Stability of pony-truss bridges

Stability of Pony-Truss Bridges

Stabilité des ponts en treillis ouverts à la partie supérieure

Stabilität von oben offenen Fachwerkbrücken

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Introduction

A pony-truss bridge is a low-truss bridge in which vertical roadway clearance requirements necessitate the elimination of the top lateral bracing (fig. 1). In a through span, the capacity of the trusses is governed mainly by yielding of the tension members, and buckling of the individual compression members. The absence of the top lateral bracing in a pony-truss bridge introduces an additional problem, the lateral buckling of the entire compression chord as a unit.

The purpose of this paper is to present a general method of analysis of the pony-truss problem taking into consideration the following secondary effects:

a) varying axial compression along the chord,
b) non-uniform chord sections,
c) chord curvature,

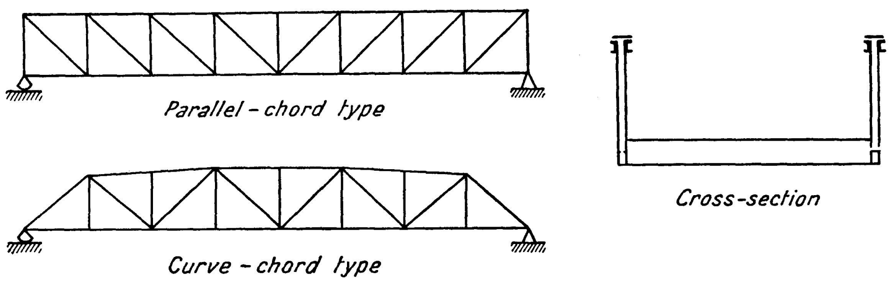

Fig. 1. Typical Pony-Truss Bridges.

d) various end connections,
e) torsional stiffness of the members,
f) support provided by the diagonals,
g) axial thrust in web members,
h) continuity between chord and web members,
i) unequal joint stiffnesses,
j) bending of floor beams,
k) initial eccentricities.

The method lends itself not only to the solution of the buckling problem, but also to the determination of the secondary stresses due to lateral displacements developed in the members under loads less than the critical load. These secondary stresses are distinct from those caused by plane bending of the members arising from the translation of the panel points in the truss plane.

Theory of Lateral Stability

Fundamental Considerations

Fundamentally, a pony-truss bridge is a space framework composed of an assemblage of beam-columns subjected to axial compression or tension and, in the case of the floor beams, the action of transverse loads. The top chord, including the two end posts, may be treated as a continuous beam-column elastically supported at intermediate joints by the diagonals and by U-shape crossframes each composed of two opposite vertical posts and the floor beam running between them. Restraints at the two end joints depend upon the stiffnesses of the end floor beams and the bottom chords as well as on the design of the supports.

In general, any joint in the truss has six degrees of freedom: its displacement in space may be described by three translational components along and three rotational components about three mutually perpendicular axes. In this discussion the truss plane will be defined by the xy-plane of a rectangular coordinate system, taking any joint along the top chord, say n, as the origin, y-axis vertical and the positive directions of the other axes as shown in fig. 2. When loads are applied on the roadway, vertical reactions at the lower panel points cause the trusses to distort in their planes. During this plane deformation,

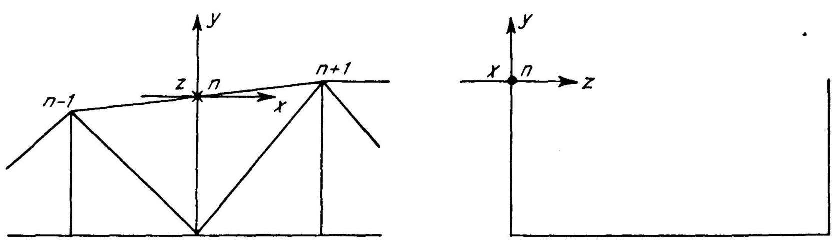

Fig. 2.

the joints undergo translation along the x and y axes and rotation about the z axis. Simultaneously, even in the absence of initial eccentricities, bending of the floor beams tilts the verticals which in turn displace the intermediate joints along the top chords out of the truss planes. This lateral displacement may be resolved into a translational component along the z axis and rotational components about the x and y axes.

If the cross-section of the truss members is symmetrical with respect to the truss plane, which is usually the case, the three displacement components associated with plane deformation have no influence on the lateral displacement of the truss. Hence, in the analysis of lateral stability, only the three lateral displacement components need be taken into consideration: deflection along the z axis and rotations about the x and y axes which, hereafter, will be denoted respectively by δ, α and β.

Several assumptions have been made in the analysis which follows. First, the diagonals are assumed to be fixed at their lower ends rather than elastically restrained by the bottom chords and the crossframes. To take their interaction into account would so complicate the analysis that the determination of the stiffnesses would be all but impossible. Perhaps the fact that this assumption has negligible effect on the end result alone suffices to justify this simplification [1]). The lower extremities of the verticals, on the other hand, are assumed to be fixed only in torsion but elastically restrained by the floor beams in bending, the influence of the bottom chords and the diagonals on the bending of the cross-frames being neglected.

An additional assumption is made that the cross-section of the members is symmetrical about both principal axes. In a section which does not satisfy this last condition, the center of twist does not coincide with the centroid, and lateral bending is accompanied by twisting. This effect will not be considered.

Generalized Slope-Deflection Equation

The method of analysis utilized in this presentation consists essentially of formulating and solving a system of simultaneous equations which express the equilibrium of the top panel points of the truss. The forces developed at the joints of the top chord are expressed in terms of the joint displacements and then the sum of the forces acting in a given direction at a given joint is equated to zero.

The forces developed in the top chord of the pony-truss may be related to the displacements of the joints by means of a generalized slope-deflection equation. Let $n-1$, n, and $n+1$ in fig. 3 be three consecutive intermediate joints along the top chord. The displacements of and the forces acting on the ends of the chord members are shown respectively in fig. 3a and b. All quanti-

1) See [1], p. 110. Numerals in [ ] refer to corresponding items in the Bibliography (see Appendix B).

ties are positive as indicated, the positive senses of z and δ being directed toward the paper. The moments and torques expressed in terms of the displacements are

$$M'_n = -(\delta_{n+1}-\delta_n)\frac{K'_n}{L_n} - m_n C_n K_n + m'_n K_n + (\delta'_{n+1}-\delta'_n)\frac{K'_n}{L_n}, \tag{1}$$

$$M_n = (\delta_{n+1}-\delta_n)\frac{K'_n}{L_n} + m_n K_n - m'_n C_n K_n - (\delta'_{n+1}-\delta'_n)\frac{K'_n}{L_n}, \tag{2}$$

$$M'_{n-1} = -(\delta_n-\delta_{n-1})\frac{K'_{n-1}}{L_{n-1}} - m_{n-1} C_{n-1} K_{n-1} + m'_{n-1} K_{n-1} + (\delta'_n-\delta'_{n-1})\frac{K'_{n-1}}{L_{n-1}}, \tag{3}$$

$$M_{n-1} = (\delta_n-\delta_{n-1})\frac{K'_{n-1}}{L_{n-1}} + m_{n-1} K_{n-1} - m'_{n-1} C_{n-1} K_{n-1} - (\delta'_n-\delta'_{n-1})\frac{K'_{n-1}}{L_{n-1}}, \tag{4}$$

$$T_n = R_n(t_n + t'_n), \tag{5}$$

$$T_{n-1} = R_{n-1}(t_{n-1} + t'_{n-1}), \tag{6}$$

in which

L = length of member,
M = moment acting on right end of member,
M' = moment acting on left end of member,
T = torsional moment,
m = rotation of right end of member about axis perpendicular to axis of member,
m' = rotation of left end of member about axis perpendicular to axis of member,
t = rotation of right end of member about its axis,
t' = rotation of left end of member about its axis,
$n-1$, n, $n+1$ = subscripts denoting joints or members,
δ' = initial eccentricity or initial deflection of a joint along the z axis (relative to the vertical plane passing through the two end joints),
C = carry-over factor,
R = torsional rigidity of member,
K = bending stiffness of member with far end fixed,
K' = modified stiffness of member in anti-symmetrical bending.

