delta \frac{\partial w_0}{\partial y} \qquad (14)$$
$$+ N_{12}^i \left(\delta \frac{\partial v_0}{\partial x} + \delta \frac{\partial u_0}{\partial y} + \frac{\partial w_0}{\partial y} \delta \frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial x} \delta \frac{\partial w_0}{\partial y} \right)$$
$$\left. - 2\, M_{12}^i\, \delta \frac{\partial^2 w_0}{\partial x\, \partial y} \right] dx\, dy.$$

Since $\delta \frac{\partial u_0}{\partial x} = \frac{\partial}{\partial x} \delta u_0$, $\delta \frac{\partial^2 w_0}{\partial x^2} = \frac{\partial^2}{\partial x} \delta w_0$, etc., the factors δu_0, δv_0, δw_0 may be brought out in the above expression by partial integration, obtaining

$$\delta \overline{W}_i = \iint\limits_{S_0} \left[-\frac{\partial^2 M_1}{\partial x^2} \delta w_0 - \frac{\partial^2 M_2}{\partial y^2} \delta w_0 - 2 \frac{\partial^2 M_{12}}{\partial x\, \partial y} \delta w_0 \right] dx\, dy$$
$$+ \iint\limits_{S_0} \left[-\frac{\partial N_1{}^i}{\partial x} \delta u_0 - \frac{\partial}{\partial x} \left(N_1{}^i \frac{\partial w_0}{\partial x} \right) \delta w_0 - \frac{\partial^2 M_1{}^i}{\partial x^2} \delta w_0 \right.$$
$$- \frac{\partial N_2{}^i}{\partial y} \delta v_0 - \frac{\partial}{\partial y} \left(N_2{}^i \frac{\partial w_0}{\partial y} \right) \delta w_0 - \frac{\partial^2 M_2{}^i}{\partial y^2} \delta w_0$$
$$- \frac{\partial}{\partial x} \left(N_3{}^i \frac{\partial w_0}{\partial x} \right) \delta w_0 - \frac{\partial}{\partial x} \left(N_3{}^i \frac{\partial w_0}{\partial y} \right) \delta w_0 \qquad (15)$$
$$- \frac{\partial N_{12}^i}{\partial x} \delta v_0 - \frac{\partial N_{12}^i}{\partial y} \delta u_0 - \frac{\partial}{\partial x} \left(N_{12}^i \frac{\partial w_0}{\partial y} \right) \delta w_0 - \frac{\partial}{\partial y} \left(N_{12}^i \frac{\partial w_0}{\partial x} \right) \delta w_0$$
$$\left. - 2 \frac{\partial^2 M_{12}^i}{\partial x\, \partial y} \delta w_0 \right] dx\, dy$$
$$+ \oint \left[N_n{}^i \delta u_n + N_{ns}^i \delta u_s + N_n{}^i \frac{\partial w_0}{\partial n} \delta w_0 + N_{ns}^i \frac{\partial w_0}{\partial s} \delta w_0 \right.$$
$$\left. - M_n{}^i \frac{\partial \delta w_0}{\partial n} + 2 \left(\frac{\partial M_{ns}^i}{\partial s} + \frac{\partial M_n{}^i}{\partial s} \right) \delta w_0 \right] ds.$$

Thereby use has been made of

$$- \oint M_{ns}^i \frac{\partial}{\partial s} \delta w_0\, ds = \oint \frac{\partial M_{ns}^i}{\partial s} \delta w_0\, ds. \qquad (16)$$

In the sequel, the stress N_3 and initial moments $M_1{}^i$, $M_2{}^i$ and M_{12}^i will be neglected.

Work of External Forces

Let f_x, f_y, f_z denote x, y, z components, respectively, of the external force (traction), per unit original area, acting at the boundary. The (virtual) work done by these external forces will be, in all generality,

$$\delta W_e = \iint_{S_0} (f_x \delta u + f_y \delta v + f_z \delta w)\, d s \tag{17}$$

the integral being extended over the whole surface of the body.

In the case of a plate, the total surface consists of the two plane faces $z = \pm\frac{h}{2}$ and the cylindrical surfaces bounding the plate. The expression for the work is therefore

$$\begin{aligned} \delta W_e = & \left[\iint_{S_0} (f_x \delta u + f_y \delta v + f_z \delta w)\, d x\, d y\right]_{z=-h/2}^{z=h/2} \\ & + \oint\int_{-h/2}^{h/2} (f_n \delta u_n + f_s \delta u_s + f_z \delta w)\, d z\, d s . \end{aligned} \tag{18}$$

The line integral is to be taken around all (external and internal) cylindrical boundaries of the plate. Subscripts n and s indicate components referred to co-ordinates n and s measured normal to and along the boundary.

