Zeitschrift: IABSE publications = Mémoires AIPC = IVBH Abhandlungen

Band: 18 (1958)

Artikel: Stability of rib-reinforced cylindrical shells under lateral pressure

Autor: Moe, Johannes

DOI: https://doi.org/10.5169/seals-16510

Stability of Rib-Reinforced Cylindrical Shells Under Lateral Pressure

Stabilité sous charge latérale des voiles cylindriques avec nervures de renforcement

Stabilität von zylindrischen Schalen mit Verstärkungsrippen unter seitlicher Belastung

JOHANNES MOE, Lic. techn., Trondheim, Norway. At present: Visiting Engineer, Portland Cement Association, Chicago, Ill., U.S.A.

1. Introduction

It is the purpose of the present paper to investigate the stiffening effect of ribs on cylindrical shells under purely lateral pressure.

During the last decades several authors have devoted a great amount of theoretical and experimental work to the problem of the buckling of cylindrical shells.

When considering the special case of shells subject to lateral pressure, some of the earliest theoretical investigations are due to LORENZ [1], SOUTHWELL [2, 3] and v. MISES [4].

Valuable contributions were also made by SANDEN, TØLKE [5] and by FLÜGGE [6].

WINDENBURG and TRILLING [7] compared various instability formulas and results obtained at the U.S. Experimental Model Basin.

Further theoretical and experimental investigations were accomplished by R. G. STURM [8].

Most of the earlier investigators arrived at a set of three homogeneous partial differential equations relating the three displacement components of the median surface of the shell, which expressed the buckling condition.

In 1934, however, DONNELL [9] succeeded in establishing one differential equation of the eighth order which expressed entirely the buckling condition. The derivation was based upon certain approximations which in most cases are of very little importance. Certain limitations of the equation will be discussed in Section 12.

On the basis of Donnell's equation Batdorf [10] demonstrated that the small deflection buckling theory may be considerably simplified.

The experimental works reported in the papers mentioned above are carried out on metallic cylinders. The only experiments known to the author, which have been performed on reinforced concrete shells failing by instability, are due to Lundgren [11] who tested five short shell roof models under lateral pressure.

Most of the papers mentioned deal with buckling problems of isotropic shells. In a recent report Stein, Sanders and Crate [12] have demonstrated that Donnell's equation in a modified form may be applied to shells stiffened by ribs and loaded with pure shear.

The present paper investigates in a similar manner the buckling of stiffened cylindrical shells subject to lateral pressure.

2. Symbols

L	length of shell.
b	width of curved panel.
r	radius of shell.
t	thickness of shell.
q	number of stiffeners.
d	stiffener spacing $\left(\frac{L}{q+1}\right)$.
c	width of stiffener
h	height of stiffener.
A	area of stiffener $c(h-t)$.
Z	shell parameter $\left(\frac{L^2}{r\,t}\sqrt{1-\mu^2}\right)$.
D	flexural rigidity of shell $\left(\frac{E\,t^3}{12\,(1-\mu^2)}\right)$.
$E\,J$	flexural rigidity of stiffener.
E	Young's modulus of elasticity.
μ	Poisson's ratio.
γ	ratio of rib stiffness to shell stiffness $\left(\frac{E\,J}{D\,d}\right)$.
α	ratio of rib area to shell area $\left(\frac{c\,(h-t)}{t\,d}\right)$.
λ	distance between nodes measured circumferentially.
β	wave length ratio $\left(\frac{L}{\lambda}\right)$.
x	axial coordinate.
y	circumferential coordinate.
u	axial displacement of the median surface during buckling.