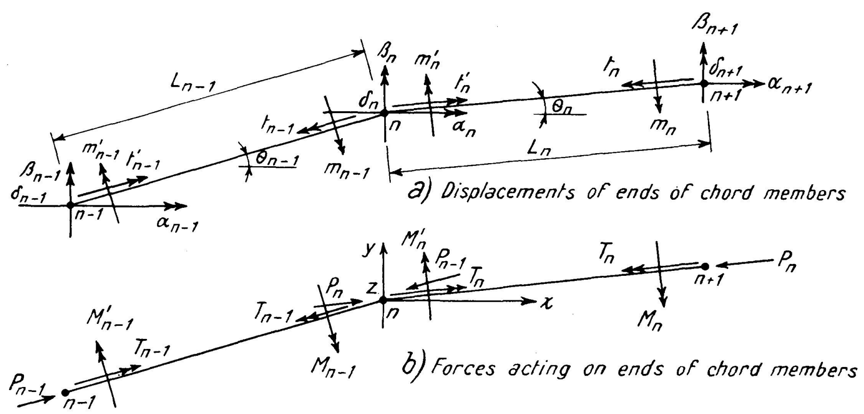

Fig. 3.

It can be shown that[2])

$$C = \frac{\zeta}{2\psi}, \tag{7}$$

$$K = \frac{EI}{L}\frac{12\psi}{4\psi^2-\zeta^2}, \tag{8}$$

$$K' = (1+C)K = \frac{EI}{L}\frac{6}{2\psi-\zeta}, \tag{9}$$

where E = modulus of elasticity,
I = moment of inertia (about axis lying in plane of truss).

For compression members

$$\zeta = \frac{6(\phi\csc\phi - 1)}{\phi^2}, \tag{10}$$

$$\psi = \frac{3(1-\phi\cot\phi)}{\phi^2} \tag{11}$$

and for tension members

$$\zeta = \frac{6(1-\phi\,\mathrm{csch}\,\phi)}{\phi^2}, \tag{12}$$

$$\psi = \frac{3(\phi\coth\phi - 1)}{\phi^2}, \tag{13}$$

where

$$\phi = L\sqrt{\frac{P}{EI}} \tag{14}$$

and P = axial force in member.

Displacement Components

In order to apply these equations in evaluating the stability of the truss, the displacements of the ends of the members are expressed in terms of the joint displacement components and the angles of inclination of the members, as follows:

$$m'_n = -\alpha_n\sin\theta_n + \beta_n\cos\theta_n, \tag{15}$$

$$t'_n = \alpha_n\cos\theta_n + \beta_n\sin\theta_n, \tag{16}$$

$$m'_{n-1} = -\alpha_{n-1}\sin\theta_{n-1} + \beta_{n-1}\cos\theta_{n-1}, \tag{17}$$

$$t'_{n-1} = \alpha_{n-1}\cos\theta_{n-1} + \beta_{n-1}\sin\theta_{n-1}, \tag{18}$$

$$m_{n-1} = \alpha_n\sin\theta_{n-1} - \beta_n\cos\theta_{n-1}, \tag{19}$$

$$t_{n-1} = -\alpha_n\cos\theta_{n-1} - \beta_n\sin\theta_{n-1}, \tag{20}$$

$$m_n = \alpha_{n+1}\sin\theta_n - \beta_{n+1}\cos\theta_n, \tag{21}$$

$$t_n = -\alpha_{n+1}\cos\theta_n - \beta_{n+1}\sin\theta_n. \tag{22}$$

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[2]) See [2], [3], and [4]. Values of the functions in the expressions for stiffness and carry-over factor are given in [5], [6], [7], [8], and [9] in different forms.

in which θ is the angle of inclination, being positive if the slope of the member is positive.

Equilibrium Equations

The lateral equilibrium of any joint n is governed by the three conditions that the summations of moments about the x and y axes and forces along the z axis are equal to zero. Thus (see fig. 3)

$\Sigma M_x = 0$, or

$$\begin{aligned} &M_{n-1}\sin\theta_{n-1} - T_{n-1}\cos\theta_{n-1} - M'_n\sin\theta_n + T_n\cos\theta_n + ({}_nS'_{xz} + {}_nS''_{xz})(\delta_n - \delta'_n) \\ &\quad + ({}_nS'_{xx} + {}_nS''_{xx})\,\alpha_n + ({}_nS'_{xy} + {}_nS''_{xy})\,\beta_n + {}_nS_x = 0. \end{aligned} \tag{23}$$

$\Sigma M_y = 0$, or

$$\begin{aligned} &-M_{n-1}\cos\theta_{n-1} - T_{n-1}\sin\theta_{n-1} + M'_n\cos\theta_n + T_n\sin\theta_n + ({}_nS'_{yz} + {}_nS''_{xy})(\delta_n - \delta'_n) \\ &\quad + ({}_nS'_{xy} + {}_nS''_{xy})\,\alpha_n + ({}_nS'_{yy} + {}_nS''_{yy})\,\beta_n = 0. \end{aligned} \tag{24}$$

$\Sigma F_z = 0$, or

$$\begin{aligned} &-\frac{P_{n-1}}{L_{n-1}}(\delta_n - \delta_{n-1}) + \frac{P_n}{L_n}(\delta_{n+1} - \delta_n) + \frac{M_{n-1} - M'_{n-1}}{L_{n-1}} + \frac{M'_n - M_n}{L_n} \\ &\quad + ({}_nS'_{zz} + {}_nS''_{zz})(\delta_n - \delta'_n) + ({}_nS'_{xz} + {}_nS''_{xz})\,\alpha_n + ({}_nS'_{yz} + {}_nS''_{yz})\,\beta_n + {}_nS_z = 0. \end{aligned} \tag{25}$$

Effects of Web System

The first four terms in each of eqs. (23) to (25) represent the forces developed in the chord members at joint n. The remaining terms represent the influence of the web system on the equilibrium of the joint. The symbols used in these terms are defined as follows:

${}_nS_x$ and ${}_nS_z$ are forces acting at the top of the vertical post due to bending of the floor beam under the action of vertical loads; ${}_nS'_{xx}$, ${}_nS'_{xy}$, ${}_nS'_{xz}$, ${}_nS'_{yy}$, ${}_nS'_{yz}$, and ${}_nS'_{zz}$ are the stiffnesses of the cross-frame at joint n and ${}_nS''_{xx}$, ${}_nS''_{xy}$, ${}_nS''_{xz}$, ${}_nS''_{yy}$, ${}_nS''_{yz}$, and ${}_nS''_{zz}$ are the stiffnesses of the diagonals. The two subscripts in the stiffnesses are interchangeable, one denoting the direction of the force and the other that of the displacement component producing the force. For instance, S_{xz} is the moment about x axis produced by unit deflection along the z axis or the thrust along the z axis produced by unit rotation about the x axis. In eq. (25), P_{n-1} and P_n are axial compressive forces. These web system effects are treated separately in the following paragraphs.