Substituting for the displacements u, v, w their approximations $\bar{u}, \bar{v}, \bar{w}$ from (8), the work of boundary forces acting on the plate is given by

$$\begin{aligned} \delta \overline{W}_e = & \left[\iint \left[f_x\left(\delta u_0 - z\,\delta\frac{\partial w_0}{\partial x}\right) + f_y\left(\delta v_0 - z\,\delta\frac{\partial w_0}{\partial y}\right) + f_z\, \delta w_0\right] d x\, d y\right]_{z=-h/2}^{z=h/2} \\ & + \oint\int_{-h/2}^{h/2} \left[f_n\left(\delta u_{0n} - z\,\delta\frac{\partial w_0}{\partial n}\right) + f_s\left(\delta u_{0s} - z\,\delta\frac{\partial w_0}{\partial s}\right) + f_z\, \delta w_0\right] d z\, d s . \end{aligned} \tag{19}$$

Employing the notation

$$\begin{aligned} & F_x = f_x\Big|_{-h/2}^{h/2}; \qquad F_y = f_y\Big|_{-h/2}^{h/2}; \qquad q = f_z\Big|_{-h/2}^{h/2}; \\ & m_x = z f_x\Big|_{-h/2}^{h/2}; \qquad m_y = z f_y\Big|_{-h/2}^{h/2}; \\ & N_n{}^* = \int_{-h/2}^{h/2} f_n\, d z; \qquad N_{ns}^* = \int_{-h/2}^{h/2} f_s\, d z; \\ & M_n{}^* = \int_{-h/2}^{h/2} f_n\, z\, d z; \qquad Q_n = \int_{-h/2}^{h/2} f_z\, d z \end{aligned} \tag{20}$$

expression (19) may be rewritten as

$$\begin{aligned} \delta \overline{W}_e = & \iint \left[F_x \delta u_0 - m_x \delta\frac{\partial w_0}{\partial x} + F_y \delta v_0 - m_y \delta\frac{\partial w_0}{\partial y} + q\, \delta w_0\right] d x\, d y \\ & + \oint \left(N_n{}^* \delta u_{0n} - M_n{}^* \delta\frac{\partial w_0}{\partial n} + N_{ns}^* \delta u_{0s} - M_{ns}^* \delta\frac{\partial w_0}{\partial s} + Q_n\, \delta w_0\right) d s . \end{aligned} \tag{21}$$

Noting that

$$\iint \left(m_x \delta \frac{\partial w_0}{\partial x} + m_y \delta \frac{\partial w_0}{\partial y}\right) dx\,dy = \oint m_n \delta w_0\, ds - \iint \left(\frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y}\right) \delta w_0\, dx\,dy \tag{22}$$

the work of boundary tractions may be expressed as

$$\begin{aligned} \delta \overline{W}_e = & \iint \left[F_x \delta u_0 + F_y \delta v_0 + \left(q + \frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y}\right) \delta w_0\right] dx\,dy \\ & + \oint \left(N_n{}^* \delta u_{0n} + N^*_{ns} \delta u_s - M_n{}^* \delta \frac{\partial w_0}{\partial n} + \frac{\partial M^*_{ns}}{\partial s} \delta w_0 + m_n \delta w_0 + Q_n \delta w_0\right) ds. \end{aligned} \tag{23}$$

In addition to the work done by increments of boundary tractions, work will also be done by increments in body forces. Let X, Y, Z designate the x, y, z components, respectively, of the body force per unit original volume

$$\delta W_B = \iiint_{V_0} (X \delta u + Y \delta v + Z \delta w)\, dx\,dy\,dz. \tag{24}$$

Substituting for the displacements u, v, w again their approximations $\bar{u}$, $\bar{v}$, $\bar{w}$ from (8) we obtain

$$\delta \overline{W}_B = \iiint_{V_0} \left[X\left(\delta u_0 - z\,\delta \frac{\partial w_0}{\partial x}\right) + Y\left(\delta v_0 - z\,\delta \frac{\partial w_0}{\partial y}\right) + Z \delta w_0\right] dx\,dy\,dz. \tag{25}$$

Assuming, for the sake of simplicity,

$$\int_{-h/2}^{h/2} X z\, dz = 0; \qquad \int_{-h/2}^{h/2} Y z\, dz = 0; \qquad \int_{-h/2}^{h/2} Z\, dz = 0 \tag{26}$$

the expression (25) reduces to

$$\delta \overline{W}_B = \iint_{S_0} [\overline{X} \delta u_0 + \overline{Y} \delta v_0]\, dx\,dy \tag{27}$$

where

$$\overline{X} = \int_{-h/2}^{h/2} X\, dz; \qquad \overline{Y} = \int_{-h/2}^{h/2} Y\, dz. \tag{28}$$