v	circumferential displacement of the median surface during buckling.
w	radial displacement of the median surface during buckling, positive inward.
a_m, B_{mp}, C_m, D_m	coefficients.
ϵ_y	elastic strain in the circumferential direction.
κ_y	change in curvature in the circumferential direction.
p	lateral pressure, positive inward.
σ_y	applied circumferential stress, positive in tension.
σ_x	applied axial stress, positive in tension.
τ	applied shear stress.
$\sigma'_y, \sigma'_x, \tau'$	corresponding additional stresses due to buckling.
N_y	applied circumferential force, positive in tension.
N_x	applied axial force, positive in tension.
N_{xy}	applied shear force.
N'_y, N'_x, N'_{xy}	additional forces due to buckling.
M'_y, M'_x, M'_{xy}	additional moments due to buckling.
Q'_x, Q'_y	additional transverse shear forces due to buckling.
k_y	stress parameter $\left(-\frac{\sigma_y t L^2}{D \pi^2}\right)$.
k_x	stress parameter $\left(-\frac{\sigma_x t L^2}{D \pi^2}\right)$.
F	Airy's stress function for the additional median surface stresses produced during buckling. $\left(\sigma'_y = \frac{\partial^2 F}{\partial x^2}, \quad \sigma'_x = \frac{\partial^2 F}{\partial y^2}, \quad \tau' = -\frac{\partial^2 F}{\partial x \, \partial y}\right).$
U_e	strain energy in ribs due to extensional stresses produced during buckling.
U_b	strain energy in ribs due to bending stresses produced during buckling.
η	relative importance of the extensional forces $\left(\frac{U_e}{U_b}\right)$.
$f(y), g_m(x)$	deflection functions.
$Q()$	differential operator.
∇^2	Laplace operator $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.
$\nabla^4 = \nabla^2 \cdot \nabla^2$	$\nabla^8 = \nabla^4 \nabla^4$.
∇^{-4}	inverse operator defined by the equation $\nabla^{-4}(\nabla^4 f) = f$.
$\delta(x - i d)$	delta function defined by the equation.

$$\int_{-\infty}^{+\infty} f(x)\, \delta(x - i d)\, dx = f(i d),$$

$$\delta_{qmp} = \frac{2}{q+1} \sum_{i=1}^{q} \sin \frac{m \pi i}{q+1} \sin \frac{p \pi i}{q+1}.$$

3. Assumptions

The derivations below are based upon the following assumptions:

1. The ribs are placed symmetrically with respect to the median surface of the shell, as shown in fig. 1.
2. The torsional stiffness of the ribs may be neglected, and the ribs are acting along lines in the median surface of the shell.
3. Ribs as well as shell are in a state of uniform compression before buckling.
4. The shell is made of a perfectly elastic material.
5. Plane sections perpendicular to the median surface of the shell remain plane and normal to the deformed median surface.
6. Thickness of shell is constant during buckling.
7. Shell thickness is small compared to radius of curvature and length of shell.
8. Deflections are small compared to radius of curvature and length of shell.

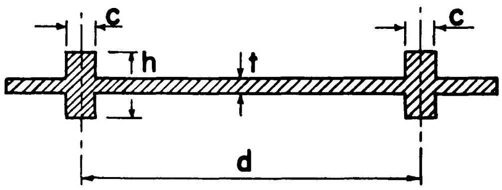

Fig. 1. Section Through the Shell.

Regarding the first assumption, it is common practice to provide the shell with ribs on one side only. In such cases it is probably correct to include the effective width of the shell when calculating the rigidity of the ribs.

The torsional rigidity of the ribs is of little importance if the shell buckles into one half wave in the longitudinal direction. If, however, the panels between the ribs buckle separately, the second assumption involves some inaccuracy on the safe side.

Assumptions 5 to 8 are well known as the basis of the "small deflection" buckling theory.

4. Basic Equations

In this section the Donnell-equation, which is the basis of the further derivations, is established.

The reader should note that for the present problem we are interested only in the deflections u, v and w that develop during buckling. Deflections are therefore measured from the position of the deformed surface just before buckling starts.

Correspondingly, we are interested in the *additional* forces and moments produced in the shell during buckling. Those quantities are denoted, N'_x, N'_y, Q'_x, M'_x, etc.