1. *Bending of floor beams.* It can be shown[3]) that the forces developed at the top of the vertical post due to bending of the floor beam are given by:

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[3]) See [5] for the derivation of the equations.

$$S_x = -M_f \frac{C_v}{1+r}, \tag{26}$$

$$S_z = -\frac{M_f}{L_v} \frac{1+C_v}{1+r}. \tag{27}$$

M_f is the fixed-end moment acting on the floor beam due to vertical loads assumed symmetrically applied on the roadway; C_v and L_v are respectively the carry-over factor and the length of the vertical at joint n and

$$r = \frac{K_b''}{K_v}. \tag{28}$$

K_b'' is the modified stiffness of the floor beam in symmetrical bending and K_v is the bending stiffness of the vertical with the far end fixed. The modified stiffness of a member in symmetrical bending is given by

$$K'' = (1-C)\,K = \frac{EI}{L} \frac{6}{2\psi+\zeta}. \tag{29}$$

2. *Stiffness of cross-frames.* The stiffnesses of the cross-frame [4]) are

$$S'_{xx} = K_v\left(1 - \frac{C_v^2}{1+r}\right), \tag{30}$$

$$S'_{xy} = 0, \tag{31}$$

$$S'_{xz} = \frac{K'_v}{L_v}\left(1 - \frac{C_v}{1+r}\right), \tag{32}$$

$$S'_{yy} = R_v, \tag{33}$$

$$S'_{yz} = 0, \tag{34}$$

$$S'_{zz} = \frac{K'_v}{L_v^2}\left(2 - \frac{1+C_v}{1+r}\right) \pm \frac{P_v}{L_v}, \tag{35}$$

in which P_v is the axial force in the vertical, K'_v its modified stiffness in anti-symmetrical bending and R_v its torsional rigidity. In eq. (35), the plus sign should be used if the vertical is in tension, the minus sign if in compression.

3. *Stiffness of diagonals.* If the diagonal is on the right side of the vertical, its stiffnesses are

$$S''_{xx} = K_d \sin^2\theta_d + R_d \cos^2\theta_d, \tag{36}$$

$$S''_{xy} = (K_d - R_d)\sin\theta_d \cos\theta_d, \tag{37}$$

$$S''_{xz} = \frac{K'_d}{L_d}\sin\theta_d, \tag{38}$$

$$S''_{yy} = K_d \cos^2\theta_d + R_d \sin^2\theta_d, \tag{39}$$

$$S''_{yz} = \frac{K'_d}{L_d}\cos\theta_d, \tag{40}$$

$$S''_{zz} = \frac{2K'_d}{L_d^2} \pm \frac{P_d}{L_d}, \tag{41}$$

[4]) See Appendix A.

in which the subscript d denotes the diagonal. If the diagonal is on the left side of the vertical, eqs. (36), (38), (39) and (41) remain unchanged but eqs. (37) and (40) become

$$S''_{yy} = -(K_d - R_d)\sin\theta_d\cos\theta_d, \tag{42}$$

$$S''_{yz} = -\frac{K'_d}{L_d}\cos\theta_d. \tag{43}$$

In eqs. (36) to (43), the absolute value of θ_d should be used. The plus sign in eq. (41) should be used if the diagonal is in tension, the minus sign if in compression.

General Equations

Now expressing the forces and moments of eqs. (23) to (25) in terms of member displacements by means of eqs. (1) to (6), and transforming the member displacements into joint displacements by means of eqs. (15) to (22), the general equations of equilibrium of joint n are obtained:

$$\begin{aligned}
&\delta_{n-1}\left[-\frac{K'_{n-1}}{L_{n-1}}\sin\theta_{n-1}\right] + \alpha_{n-1}\left[C_{n-1}K_{n-1}\sin^2\theta_{n-1} - R_{n-1}\cos^2\theta_{n-1}\right]\\
&\quad + \beta_{n-1}\left[-(C_{n-1}K_{n-1} + R_{n-1})\sin\theta_{n-1}\cos\theta_{n-1}\right]\\
&\quad + \delta_n\left[\frac{K'_{n-1}}{L_{n-1}}\sin\theta_{n-1} - \frac{K'_n}{L_n}\sin\theta_n + {}_nS'_{xz} + {}_nS''_{xz}\right]\\
&\quad + \alpha_n\left[K_{n-1}\sin^2\theta_{n-1} + R_{n-1}\cos^2\theta_{n-1} + K_n\sin^2\theta_n + R_n\cos^2\theta_n + {}_nS'_{xx} + {}_nS''_{xx}\right]\\
&\quad + \beta_n\left[(-K_{n-1} + R_{n-1})\sin\theta_{n-1}\cos\theta_{n-1} + (-K_n + R_n)\sin\theta_n\cos\theta_n + {}_nS''_{xy}\right]\\
&\quad + \delta_{n+1}\left[\frac{K'_n}{L_n}\sin\theta_n\right] + \alpha_{n+1}\left[C_nK_n\sin^2\theta_n - R_n\cos^2\theta_n\right]\\
&\quad + \beta_{n+1}\left[-(C_nK_n + R_n)\sin\theta_n\cos\theta_n\right]\\
&= \left[(\delta'_n - \delta'_{n-1})\frac{K'_{n-1}}{L_{n-1}}\sin\theta_{n-1} + (\delta'_{n-1} - \delta'_n)\frac{K'_n}{L_n}\sin\theta_n + \delta'_n\left({}_nS'_{xz} + {}_nS''_{xz}\right) - {}_nS_x\right].
\end{aligned} \tag{44}$$

$$\begin{aligned}
&\delta_{n-1}\left[\frac{K'_{n-1}}{L_{n-1}}\cos\theta_{n-1}\right] + \alpha_{n-1}\left[-(C_{n-1}K_{n-1} + R_{n-1})\sin\theta_{n-1}\cos\theta_{n-1}\right]\\
&\quad + \beta_{n-1}\left[C_{n-1}K_{n-1}\cos^2\theta_{n-1} - R_{n-1}\sin^2\theta_{n-1}\right]\\
&\quad + \delta_n\left[-\frac{K'_{n-1}}{L_{n-1}}\cos\theta_{n-1} + \frac{K'_n}{L_n}\cos\theta_n + {}_nS''_{yz}\right]\\
&\quad + \alpha_n\left[(-K_{n-1} + R_{n-1})\sin\theta_{n-1}\cos\theta_{n-1} + (-K_n + R_n)\sin\theta_n\cos\theta_n + {}_nS''_{xy}\right]\\
&\quad + \beta_n\left[K_{n-1}\cos^2\theta_{n-1} + R_{n-1}\sin^2\theta_{n-1} + K_n\cos^2\theta_n + R_n\sin^2\theta_n + {}_nS'_{yy} + {}_nS''_{yy}\right]\\
&\quad + \delta_{n+1}\left[-\frac{K'_n}{L_n}\cos\theta_n\right] + \alpha_{n+1}\left[-(C_nK_n + R_n)\sin\theta_n\cos\theta_n\right]\\
&\quad + \beta_{n+1}\left[C_nK_n\cos^2\theta_n - R_n\sin^2\theta_n\right]\\
&= \left[-(\delta'_n - \delta'_{n-1})\frac{K'_{n-1}}{L_{n-1}}\cos\theta_{n-1} - (\delta'_{n+1} - \delta'_n)\frac{K'_n}{L_n}\cos\theta_n + \delta'_n\,{}_nS''_{yz}\right].
\end{aligned} \tag{45}$$

$$
\begin{aligned}
&\delta_{n-1}\left[\frac{P_{n-1}}{L_{n-1}}-\frac{2K'_{n-1}}{L^2_{n-1}}\right]+\alpha_{n-1}\left[\frac{K'_{n-1}}{L_{n-1}}\sin\theta_{n-1}\right]+\beta_{n-1}\left[-\frac{K'_{n-1}}{L_{n-1}}\cos\theta_{n-1}\right]\\
&\quad+\delta_n\left[-\frac{P_{n-1}}{L_{n-1}}-\frac{P_n}{L_n}+\frac{2K'_{n-1}}{L^2_{n-1}}+\frac{2K'_n}{L^2_n}+{}_nS'_{zz}+{}_nS''_{zz}\right]\\
&\quad+\alpha_n\left[\frac{K'_{n-1}}{L_{n-1}}\sin\theta_{n-1}-\frac{K'_n}{L_n}\sin\theta_n+{}_nS'_{xz}+{}_nS''_{xz}\right]\\
&\quad+\beta_n\left[-\frac{K'_{n-1}}{L_{n-1}}\cos\theta_{n-1}+\frac{K'_n}{L_n}\cos\theta_n+{}_nS''_{yz}\right]+\delta_{n+1}\left[\frac{P_n}{L_n}-\frac{2K'_n}{L^2_n}\right]\\
&\quad+\alpha_{n+1}\left[-\frac{K'_n}{L_n}\sin\theta_n\right]+\beta_{n+1}\left[\frac{K'_n}{L_n}\cos\theta_n\right]\\
&=\left[2(\delta'_n-\delta'_{n-1})\frac{K'_{n-1}}{L^2_{n-1}}-2(\delta'_{n+1}-\delta'_n)\frac{K'_n}{L^2_n}+\delta'_n({}_nS'_{zz}+{}_nS''_{zz})-{}_nS_z\right].
\end{aligned}
\tag{46}
$$

These equations represent respectively the conditions of equilibrium of moments about the x and y axes, and equilibrium of forces along the z axis, all expressed in terms of the joint displacements.