Kinetic Energy

The kinetic energy T of a body occupying before the deformation a volume V_0 and having a mass density ρ is given by

$$T = \tfrac{1}{2} \iiint_{V_0} \rho\, (\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\, dx\,dy\,dz. \tag{29}$$

The dot indicates differentiation with respect to time, t. Using for the velocities the approximate expressions based on displacements (8), we obtain an approximate form of the kinetic energy $\overline{T}$ in a plate

$$\overline{T} = \frac{1}{2}\iiint\limits_{V_0} \rho \left[\left(z\frac{\partial \dot{w}_0}{\partial x}\right)^2 + \left(z\frac{\partial \dot{w}_0}{\partial y}\right)^2 + \dot{w}_0^2\right] dx\,dy\,dz \tag{30}$$

which can be integrated through the thickness h of the plate, yielding, under the simplifying assumption that ρ is independent of z

$$\overline{T} = \frac{1}{2}\iint\limits_{S_0} \rho \left[\frac{h^3}{12}\left(\frac{\partial \dot{w}_0}{\partial x}\right)^2 + \frac{h^3}{12}\left(\frac{\partial \dot{w}_0}{\partial y}\right)^2 + h\,\dot{w}_0^2\right] dx\,dy. \tag{31}$$

The two terms with $\frac{h^3}{12}$ represent the rotatory inertia and will be omitted in the sequel. The variation of the kinetic energy $\overline{T}, \delta\,\overline{T}$ is then found to be

$$\delta\,\overline{T} = \iint\limits_{S_0} \rho\,h\,\dot{w}_0\,\delta\,\dot{w}_0\,dx\,dy. \tag{32}$$

We may integrate the above expression by parts with respect to time, setting, as usual, the variations at the beginning and end of the time interval equal to zero,

$$\delta\,\overline{T} = -\iint\limits_{S_0} \rho\,h\,\ddot{w}_0\,\delta\,w_0\,dx\,dy. \tag{33}$$

Equations of Motion

Hamilton's principle is now applied to derive the equations of motion. The principle states that for an arbitrary time interval $t_1 - t_2$

$$\delta\int_{t_1}^{t_2} L\,dt = 0 \tag{34}$$

where L is the kinetic potential, in general

$$L = T - U. \tag{35}$$

U being the potential energy. In the present case of the plate, the potential energy consists of the two parts, namely, the internal strain energy $\overline{W}_i$ and the potential of the external forces $-\overline{W}_e - \overline{W}_B$. Hence

$$\delta\,L = \delta\,\overline{T} - \delta\,\overline{W}_i + \delta\,\overline{W}_e + \delta\,\overline{W}_B. \tag{36}$$

This variation of the kinetic potential $\delta\,L$ must be equal to zero for arbitrary values of $\delta\,u_0$, etc. Using eqs. (33), (15), (23) and (27) for $\delta\,\overline{T}$, $\delta\,\overline{W}_i$, $\delta\,\overline{W}_e$ and $\delta\,\overline{W}_B$ respectively, we obtain three equations from the integrand of the double integral in the expression for $\delta\,L$, by equating to zero the coefficients of $\delta\,u_0$, etc.

$$
\begin{gathered}
\frac{\partial N_1{}^i}{\partial x}+\frac{\partial N_{12}^i}{\partial y}+F_x+\bar{X}=0, \\
\frac{\partial N_2{}^i}{\partial y}+\frac{\partial N_{12}^i}{\partial x}+F_y+\bar{Y}=0, \\
\frac{\partial^2 M_1}{\partial x^2}+\frac{\partial^2 M_2}{\partial y^2}+2\frac{\partial^2 M_{12}}{\partial x\,\partial y}+\frac{\partial}{\partial x}\left(N_1{}^i\frac{\partial w_0}{\partial x}\right)+\frac{\partial}{\partial y}\left(N_2{}^i\frac{\partial w_0}{\partial y}\right) \\
+\frac{\partial}{\partial x}\left(N_{12}^i\frac{\partial w_0}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_{12}^i\frac{\partial w_0}{\partial x}\right)+q+\frac{\partial m_x}{\partial x}+\frac{\partial m_y}{\partial y}=\rho h \ddot{w}_0.
\end{gathered}
\tag{37}
$$

From the integrand of the line integral we obtain, apart from identities, a relationship between the transverse shear force Q_n and other plate stresses

$$Q_n=\frac{\partial M_n}{\partial n}+\frac{\partial M_{ns}}{\partial s}+m_n. \tag{38}$$

The first two equations of the system (37) represent the equilibrium equations to be satisfied by the initial plate stresses $N_1{}^i$, $N_2{}^i$, N_{12}^i. The third equation describes the transverse motions of the initially stressed plate. For equilibrium ($\ddot{w}_0=0$) and in the absence of surface moments m_x, m_y the system (37) was derived, in quite a different fashion, by SAINT-VENANT [4, 5].