DONNELL [9] introduced the following simplified relations between forces, moments and deflections of the median surface of the shell:

$$
\begin{aligned}
N'_x &= \frac{E\,t}{1-\mu^2}\left[\frac{\partial u}{\partial x}+\mu\left(\frac{\partial v}{\partial y}-\frac{w}{r}\right)\right],\\
N'_y &= \frac{E\,t}{1-\mu^2}\left(\frac{\partial v}{\partial y}-\frac{w}{r}+\mu\frac{\partial u}{\partial x}\right),\\
N'_{xy} &= N'_{yx} = \frac{E\,t}{2(1+\mu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\\
M'_x &= -D\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right),\\
M'_y &= -D\left(\mu\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right),\\
M'_{xy} &= M'_{yx} = -D\,(1-\mu)\frac{\partial^2 w}{\partial x\,\partial y}.
\end{aligned}
\tag{4.1}
$$

Eqs. (4.1) are the same as those of the general bending theory established by Donnell.

The same equations are used by JENKINS [13]. HOLAND [14] and MOE [16] have shown that for most practical cases the Donnell-theory is of sufficient accuracy.

Before buckling starts, the shell is assumed to be in a membrane state of stresses, expressed by the equations:

$$
\frac{\partial N_x}{\partial x}+\frac{\partial N_{yx}}{\partial y}=0,\qquad \frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}=0,\qquad \frac{1}{r}N_y+p=0. \tag{4.2}
$$

Eqs. (4.2) express a stress condition which exists in a long cylindrical tube loaded with constant axial load and twist at the ends, and constant lateral pressure.

When the membrane forces reach the critical value, buckling starts. Due to the buckling, secondary forces and moments arise, and the equations of equilibrium take the following form:

$$
\begin{aligned}
&\frac{\partial (N_x+N'_x)}{\partial x}+\frac{\partial (N_{yx}+N'_{yx})}{\partial y}=0,\\
&\frac{\partial (N_y+N'_y)}{\partial y}+\frac{\partial (N_{xy}+N'_{xy})}{\partial x}=0.\\
&\frac{\partial Q'_x}{\partial x}+\frac{\partial Q'_y}{\partial y}+\frac{1}{r}(N_y+N'_y)+p+N_y\frac{\partial^2 w}{\partial y^2}+N_x\frac{\partial^2 w}{\partial x^2}+2\,N_{xy}\frac{\partial^2 w}{\partial x\,\partial y}=0,\\
&\frac{\partial M'_x}{\partial x}+\frac{\partial M_{yx}}{\partial y}-Q'_x=0,\\
&\frac{\partial M'_y}{\partial y}+\frac{\partial M'_{xy}}{\partial x}-Q'_y=0,
\end{aligned}
\tag{4.3}
$$

where $N'_x N'_y$ etc. denotes the secondary forces.

Only the third of eqs. (4.3) differs appreciably from the corresponding equation in the bending theory. This equation expresses the equilibrium of forces in the radial direction.

The last two terms take into account the radial components of N_x, N_{xy} and N_{yx} due to the distortion of the shell element. The term $N_y \frac{\partial^2 w}{\partial y^2}$ correspondingly takes into account the increased radial component of N_y due to the change of angle $\left(\frac{\partial^2 w}{\partial y^2}\right)$ between the faces of the shell element.

The secondary forces N'_x N'_y are of an order of magnitude smaller than N_x, N_y and N_{xy}. This is the reason why the secondary effects of N_x, N_y and N_{xy} must be taken into account when establishing the equations of equilibrium.

There are several other secondary terms which enter into the equations of equilibrium (see for inst. [15]), but those terms are of minor importance and are not considered here.