Special Equations for Joints Adjacent to the End Joints

Since the displacements of the top chord are restricted at the end joints, the equations of equilibrium for the joints adjacent to the end joints may be modified.

At the two end joints,

$$
\begin{aligned}
\delta_0 &= 0,\\
\delta'_0 &= 0,\\
\delta_{i+1} &= 0,\\
\delta'_{i+1} &= 0,
\end{aligned}
$$

where 0 and $i+1$ denote respectively the left and the right end joint; i being numerically equal to the number of intermediate joints. It will be observed that if the ends of the top chord are free to rotate in bending,

$$
\begin{aligned}
M'_0 &= 0,\\
M_i &= 0.
\end{aligned}
$$

If they are fixed in bending,

$$
\begin{aligned}
m'_0 &= 0,\\
m_i &= 0.
\end{aligned}
$$

Also, if they are free to rotate in torsion,

$$
\begin{aligned}
T_0 &= 0,\\
T_i &= 0.
\end{aligned}
$$

If they are fixed in torsion,

$$
\begin{aligned}
t'_0 &= 0,\\
t_i &= 0.
\end{aligned}
$$

Applying these boundary conditions in eqs. (44), (45), and (46) the following special equations are obtained for joint 1,

$$\begin{aligned}
&\delta_1\left[\frac{K_0'}{L_0}(1-a\,C_0)\sin\theta_0-\frac{K_1'}{L_1}\sin\theta_1+{}_1S_{xz}'+{}_1S_{xz}''\right]\\
&\quad+\alpha_1\left[K_0(1-a\,C_0^2)\sin^2\theta_0+R_0(1-b)\cos^2\theta_0+K_1\sin^2\theta_1+R_1\cos^2\theta_1+{}_1S_{xx}'+{}_1S_{xx}''\right]\\
&\quad+\beta_1\left[\{-K_0(1-a\,C_0^2)+R_0(1-b)\}\sin\theta_0\cos\theta_0+(-K_1+R_1)\sin\theta_1\cos\theta_1+{}_1S_{xy}''\right]\\
&\quad+\delta_2\left[\frac{K_1'}{L_1}\sin\theta_1\right]+\alpha_2\left[C_1K_1\sin^2\theta_1-R_1\cos^2\theta_1\right]+\beta_2\left[-(C_1K_1+R_1)\sin\theta_1\cos\theta_1\right]\\
&\quad=\left[\delta_1'\frac{K_0'}{L_0}(1-a\,C_0)\sin\theta_0+(\delta_2'-\delta_1')\frac{K_1'}{L_1}\sin\theta_1+\delta_1'\,({}_1S_{xz}'+{}_1S_{xz}'')-{}_1S_x\right].
\end{aligned}\tag{47}$$

$$\begin{aligned}
&\delta_1\left[-\frac{K_0'}{L_0}(1-a\,C_0)\cos\theta_0+\frac{K_1'}{L_1}\cos\theta_1+{}_1S_{yz}''\right]\\
&\quad+\alpha_1\left[\{-K_0(1-a\,C_0^2)+R_0(1-b)\sin\theta_0\cos\theta_0+(-K_1+R_1)\sin\theta_1\cos\theta_1+{}_1S_{xy}''\right]\\
&\quad+\beta_1\left[K_0(1-a\,C_0^2)\cos^2\theta_0+R_0(1-b)\sin^2\theta_0+K_1\cos^2\theta_1+R_1\sin^2\theta_1+{}_1S_{yy}'+{}_1S_{yy}''\right]\\
&\quad+\delta_2\left[-\frac{K_1'}{L_1}\cos\theta_1\right]+\alpha_2\left[-(C_1K_1+R_1)\sin\theta_1\cos\theta_1\right]+\beta_2\left[C_1K_1\cos^2\theta_1-R_1\sin^2\theta_1\right]\\
&\quad=\left[-\delta_1'\frac{K_0'}{L_0}(1-a\,C_0)\cos\theta_0-(\delta_2'-\delta_1')\frac{K_1'}{L_1}\cos\theta_1+\delta_1'\,{}_1S_{yz}''\right].
\end{aligned}\tag{48}$$

$$\begin{aligned}
&\delta_1\left[-\frac{P_0}{L_0}-\frac{P_1}{L_1}+\{2-a\,(1+C_0)\}\frac{K_0'}{L_0^2}+\frac{2\,K_1'}{L_1^2}+{}_1S_{zz}'+{}_1S_{zz}''\right]\\
&\quad+\alpha_1\left[\frac{K_0'}{L_0}(1-a\,C_0)\sin\theta_0-\frac{K_1'}{L_1}\sin\theta_1+{}_1S_{xz}'+{}_1S_{xz}''\right]\\
&\quad+\beta_1\left[-\frac{K_0'}{L_0}(1-a\,C_0)\cos\theta_0+\frac{K_1'}{L_1}\cos\theta_1+{}_1S_{yz}''\right]+\delta_2\left[\frac{P_1}{L_1}-\frac{2\,K_1'}{L_1^2}\right]\\
&\quad+\alpha_2\left[-\frac{K_1'}{L_1}\sin\theta_1\right]+\beta_2\left[\frac{K_1'}{L_1}\cos\theta_1\right]\\
&\quad=\left[\{2-a\,(1+C_0)\}\,\delta_1'\frac{K_0'}{L_0^2}-2\,(\delta_2'-\delta_1')\frac{K_1'}{L_1^2}+\delta_1'\,({}_1S_{zz}'+{}_1S_{zz}'')-{}_1S_z\right].
\end{aligned}\tag{49}$$

and for joint i,

$$\begin{aligned}
&\delta_{i-1}\left[-\frac{K_{i-1}'}{L_{i-1}}\sin\theta_{i-1}\right]+\alpha_{i-1}\left[C_{i-1}K_{i-1}\sin^2\theta_{i-1}-R_{i-1}\cos^2\theta_{i-1}\right]\\
&\quad+\beta_{i-1}\left[-(C_{i-1}K_{i-1}+R_{i-1})\sin\theta_{i-1}\cos\theta_{i-1}\right]\\
&\quad+\delta_i\left[\frac{K_{i-1}'}{L_{i-1}}\sin\theta_{i-1}-\frac{K_i'}{L_i}(1-a\,C_i)\sin\theta_i+{}_iS_{xz}'+{}_iS_{xz}''\right]\\
&\quad+\alpha_i\left[K_{i-1}\sin^2\theta_{i-1}+R_{i-1}\cos^2\theta_{i-1}+K_i(1-a\,C_i^2)\sin^2\theta_i+R_i(1-b)\cos^2\theta_i+{}_iS_{xx}'+{}_iS_{xx}''\right]\\
&\quad+\beta_i\left[(-K_{i-1}+R_{i-1})\sin\theta_{i-1}\cos\theta_{i-1}+\{-K_i(1-a\,C_i^2)+R_i(1-b)\}\sin\theta_i\cos\theta_i+{}_iS_{xy}''\right]\\
&\quad=\left[(\delta_i'-\delta_{i-1}')\frac{K_{i-1}'}{L_{i-1}}\sin\theta_{i-1}-\delta_i'\frac{K_i'}{L_i}(1-a\,C_i)\sin\theta_i+\delta_i'\,({}_iS_{xz}'+{}_iS_{xz}'')-{}_iS_x\right].
\end{aligned}\tag{50}$$