Stress-Displacement Relations

In a purely elastic material the plate moments M_1, M_2, M_{12} are connected with the plate displacement w_0 by the well known relations

$$
\begin{aligned}
M_1&=-D\left(\frac{\partial^2 w_0}{\partial x^2}+\nu\frac{\partial^2 w_0}{\partial y^2}\right), \\
M_2&=-D\left(\frac{\partial^2 w_0}{\partial y^2}+\nu\frac{\partial^2 w_0}{\partial x^2}\right), \\
M_{12}&=-D(1-\nu)\frac{\partial^2 w_0}{\partial x\,\partial y}.
\end{aligned}
\tag{39}
$$

D is the plate modulus

$$D=\frac{E h^3}{12(1-\nu^2)}. \tag{40}$$

Where E is Young's modulus and ν Poisson's ratio. The initial plate stresses $N_1{}^i$, $N_2{}^i$, N_{12}^i are not related to plate displacements w_0, on the basis of what was said above.

If the material exhibits internal damping (creep), or retarded elasticity, it may be assumed to consist of Voigt (or Kelvin) elements, that is elements made of spring and dashpot in parallel. Uniaxial Hooke's law, $\sigma=E\epsilon$, where ϵ is the strain, is then replaced by Stoke's law, $\sigma=E\epsilon+\eta\dot{\epsilon}$, where η is the coefficient of viscosity and $\dot{\epsilon}$ is the strain rate. To pass from the elastic to the viscoelastic case described above it is only necessary, in the uniaxial state of stress, to replace E by $\left(E+\eta\frac{\partial}{\partial t}\right)$. We shall suppose presently, that for a plate of Voigt-type viscoelastic material it is sufficient to replace E, in the plate modulus D, by $\left(E+\eta\frac{\partial}{\partial t}\right)$ without altering Poisson's ratio ν in the relations (39). The precise assumptions underlying this simplified treatment will be discussed in a separate report.

Plate under Initial Thrust

As a first application of the equations derived above, we investigate free vibrations of a visco-elastic rectangular plate, simply supported along all four edges and subjected to initial uniformly distributed uniaxial thrust, say $-N_1^i$. We shall assume that this thrust has been acting for a sufficiently long time, such that creep caused by it may be neglected. In the absence of forces F_x, $\bar{X}$, F_y, $\bar{Y}$, q, N_2^i, N_{12}^i and moments m_x, m_y, the first two equations (of equilibrium) of the set (37) are satisfied identically and the third becomes, with $-N_1^i = P_x$,

$$\frac{\partial^2 M_1}{\partial x^2}+\frac{\partial^2 M_2}{\partial y^2}+2\frac{\partial^2 M_{12}}{\partial x\,\partial y}-P_x\frac{\partial^2 w_0}{\partial x^2}=\rho h \ddot{w}_0. \tag{41}$$

For a plate material exhibiting retarded elasticity, the displacement eqs. of motion are

$$\bar{D}\,(\nabla^4 w_0) = P_x\frac{\partial^2 w_0}{\partial x^2}+\rho h \ddot{w}_0 \tag{42}$$

where

$$\bar{D} = D+\frac{\eta h^3}{12\,(1-\nu^2)}\frac{\partial}{\partial t} \tag{43}$$

and

$$\nabla^4 = \frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\,\partial y^2}+\frac{\partial^4}{\partial y^4}. \tag{44}$$

Designating the edge lengths by a (along x-axis) and b (along y-axis), respectively, the solution may be assumed in the form

$$w_0\,(x,y,t) = T\,(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \tag{45}$$

The substitution into the equation of motion (41) yields

$$\bar{D}\left(\frac{\pi^4}{a^4}+2\frac{\pi^4}{a^2 b^2}+\frac{\pi^4}{b^4}\right)T = P_x\frac{\pi^2}{a^2}T+\rho h \ddot{T} \tag{46}$$

or

$$\frac{\pi^2}{\rho h a^2}\,(P_{xcr}-P_x)\,T+\frac{\eta h^2 \pi^4}{12\,(1-\nu^2)\,a^2 b^2 \rho}\left(\frac{b}{a}+\frac{a}{b}\right)^2\dot{T}+\ddot{T} = 0. \tag{47}$$