$$\delta_{i-1}\left[\frac{K'_{i-1}}{L_{i-1}}\cos\theta_{i-1}\right]+\alpha_{i-1}\left[-(C_{i-1}K_{i-1}+R_{i-1})\sin\theta_{i-1}\cos\theta_{i-1}\right]$$
$$+\beta_{i-1}\left[C_{i-1}K_{i-1}\cos^2\theta_{i-1}-R_{i-1}\sin^2\theta_{i-1}\right]$$
$$+\delta_i\left[-\frac{K'_{i-1}}{L_{i-1}}\cos\theta_{i-1}+\frac{K'_i}{L_i}(1-a\,C_i)\cos\theta_i+{}_iS''_{yz}\right] \tag{51}$$
$$+\alpha_i\left[(-K_{i-1}+R_{i-1})\sin\theta_{i-1}\cos\theta_{i-1}+\{-K_i(1-a\,C_i^2)+R_i(1-b)\}\sin\theta_i\cos\theta_i+{}_iS''_{xy}\right]$$
$$+\beta_i\left[K_{i-1}\cos^2\theta_{i-1}+R_{i-1}\sin^2\theta_{i-1}+K_i(1-a\,C_i^2)\cos^2\theta_i+R_i(1-b)\sin^2\theta_i+{}_iS'_{yy}+{}_iS''_{yy}\right]$$
$$=\left[-(\delta'_i-\delta'_{i-1})\frac{K'_{i-1}}{L_{i-1}}\cos\theta_{i-1}+\delta'_i\frac{K'_i}{L_i}(1-a\,C_i)\cos\theta_i+\delta'_i\,{}_iS''_{yz}\right].$$

$$\delta_{i-1}\left[\frac{P_{i-1}}{L_{i-1}}-\frac{2K'_{i-1}}{L^2_{i-1}}\right]+\alpha_{i-1}\left[\frac{K'_{i-1}}{L_{i-1}}\sin\theta_{i-1}\right]+\beta_{i-1}\left[-\frac{K'_{i-1}}{L_{i-1}}\cos\theta_{i-1}\right]$$
$$+\delta_i\left[-\frac{P_{i-1}}{L_{i-1}}-\frac{P_i}{L_i}+\frac{2K'_{i-1}}{L^2_{i-1}}+\{2-a(1+C_i)\}\frac{K_i}{L_i^2}+{}_iS'_{zz}+{}_iS''_{zz}\right]$$
$$+\alpha_i\left[\frac{K'_{i-1}}{L_{i-1}}\sin\theta_{i-1}-\frac{K_i}{L_i}(1-a\,C_i)\sin\theta_i+{}_iS'_{xz}+{}_iS''_{xz}\right] \tag{52}$$
$$+\beta_i\left[-\frac{K'_{i-1}}{L_{i-1}}\cos\theta_{i-1}+\frac{K'_i}{L_i}(1-a\,C_i)\cos\theta_i+{}_iS''_{yz}\right]$$
$$=\left[2(\delta'_i-\delta'_{i-1})\frac{K'_{i-1}}{L^2_{i-1}}+\{2-a(1-C_i)\}\,\delta'_i\frac{K'_i}{L_i^2}+\delta'_i({}_iS'_{zz}+{}_iS''_{zz})-{}_iS_z\right].$$

In eqs. (47) to (52), a and b are numerical constants defined as follows: if the ends of chord are free to rotate in bending, $a=1$; if fixed in bending, $a=0$; if free to rotate in torsion, $b=1$; and if fixed in torsion, $b=0$. Observe that when $a=1$,

$$K'(1-aC)=K(1-aC^2)=K'\{2-a(1+C)\}=K'''=\frac{EI}{L}\frac{3}{\psi}. \tag{53}$$

K''' being the modified stiffness of a member with the far end pinned. Also when $b=1$, the factor $(1-b)$ becomes zero in which case the torsional stiffness of the end post does not enter into the problem. For the special case where the end posts are vertical, $\theta_0=90°$ and $\theta_i=-90°$.

Characteristic Equations

For a truss with i intermediate joints, eqs. (44) to (52) represent $3\,i$ simultaneous linear equations involving the same number of unknowns, i.e., $\delta_1,\alpha_1,\beta_1,\delta_2,\alpha_2,\ldots\beta_{i-1},\delta_i,\alpha_i,\beta_i$. It will be noted that these equations do not involve the displacement components of the end joints. Letting these unknown displacements be denoted respectively by $X_1, X_2,\ldots X_{j-1}$ and X_j where $j=3\,i$, the system of equilibrium equations may be written:

$$\left.\begin{array}{l} A_{11}X_1+A_{12}X_2 \dots\dots\dots A_{1(j-1)}X_{j-1}+A_{1j}X_j = B_1, \\ A_{21}X_1+A_{22}X_2 \dots\dots\dots A_{2(j-1)}X_{j-1}+A_{2j}X_j = B_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ A_{(j-1)1}X_1+A_{(j-1)2}X_2 \dots A_{(j-1)(j-1)}X_{j-1}+A_{(j-1)j}X_j = B_{j-1}, \\ A_{j1}X_1+A_{j2}X_2 \dots A_{j(j-1)}X_{j-1}+A_{jj}X_j = B_j, \end{array}\right\} \quad (54)$$

in which the constants A represent the forces associated with unit values of the displacements. These quantities are called stiffness influence coefficients, and are given by the expressions contained in brackets on the left side of eqs. (44) to (52), as follows:

$$A_{11} = \frac{K_0'}{L_0}(1-a\,C_0)\sin\theta_0 - \frac{K_1'}{L_1}\sin\theta_1 + {}_1S_{xz}' + {}_1S_{xz}'',$$

$$A_{12} = K_0(1-a\,C_0^2)\sin^2\theta_0 + R_0(1-b)\cos^2\theta_0 + K_1\sin^2\theta_1 + R_1\cos^2\theta_1 + {}_1S_{xx}' + {}_1S_{xx}'',$$

and so on for the other terms.

It should be noted that these coefficients are dependent upon the geometry of the structure, the cross-sectional dimensions and physical properties of the members, and the magnitude of the applied load.

The constants B in eqs. (54) represent the effect of initial eccentricities of the top chord and bending of the floor beams on the equilibrium of the joints. They are given by the expressions on the right side of eqs. (44) to (52) as follows:

$$B_j = 2(\delta_i' - \delta_{i-1}')\frac{K_{i-1}'}{L_{i-1}^2} + \{2 - a(1+C_i)\}\,\delta_i'\frac{K_i'}{L_i^2} + \delta_i'({}_iS_{zz}' + {}_iS_{zz}'') - {}_iS_z, \text{ etc.}$$

Buckling Condition

For any given magnitude of applied load W and initial eccentricities of the top chord, the deflected shape of the structure may be determined by solving the system of simultaneous eqs. (54). The value of the displacements X_j will increase as the determinant of the stiffness influence coefficients A_{ij} decreases. The buckling condition of the structure is indicated by the displacements approaching infinity, which condition is reached when the determinant vanishes. In other words, if the determinant of the stiffness coefficients is denoted by D, the buckling load W_b of the structure may be determined as the load for which

$$D = 0. \quad (55)$$

In practice this critical load may be evaluated by assuming various values of W, calculating the corresponding values of D and plotting a $D-W$ curve. The critical load is given by the intercept of this curve on the $D=0$ axis. It will be observed that eq. (55) does not involve the B terms in eq. (54), hence the buckling load is independent of the effect of initial eccentricities and bending of the floor beams.