$P_{x\,cr}$ stands for the critical buckling load of the purely elastic plate, assuming the instability to occur in one half-wave in each direction, that is assuming $\frac{a}{b} \leqq \sqrt{2}$,

$$P_{xcr} = D\frac{\pi^2}{b^2}\left(\frac{b}{a}+\frac{a}{b}\right)^2. \tag{48}$$

The problem is thus reduced to free, damped, linear vibrations of a single mass, since the factors of T and $\dot{T}$ are constants in time. Rewriting eq. (47), for convenience, as

$$\ddot{T}+2\beta^*\omega\,\dot{T}+\omega^2\,T = 0 \tag{49}$$

where ω is the circular frequency of the undamped vibration and β^* is the fraction of critical damping, defined by

$$\omega^2 = \frac{\pi^2}{\rho\,h\,a^2}(P_{xcr}-P_x), \tag{50}$$

$$\beta^* = \frac{\eta\,h^2\,\pi^4\left(\frac{b}{a}+\frac{a}{b}\right)^2}{24\,(1-\nu^2)\,a^2\,b^2\,\rho\,\omega} \tag{51}$$

the motion may be readily discussed.

The well-known fact is noted, first, that the frequency of undamped vibration ω^2 decreases with increasing compression P_x and approaches zero as P_x approaches the buckling load P_{xcr}.

In the presence of damping, $\eta \neq 0$, it may be of particular interest to determine that value of the initial thrust P_x, which separates periodic motions, $\beta^* < 1$, from aperiodic motions, $\beta^* \geqq 1$, that is which separates stable from unstable equilibrium (buckling load). Setting $\beta^* = 1$, we obtain the following expression for P_x from eqs. (50) and (51)

$$P_x = P_{xcr} - \frac{\eta^2\,h^5\,\pi^6}{576\,(1-\nu^2)^2 a^2 b^4 \rho}\left(\frac{b}{a}+\frac{a}{b}\right)^4. \tag{52}$$

Designating by β_0 the fraction of critical damping of the plate in the absence of any initial loading, $P_x = 0$, one obtains from eq. (51)

$$\beta_0 = \frac{\eta\,\omega_0}{2\,E} \tag{53}$$

where ω_0 is the natural frequency for $P_x = 0$

$$\omega_0{}^2 = \frac{D\,\pi^4}{\rho\,h\,a^2 b^2}\left(\frac{b}{a}+\frac{a}{b}\right)^2. \tag{54}$$

Relation (52) may then be put into the more compact form

$$P_x = P_{x\,cr}(1-\beta_0{}^2). \tag{55}$$

Thus, in the presence of a dissipation mechanism represented by a Kelvin element, the buckling load is decreased as compared to the corresponding elastic buckling load. This decrease depends on the fraction of critical damping in precisely the same fashion as the decrease of the square of the natural frequency (in the absence of initial stress).

Dropping now the restricting assumption concerning the buckling shape of one half-wave length in the direction of the thrust, any aspect ratio $\frac{a}{b}$ is considered. For the sake of simplicity, only an elastic plate shall be investigated. Seeking a solution in the form

$$w_0(x, y, t) = A \sin \omega t \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \tag{56}$$

for a given aspect ratio $\frac{a}{b}$ and a given initial thrust P_x, the lowest natural frequency ω of free vibration shall be determined.

Substitution of the form (56) into the equation of motion (41), together with the relations (39), results in

$$\rho h \omega^2 = D \pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 - P_x \frac{m^2 \pi^2}{a^2}. \tag{57}$$

It is apparent that the lowest frequency will occur with $n = 1$, but not necessarily with $m = 1$, as in the case of zero initial thrust. It is convenient to express the initial thrust P_x as a fraction of the lowest buckling load $(P_{cr})_{min}$

$$P_x = \mu (P_{cr})_{min} \tag{58}$$

and μ may be given the name "fraction of critical loading", by analogy to the concept of the fraction of critical damping. In stability investigations of plates it is customary to express the buckling load as a multiple of the Euler buckling load P_e, of a plate strip of unit width. In the present case this load is $\frac{\pi^2 D}{b^2}$, such that

$$(P_{cr})_{min} = k_{min} \frac{\pi^2 D}{b^2} = k_{min} P_e \tag{59}$$

and

$$k_{min} = 4 \tag{60}$$

by eq. (48). The frequency ω, from eq. (57), may then be given in the form

$$\omega^2 = \frac{D \pi^4}{\rho h a^2 b^2} \left[\left(\frac{b}{a} m^2 + \frac{a}{b}\right)^2 - \mu k_{min} m^2\right]. \tag{61}$$

It is thus seen, that the lowest natural frequency cannot be related directly to the buckling load, because for large values of the aspect ratio the mode shape of the lowest frequency, determined by the number of half-waves m, will depend on the fraction of critical loading μ. For aspect ratios $\frac{a}{b} = 1, 2, 3,$ 4 and 5, the square of a nondimensional frequency

$$\frac{\omega^2}{\omega_s{}^2} = \frac{\omega^2}{4\,D\,\pi^4/\rho\,h\,a^2\,b^2}$$

is plotted as a function of the fraction of critical loading μ, for various values of the mode shape m (fig. 1). While for small enough values of initial thrust, the lowest frequency is obtained always with $m = 1$, whatever the aspect ratio, for a large thrust, near the buckling load, the mode of vibration associated with the lowest frequency is always the same as the buckling mode.

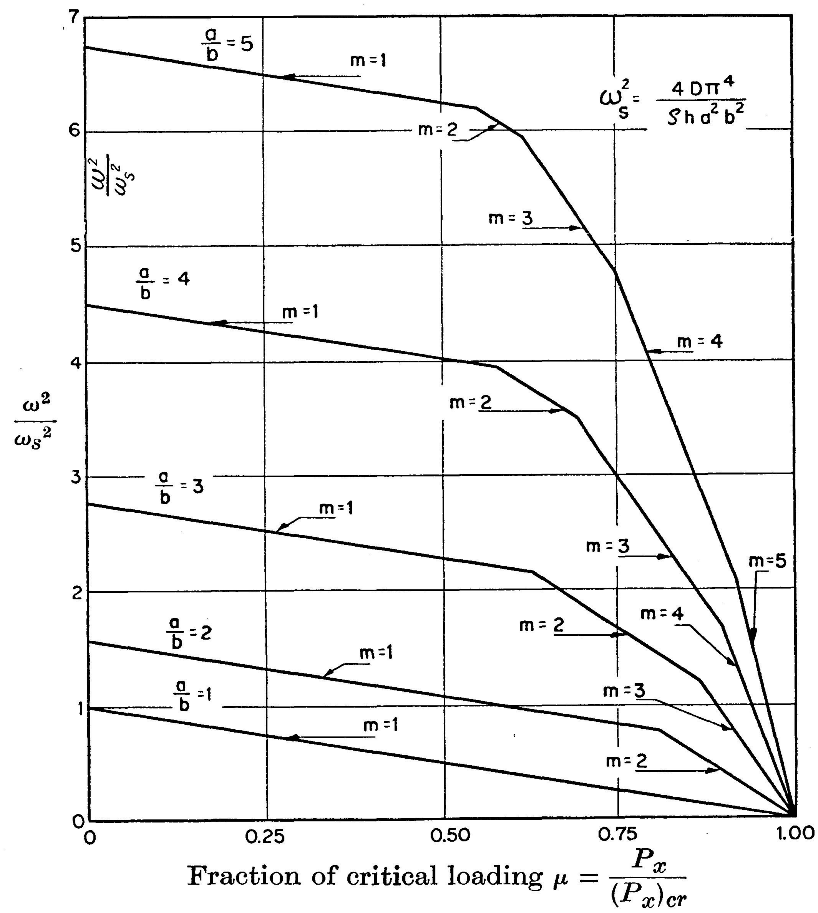


Fig. 1. Influence of initial thrust P_x on the lowest natural frequency ω for plate aspect ratios $\frac{a}{b} = 1, 2, 3, 4, 5$.

Heavy Plate under Edge Thrust

As a second problem, we shall investigate the influence of a body force, in addition to uniaxial compression, on the dynamic behaviour of an elastic, rectangular, simply supported plate. The body force, which is assumed to be acting in the direction of the thrust, may be due to the influence of the weight of the plate, in case it is in vertical position, or else it may arise in case the plate is accelerated in its own plane. Since for this initial loading, the differential equation governing the bending cannot be readily solved, Rayleigh's method shall be used to calculate the first approximation to the fundamental frequency of free vibration. The initial forces in the plate will be given by

$$N_2{}^i = N_{12}^i = 0, \qquad N_1{}^i = -P_x - \gamma h x \tag{62}$$

such that the first two equations of the set (37) are satisfied. γ is the specific weight and P_x is the compressive force at the upper edge of the plate.