Secondary Stresses

For known values of the terms B, the displacement components of the intermediate joints for loads less then W_b can be obtained by solving eq. (54). The rotations of the ends of the members are then given by eqs. (15), (16), (21), and (22) and the moments and the torques acting on the members by eqs. (1), (2), and (5). For the particular case where the end joints are free to rotate in bending, eqs. (1) and (2) applied to the end posts yield respectively

$$M_i' = (\delta_i - \delta_i')\frac{K_i'''}{L_i} + m_i' K_i''', \tag{56}$$

$$M_0 = (\delta_1 - \delta_1')\frac{K_0'''}{L_0} + m_0 K_0'''. \tag{57}$$

If the end joints are free to rotate in torsion, $t_0 = t_0'$ and $t_i = t_i'$, hence the torque in the end posts is zero. When the moments and the torques are known, the secondary stresses can be determined readily.

Symmetry and Anti-symmetry

If the truss is symmetrical about the center-line, and is loaded symmetrically, the buckling mode of the top chord will be either symmetrical or anti-symmetrical. By taking advantage of this fact, the number of unknown joint displacements which must be considered in obtaining the critical load for either the symmetrical or the anti-symmetrical case may be reduced by half. However, it must be recognized that solutions will have to be obtained for both cases since it is not possible, in general, to predict whether the symmetrical or the anti-symmetrical case will be critical.

For a structure having i intermediate joints in the top chord, the number of unknown joint displacements is reduced by applying the following conditions:

For symmetrical buckling:

$$\begin{aligned} \delta_1 &= \delta_i, \ \delta_2 = \delta_{i-1}, \text{ etc.} \\ \alpha_1 &= \alpha_i, \ \alpha_2 = \alpha_{i-1}, \text{ etc.} \\ \beta_1 &= -\beta_i, \ \beta_2 = -\beta_{i-1}, \text{ etc.} \end{aligned} \tag{58}$$

For anti-symmetrical buckling:

$$\begin{aligned} \delta_1 &= -\delta_i, \ \delta_2 = -\delta_{i-1}, \text{ etc.} \\ \alpha_1 &= -\alpha_i, \ \alpha_2 = -\alpha_{i-1}, \text{ etc.} \\ \beta_1 &= \beta_i, \ \beta_2 = \beta_{i-1}, \text{ etc.} \end{aligned} \tag{59}$$

In addition, at the center joint h in a truss having an odd number of top chord joints, the following special conditions are developed:

for symmetrical buckling: $\beta_h = 0,$ (60)

for anti-symmetrical buckling: $\delta_h = \alpha_h = 0.$ (61)

Numerical Example

Critical Buckling Load

As an example of the application of the method of analysis in a practical case, the critical buckling load of the structure shown in fig. 4 will be determined. For this calculation, the critical load is determined in terms of a "safety factor" which is the ratio of the buckling load to the design load. This example was analyzed by Holt [1], [10], who also worked out results by means of procedures advanced by Bleich [11], Chwalla [12], Engesser [13], [14], [15], Schweda [16], Timoshenko [17], and Zimmermann [18], [19], [20].

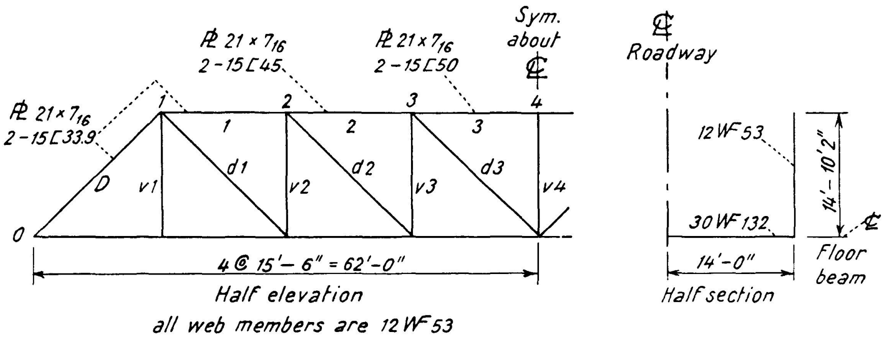

Fig. 4.

Table 1. Design Dimensions and Data, all Quantities in Inches and Kips

Member	0	1	2	3	v_1	d_1	v_2	d_2	v_3	d_3	v_4
L	263	186	186	186	178.5	252	178.5	252	178.5	252	178.5
I	1388	1388	1732	1893	426	426	426	426	426	426	426
J	682	682	765	793	1.434	1.434	1.434	1.434	1.434	1.434	1.434
P	-324	-397	-504	-550	+61	+238	-107	+151.3	-46	+65.1	0
A	29.0	29.0	35.5	38.5	15.59	15.59	15.59	15.59	15.59	15.59	15.59
σ_a	11.18	13.70	14.20	14.28	3.91	15.28	6.86	9.71	2.95	4.18	0

The pertinent dimensions and design data are given in table 1, and the various quantities required to compute the elements of the determinant of eqs. (54) for a safety factor of 2.28 are given in table 2. In obtaining these quantities, the values of the torsional constants J, the effective lengths of the members, and values of the tangent modulus were taken as indicated by Holt so that results of this analysis would be comparable to his. At the end joints, the top chord was assumed to be free to rotate in bending but fixed in torsion. The determinant for the symmetrical case is

$D_s =$

$$\begin{vmatrix}
+3323 & +521800 & -201900 & 0 & -15000 & 0 & 0 & 0 & 0 & 0 & 0 \\
+1203 & -201900 & +516400 & -2372 & 0 & +15900 & 0 & 0 & 0 & 0 & 0 \\
+49.7 & +3323 & +1203 & -20.63 & 0 & +2372 & 0 & 0 & 0 & 0 & 0 \\
0 & -15000 & 0 & +2989 & +392200 & +106900 & 0 & -6120 & 0 & 0 & 0 \\
+2372 & 0 & +159000 & -495 & +106900 & +498100 & -999 & 0 & +78150 & 0 & 0 \\
-20.63 & 0 & -2372 & +57.32 & +2989 & -495 & -4.57 & 0 & +999 & 0 & 0 \\
0 & 0 & 0 & 0 & -6120 & 0 & +2991 & +382900 & +103900 & 0 & -4530 \\
0 & 0 & 0 & +999 & 0 & +78150 & +603 & +103900 & +284800 & -738 & 0 \\
0 & 0 & 0 & -4.57 & 0 & -999 & +37.9 & +2991 & +603 & -1.2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -9060 & 0 & +2141 & +279700 \\
0 & 0 & 0 & 0 & 0 & 0 & -2.4 & 0 & -1476 & +24.89 & +2141
\end{vmatrix}$$

$= -285.7 \times 10^{40}.$

Table 2. Safety Factor = 2.28, all Quantities in Inches and Kips

Member	P	L	σ_a	E	I	ϕ	C	K	K''' or K'	CK	K'''/L or K'/L
0	-739	263	25.6	30,000	1388	1.108	0.533	—	435,000	—	1654
1	-905	186	31.24	10,220	1388	1.484	0.562	283,000	441,000	159,000	2372
2	-1148	186	32.38	3,720	1732	2.48	0.726	107,700	185,800	78,150	999
3	-1253	186	32.56	2,660	1893	2.94	0.889	72,700	137,200	64,600	738
v_1	+1391	178.5	8.91	30,000	426	0.59	0.491	289,000	431,000	—	2415
d_1	+543	252	34.85	0	426	∞	0	0	0	—	0
v_2	-244	178.5	15.64	30,000	426	0.78	0.516	280,000	424,000	—	2372
d_2	+345	252	22.14	30,000	426	1.31	0.461	214,000	313,000	—	1242
v_3	-104.8	178.5	6.73	30,000	426	0.512	0.507	284,000	427,000	—	2390
d_3	+148.4	252	9.53	30,000	426	0.86	0.482	208,000	308,000	—	1222
v_4	0	178.5	0	30,000	426	0	0.5	286,000	429,000	—	2403

Member	K'''/L^2 or K'/L^2	P/L	G	J	R
0	6.29	2.81	12,000	682	31,200
1	12.75	4.87	4,090	682	15,000
2	5.37	6.17	1,488	765	6,120
3	3.96	6.73	1,062	793	4,530
v_1	13.55	0.78	12,000	1.434	338
d_1	0	2.15	0	1.434	0

Member	K'''/L^2 or K'/L^2	P/L	G	J	R
v_2	13.3	1.37	12,000	1.434	275
d_2	4.93	1.37	12,000	1.434	191
v_3	13.4	0.588	12,000	1.434	297
d_3	4.85	0.588	12,000	1.434	168
v_4	13.45	0	12,000	1.434	315

Joint	r	S'_{xx}	S'_{xz}	S'_{yy}	S'_{zz}	S''_{xx}	S''_{xy}	S''_{xz}	S''_{yy}	S''_{yz}	S''_{zz}
1	3.56	273,700	2154	338	23.44	0	0	0	0	0	2.15
2	3.67	264,000	2111	275	20.89	107,100	106,900	878	107,100	878	11.23
3	3.62	268,100	2127	297	21.84	104,100	103,900	864	104,100	864	10.29
4	3.59	270,600	2141	315	22.49	—	—	—	—	—	—

The first three rows of the determinant were given by eqs. (47) to (49) and the fourth to the ninth rows by eqs. (44) to (46). The last two rows were computed from modified equilibrium equations, taking advantage of symmetry about the centerline.