Assuming the motion to consist merely of free transverse, harmonic vibration, e. g.

$$\begin{aligned} u_0 &= v_0 = 0, \\ w_0 &= A \sin \omega t \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}. \end{aligned} \tag{63}$$

The maximum strain energy, by eq. (12), will then be given by the expression

$$\overline{W}_{i\,max} = -\frac{\pi^2 b}{8a} P_x m^2 - \frac{\pi^2}{16} \gamma h b m^2 + \frac{\pi^4 a b}{8} D \left(\frac{m^2}{a^2} + \frac{1}{b^2} \right)^2. \tag{64}$$

The maximum kinetic energy, by eq. (31), neglecting rotatory inertia terms, is

$$\overline{T}_{max} = \frac{1}{2} \rho h \omega^2 \frac{a b}{4}. \tag{65}$$

No work is done by surface forces. For convenience, the ratio

$$\xi = \frac{\gamma h a}{\pi^2 D / b^2} \tag{66}$$

is introduced, where the numerator is the weight of a vertical element of unit width and the denominator is Euler's buckling load P_e.

Equating $\overline{W}_{i\,max}$ to $\overline{T}_{max}$ and solving for ω^2 we obtain

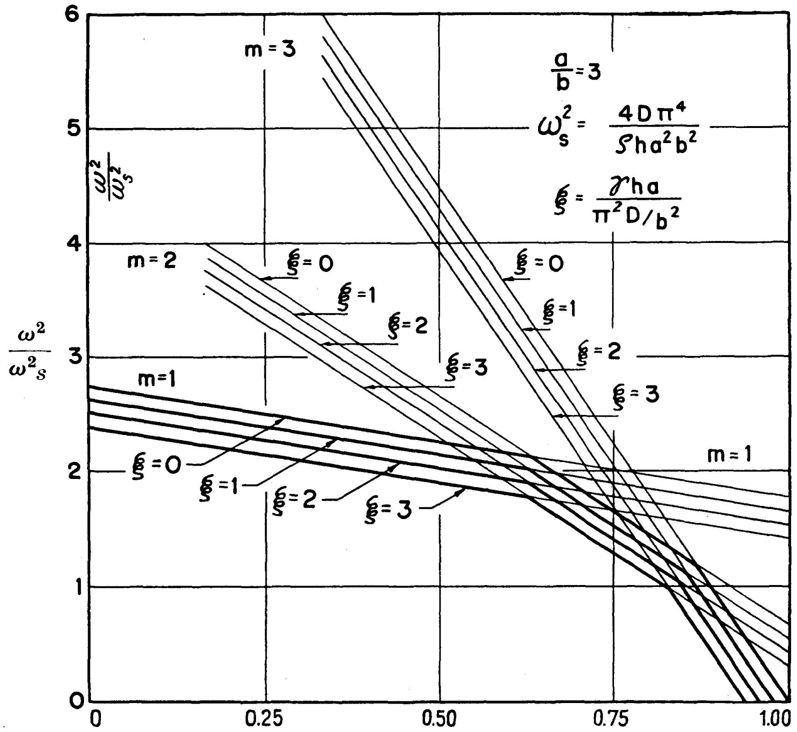
$$\omega^2 = \frac{D \pi^4}{\rho h a^2 b^2} \left[\left(\frac{b}{a} m^2 + \frac{a}{b} \right)^2 - m^2 \left(\mu k_{min} + \frac{\xi}{2} \right) \right]. \tag{67}$$

It is observed that in the absence of the body force, $\xi = 0$, the above approximate expression coincides with the exact expression for the frequency given by eq. (61). It is thus concluded, that for not too large values of the body force parameter ξ, the expression (67) will yield results of reasonable accuracy.

Furthermore, it is noted that if $\omega^2 = 0$, $\mu = 1$ and k_{min} is replaced by k, eq. (67) yields a first approximation to this buckling value k and is identical to the one given by H. FAVRE [8], who studied the influence of its own weight on the stability of a rectangular plate.

By eq. (67), the lowest natural frequency depends upon three parameters, namely the aspect ratio $\frac{a}{b}$, the fraction of critical loading μ (taken the same as in the preceding problem) and the body force parameter ξ. In fig. 2, the dimensionless frequency $\frac{\omega^2}{4 D \pi^2/\rho a^2 b^2 h}$ was plotted for $\frac{a}{b} = 3$ and $\xi = 0, 1, 2, 3$ as a function of μ, and again with $k_{min} = 4$. In first approximation, the linear term $\gamma h x$ is thus being replaced by its average value $\gamma h a/2$.

Just as in the previous example, it does not appear possible to express the lowest natural frequency in terms of the buckling load, for an arbitrary aspect ratio of the plate. For aspect ratios for which buckling occurs with one half-wave length, ($m = 1$), however, the frequency may be expressed, in the present loading case, in the same form as given by eq. (50).



Fraction of critical loading $\mu = \frac{P_x}{(P_x)_{cr}}$

Fig. 2. Influence of initial thrust P_x and body force ξ on the natural frequency ω for a plate aspect ratio $\frac{a}{b} = 3$.

Plate with one Free Edge

Finally, let us change the boundary conditions and consider a plate, under the action of a uniaxial, uniformly distributed thrust, simply supported along three edges, one edge, parallel to the loading, being free. These particular boundary conditions have been selected for consideration, because in this case of support the buckling occurs always in one half-wave length (see, for example [9]), and the lowest natural frequency may thus be related to the buckling load for any aspect ratio.

Using, again, Rayleigh's method and assuming the transverse displacement in the form

$$w = A \sin(\omega t)\, y \sin\frac{\pi x}{a} \tag{68}$$

the maximum strain energy is calculated to be, using again relation (12),

$$\overline{W}_{i\,max} = -\frac{P_x}{2} A^2 \frac{b^3}{3} \frac{\pi^2}{2a} + \frac{A^2 D}{2}\left[\frac{b^3 \pi^4}{3a^4}\frac{a}{2} + 2(1-\nu)\frac{\pi^2}{a^2}\frac{ab}{2}\right] \tag{69}$$

and the maximum kinetic energy, by eq. (31)

$$\overline{T}_{max} = \tfrac{1}{2}\rho h \omega^2 A^2 \frac{b^3}{3}\frac{a}{2}. \tag{70}$$

The buckling load $(P_x)_{cr}$ is obtained by equating to zero the right-hand side of eq. (69)

$$(P_x)_{cr} = P_e\left[\left(\pi\frac{b}{a}\right)^2 + 6(1-\nu)\right]. \tag{71}$$

The frequency is obtained by equating the right-hand sides of eqs. (69) and (70)

$$\omega^2 = \frac{\pi^2}{\rho h a^2}[(P_x)_{cr} - P_x]. \tag{72}$$

Indicating by ω_0^2 the natural frequency in the absence of any initial thrust, e. g., $P_x = 0$,

$$\omega_0^2 = \frac{\pi^2}{\rho h a^2}(P_x)_{cr}. \tag{73}$$

ω^2 may also be expressed in the form

$$\omega^2 = \omega_0^2\left[1 - \frac{P_x}{(P_x)_{cr}}\right] = \omega_0^2(1-\mu) \tag{74}$$

which holds for any aspect ratio $\frac{a}{b}$.

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Summary

Starting with the general three-dimensional equations, governing the dynamic behaviour of a deformable body under initial stress, plate equations are derived in a rigorous manner by means of a variational process.

These equations, which describe the dynamic behaviour of an initially stressed plate, are solved for a freely vibrating rectangular plate for several loading and boundary conditions. Particular attention is given to the influence of the internal damping (or creep) characteristics of the plate material and of the aspect ratio of the plate on the deflections and on the frequencies of vibration. The connection to the associated stability problem of the plate is worked out in some detail.

Résumé

En partant des équations tri-dimensionnelles générales, qui régissent le comportement dynamique d'un corps déformable soumis à une tension initiale, on déduit d'une façon rigoureuse les équations de la plaque au moyen d'un procédé de variation.

Ces équations qui décrivent le comportement dynamique d'une plaque initialement sous tension, sont résolues pour le cas d'une plaque rectangulaire vibrant librement et pour une variété de conditions de charge et aux limites. On s'attache particulièrement à l'influence des caractéristiques de fluage du matériau de la plaque et du rapport de sa largeur à sa longueur sur les déformations et les fréquences de vibration. La relation avec le problème de la stabilité associée de la plaque est traitée en détail.

Zusammenfassung

Ausgehend von den dreidimensionalen Gleichungen über das dynamische Verhalten eines deformierbaren Körpers unter Vorspannung, werden die Plattengleichungen in strenger Weise mit Hilfe eines Variationsverfahrens abgeleitet.

Diese Gleichungen beschreiben das dynamische Verhalten einer vorgespannten Platte und werden gelöst für eine freischwingende Rechteckplatte bei verschiedenen Lastfällen und Randbedingungen. Besondere Aufmerksamkeit wird dem Einfluß der inneren Dämpfungs- (oder Kriech-) Eigenschaften des Plattenmaterials geschenkt, ferner dem Einfluß des Seitenverhältnisses der Platte auf die Durchbiegungen und Schwingungsfrequenzen. Die Verbindung zu dem verwandten Problem der Plattenstabilität wird besonders herausgearbeitet.