The determinant for the anti-symmetrical case is

$$D_A = \begin{vmatrix} +3323 & +521800 & -201900 & 0 & -15000 & 0 & 0 & 0 & 0 & 0 \\ +1203 & -201900 & +516400 & -2372 & 0 & +159000 & 0 & 0 & 0 & 0 \\ +49.7 & +3323 & +1203 & -20.63 & 0 & +2372 & 0 & 0 & 0 & 0 \\ 0 & -15000 & 0 & +2989 & +392200 & +106900 & 0 & -6120 & 0 & 0 \\ +2372 & 0 & +159000 & -495 & +106900 & +498100 & -999 & 0 & +78150 & 0 \\ -20.63 & 0 & -2372 & +57.32 & +2989 & -495 & -4.57 & 0 & +999 & 0 \\ 0 & 0 & 0 & 0 & -6120 & 0 & +2991 & +382900 & +103900 & 0 \\ 0 & 0 & 0 & +999 & 0 & +78150 & +603 & +103900 & +284800 & +64600 \\ 0 & 0 & 0 & -4.57 & 0 & -999 & +37.9 & +2991 & +603 & +738 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1476 & 0 & +129200 & +145700 \end{vmatrix}$$

$$= +5.791 \times 10^{40}.$$

It will be observed that the elements in D_s are identical to those in D_a with the exception of the tenth column and the tenth row. These elements are obtained from the special conditions associated with anti-symmetrical displacements of the structure.

Following the same process for a load corresponding with a safety factor of 2.30, the values of the determinants are found to be: $D_s = +75.03 \times 10^{40}$ and $D_a = -1.719 \times 10^{40}$. By linear interpolation, the safety factor for anti-symmetrical buckling is 2.295 while that for symmetrical buckling is 2.296. Holt obtained a value of 2.283 for the anti-symmetrical case and 2.285 for the symmetrical case.

Secondary Stresses

As another illustration, the bending stresses in the chord members at a safety factor of 1.5 will be evaluated. The various quantities required to calculate the constants A and B in eq. (54) are given in table 3. It will be noted that initial eccentricities of 0.2″ have been assumed at joints 1 and 3. To simplify the numerical work, the initial eccentricities are assumed to be symmetrical about the center-line of the truss, in which case only one half of the truss need be considered. The effect of the bending of the floor beams, S_x and S_z, was obtained with the assumption that 60% of the panel loads are uniformly distributed along the length of the beams. The resulting system of simultaneous linear equations is

$x_1=\delta_1$	$x_2=\alpha_1$	$x_3=\beta_1$	$x_4=\delta_2$	$x_5=\alpha_2$	$x_6=\beta_2$	$x_7=\delta_3$	$x_8=\alpha_3$	$x_9=\beta_3$	$x_{10}=\delta_4$	$x_{11}=\alpha_4$	B
+4234	+664400	-101700	0	-44000	0	0	0	0	0	0	+1181
+6834	-101700	+1229000	-7160	0	+452000	0	0	0	0	0	+1367
+112.8	+4234	+6834	-73.8	0	+7160	0	0	0	0	0	+29.24
0	-44000	0	+3000	+465400	+104900	0	-49300	0	0	0	+340
+7160	0	+452000	+2660	+104900	+2086000	-8950	0	+564000	0	0	+358
-73.8	0	-7160	+198.1	+3000	+2660	-92.14	0	+8950	0	0	-29.02
0	0	0	0	-49300	0	+3000	+472600	+102900	0	-51200	+937
0	0	0	+8950	0	+564000	+1690	+102900	+2405000	-9780	0	+338
0	0	0	-29.14	0	-8950	+215.4	+2140	+830	-100.8	0	+48.71
0	0	0	0	0	0	0	-102400	0	+2140	+373400	+526
0	0	0	0	0	0	-201.5	0	-19560	+224.1	+2140	-33.25

Table 3. Safety Factor = 1.50, all Quantities in Inches and Kips

Member	P	L	σ_a	E	I	ϕ	C	K	K''' or K'	CK	K'''/L or K'/L
0	-486	263	16.78	30,000	1388	0.899	0.5206	—	449,000	—	1705
1	-595	186	20.55	30,000	1388	0.703	0.5127	881,000	1,332,000	452,000	7160
2	-756	186	21.3	30,000	1732	0.709	0.5130	1,100,000	1,665,000	564,000	8950
3	-825	186	21.4	30,000	1893	0.708	0.5129	1,202,000	1,820,000	616,000	9780
v_1	+91.5	178.5	5.86	30,000	426	0.477	0.4943	288,500	431,000	—	2413
d_1	+357	252	22.9	30,000	426	1.332	0.4594	214,500	313,000	—	1243
v_2	-160.5	178.5	10.3	30,000	426	0.632	0.5103	283,000	427,000	—	2392
d_2	+227	252	14.58	30,000	426	1.062	0.4733	210,000	310,000	—	1230
v_3	-69	178.5	4.42	30,000	426	0.415	0.5043	284,500	428,000	—	2400
d_3	+97.6	252	6.27	30,000	426	0.697	0.4882	206,000	306,500	—	1217
v_4	0	178.5	0	30,000	426	0	0.5000	286,000	429,000	—	2403

Member	K'''/L^2 or K'/L^2	P/L	G	J	R
0	6.49	1.85	12,000	682	31,200
1	38.5	3.20	12,000	682	44,000
2	48.1	4.06	12,000	765	49,300
3	52.6	4.43	12,000	793	51,200
v_1	13.53	0.513	12,000	1.434	330
d_1	4.93	1.417	12,000	1.434	193
v_2	13.4	0.899	12,000	1.434	290
d_2	4.88	0.901	12,000	1.434	178
v_3	13.43	0.387	12,000	1.434	303
d_3	4.83	0.388	12,000	1.434	163
v_4	13.47	0	12,000	1.434	315

Joint	r	S_x	S_z	δ'	S'_{xx}	S'_{xz}	S'_{yy}	S'_{zz}	S''_{xx}	S''_{xy}	S''_{xz}	S''_{yy}	S''_{yz}	S''_{zz}
1	3.56	-334	-5.66	+0.2	373,000	2150	330	23.12	107,300	107,200	879	107,300	879	11.28
2	3.63	-340	-5.62	0	267,000	2130	290	21.53	105,100	104,900	870	105,100	870	10.66
3	3.61	-337	-5.62	+0.2	269,000	2140	303	22.09	103,100	102,900	860	103,100	860	10.05
4	3.59	-526	-8.83	0	271,000	2140	315	22.52	—	—	—	—	—	—

the solution of which is the set of displacements

$x_1 = \delta_1 = +0.4944''$	$x_7 = \delta_3 = +0.7995''$
$x_2 = \alpha_1 = -0.001533$	$x_8 = \alpha_3 = -0.003859$
$x_3 = \beta_1 = +0.0001792$	$x_9 = \beta_3 = +0.0005028$
$x_4 = \delta_2 = +0.4180''$	$x_{10} = \delta_4 = +0.6464''$
$x_5 = \alpha_2 = -0.002820$	$x_{11} = \alpha_4 = -0.003354$
$x_6 = \beta_2 = +0.001339$	

The bending stresses at the ends of the chord members associated with these joint displacements are evaluated in table 4, in which σ_b and σ_b' are respectively the bending stresses at the right end and the left end of the member and Z its section modulus.

Table 4. Determination of Bending Stresses at Safety Factor = 1.50, all Quantities in Inches and Kips

Member	δ_n	δ_{n+1}	K_0'''/L_0 or K_n'/L_n	δ_n'	δ_{n+1}'	m_n	K_0''' or $C_n K_n$	Z	m_n'
0	0	0.4944	1705	0	0.2	-0.00121	449,000	164.2	—
1	0.4944	0.4180	7160	0.2	0	-0.001339	452,000	164.2	0.0001792
2	0.4180	0.7995	8950	0	0.2	-0.0005028	564,000	205	0.001339
3	0.7995	0.6464	9780	0.2	0	0	616,000	224	0.0005028

Member	K_n	M_n	M_n'	σ_b	σ_b'	σ_a
0	—	- 40.6	0	0.247	0	16.78
1	881,000	-376	-122	2.29	0.743	20.55
2	1,100,000	+316	+130.5	1.54	0.636	21.30
3	1,202,000	+148.5	+146	0.724	0.711	21.40

Discussion and Conclusions

Although the effects of initial eccentricities and bending of floor beams were not taken into consideration in the analysis presented by Holt, the close agreement between the value of the buckling load determined by his method and that obtained by the procedure described herein is remarkable. This should be the case since the buckling load is independent of these two effects, a point which was clearly shown by eqs. (54) and (55).

The method of approach presented in this paper is believed to offer certain advantages. The process of computing the stiffness influence coefficients and evaluating the determinants is purely mechanical and can be performed by technicians having no knowledge of the actual structural problem. In addition, this approach permits the determination of secondary stresses due to initial eccentricities and bending of the floor beams at loads less than the buckling load.

That these secondary stresses may not be negligible was demonstrated by the example considered in this paper. Maximum values of the bending stresses at a safety factor of 1.5 amounts to 10 per cent of the axial stress, and it is quite possible that eccentricities larger than those assumed might exist.

It should be noted that the computed buckling load is only an upper boundary of the capacity of the structure in cases where the stresses exceed the proportional limit of the material. Actually, local yielding caused by secondary stresses due to lateral as well as plane deformation of the trusses occurs before this load is reached. This factor was not considered in the analysis. The tangent moduli used in evaluating the critical load were computed on the basis of the axial stresses only.

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Appendix A: Stiffnesses of the Cross-Frames

By definition, S'_{xz}, S'_{yz} and S'_{zz} are the forces acting at the joint to produce the displacement $\delta = 1$, $\alpha = \beta = 0$. First give the two joints n and n' unit deflec-

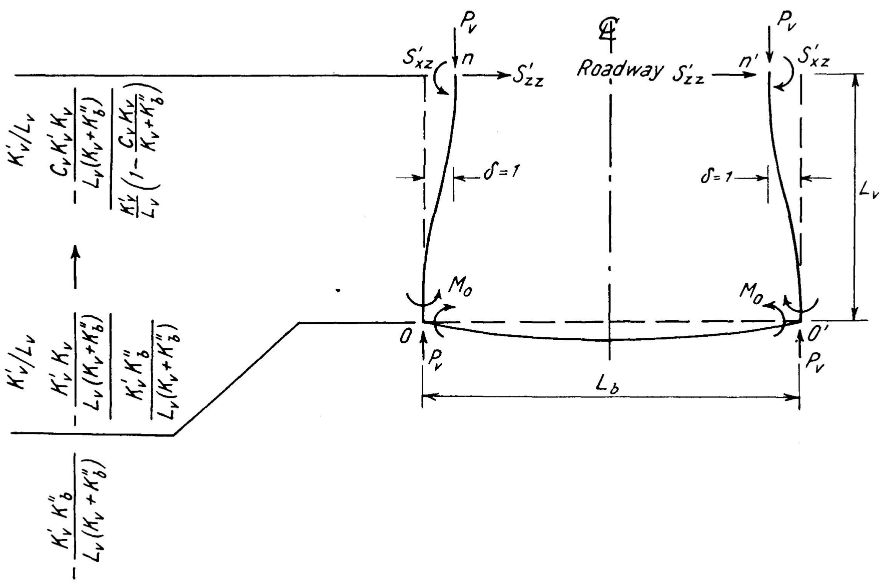

Fig. 5. Distribution of Moments Due to $\delta = 1$, $\alpha = \beta = 0$.

tion along z-axis with all joints, including n and n', locked against rotation (fig. 5). The fixed-end moments thus produced in the verticals are K'_v/L_v. Next unlock joint 0 and 0′, leaving joints n and n' locked as before. The distribution of moments is shown in the figure. Thus

$$S'_{xz} = \frac{K'_v}{L_v}\left(1 - \frac{C_v K_v}{K_v + K''_b}\right) = \frac{K'_v}{L_v}\left(1 - \frac{C_v}{1+r}\right),$$

$$S'_{zz} = \frac{1}{L_v}(S'_{xz} + M_0 \pm P_v) = \frac{K'_v}{L_v^2}\left(2 - \frac{1+C_v}{1+r}\right) \pm \frac{P_v}{L_v}.$$

Assuming that the cross-section of the verticals is symmetrical about its principal axes, $S'_{yz} = 0$.

The other stiffnesses are determined in similar manner.

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Summary

A general method of analysis of the stability of pony-truss bridges which takes into consideration many secondary effects is presented. The method lends itself not only to the solution of the buckling problem, but also to the determination of the secondary stresses developed in the members under the action of loads less than the critical load. The procedure consists of the formulation of a system of simultaneous linear equations derived by consideration of the equilibrium conditions at the joints along the top chord. The solution of these equations yields the displacement components of the joints from which the secondary stresses can be calculated. The buckling condition is obtained by equating to zero the determinant of the system of equations. Numerical examples are given to illustrate the application of the method to a practical problem.

Résumé

Les auteurs exposent une méthode générale d'étude de la stabilité dans les ponts en treillis ouverts à la partie supérieure, en tenant compte de diverses influences secondaires.

Cette méthode permet de déterminer non seulement la charge critique de flambage, mais aussi les contraintes secondaires dans les barres du treillis sous les charges inférieures à la valeur critique. Elle consiste à établir un système d'équations linéaires simultanées, résultant des considérations d'équilibre, pour la membrure supérieure. La solution de ce système donne les composantes de déplacement des nœuds, à partir desquelles il est possible de calculer les contraintes secondaires. La condition de flambage est obtenue par annulation du déterminant de ce système d'équations. Quelques exemples numériques mettent en évidence l'application de cette méthode à un problème pratique.

Zusammenfassung

Es wird hier eine allgemeine Methode der Stabilitätsuntersuchung bei oben offenen Fachwerkbrücken mit Berücksichtigung verschiedener Nebeneinflüsse angegeben.

Mit dieser Methode können nicht nur die kritische Knicklast, sondern auch die Nebenspannungen in den Fachwerkstäben bei unterkritischen Lasten bestimmt werden. Das Vorgehen besteht in der Formulierung eines Systems von simultanen linearen Gleichungen, die aus Gleichgewichtsbetrachtungen am Obergurt entstehen. Die Lösung dieses Gleichungssystems ergibt die Verschiebungskomponenten der Knoten, aus welchen dann die Nebenspannungen berechnet werden können. Man erhält die Knickbedingung durch Nullsetzen der Determinante dieses Gleichungssystems. Einige numerische Beispiele veranschaulichen die Anwendung dieser Methode an einem praktischen Problem.