Combining eqs. (4.2) and (4.3) one obtains:

$$
\begin{gathered}
\frac{\partial N'_x}{\partial x} + \frac{\partial N'_{yx}}{\partial y} = 0, \qquad \frac{\partial N'_y}{\partial y} + \frac{\partial N'_{xy}}{\partial x} = 0, \\
\frac{\partial Q'_x}{\partial x} + \frac{\partial Q'_y}{\partial y} + \frac{1}{r}(N_y + N'_y) + N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \, \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + p = 0.
\end{gathered} \tag{4.4}
$$

It is convenient at this point to introduce Airy's stress function F, defined by the following equations:

$$
\frac{\partial^2 F}{\partial y^2} = \frac{1}{t} N'_x, \qquad \frac{\partial^2 F}{\partial x^2} = \frac{1}{t} N'_y, \qquad \frac{\partial^2 F}{\partial x \, \partial y} = -\frac{1}{t} N'_{xy}. \tag{4.5}
$$

The first two of eqs. (4.4) are then identically satisfied. From the last two of eqs. (4.3) one further obtains

$$
Q'_x = \frac{\partial M'_x}{\partial x} + \frac{\partial M'_{yx}}{\partial y}, \qquad Q'_y = \frac{\partial M'_y}{\partial y} + \frac{\partial M'_{xy}}{\partial x}. \tag{4.6}
$$

Substituting expressions (4.1) for M'_x, M'_y and M'_{xy} into eqs. (4.6) yields

$$
Q'_x = -D\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \, \partial y^2}\right), \qquad Q'_y = -D\left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \, \partial y}\right). \tag{4.7}
$$

These values for Q'_x and Q'_y are substituted in the last of eqs. (4.4), yielding the following expression

$$
D \nabla^4 w - t\left(\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x \, \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2} + \frac{\sigma_y}{r} + \frac{1}{r}\frac{\partial^2 F}{\partial x^2}\right) - p = 0. \tag{4.8}
$$

Eq. (4.8) contains two unknown quantities F and w. To solve the problem, another equation relating these unknowns must be found.

This is obtained by the elimination of v and u between the first three of eqs. (4.1). Thus:

$$\begin{aligned}
N'_x - \mu N'_y &= t\left(\frac{\partial^2 F}{\partial y^2} - \mu\frac{\partial^2 F}{\partial x^2}\right) = E t\frac{\partial u}{\partial x},\\
N'_y - \mu N'_x &= t\left(\frac{\partial^2 F}{\partial x^2} - \mu\frac{\partial^2 F}{\partial y^2}\right) = E t\left(\frac{\partial v}{\partial y} - \frac{w}{r}\right),\\
\frac{\partial^2 N'_{xy}}{\partial x\,\partial y} &= -t\frac{\partial^4 F}{\partial x^2\,\partial y^2} = \frac{E t}{2(1+\mu)}\left(\frac{\partial^3 u}{\partial x\,\partial y^2} + \frac{\partial^3 v}{\partial x^2\,\partial y}\right).
\end{aligned} \tag{4.9}$$

Solving for $\frac{\partial u}{\partial x}$ from the first equation and for $\frac{\partial v}{\partial y}$ from the second and differentiating twice with respect to y and x respectively, one may eliminate u and v by substitution in the third of the equations. Thus, the following expression for F is obtained

$$\nabla^4 F = -\frac{E}{r}\frac{\partial^2 w}{\partial x^2}. \tag{4.10}$$

Eqs. (4.8) and (4.10) yield one simple differential equation of the eighth order:

$$D\nabla^8 w + \frac{E t}{r^2}\frac{\partial^4 w}{\partial x^4} - t\nabla^4\left[\sigma_x\frac{\partial^2 w}{\partial x^2} + 2\tau\frac{\partial^2 w}{\partial x\,\partial y} + \sigma_y\frac{\partial^2 w}{\partial y^2} + \frac{\sigma_y}{r}\right] - \nabla^4 p = 0. \tag{4.11}$$

Eq. (4.11) is the complete Donnell-equation. As proposed by BATDORF [10], the equation will be used in the following modified form:

$$D\nabla^4 w + \frac{E t}{r^2}\nabla^{-4}\frac{\partial^4 w}{\partial x^4} - t\left[\sigma_x\frac{\partial^2 w}{\partial x^2} + 2\tau\frac{\partial^2 w}{\partial x\,\partial y} + \sigma_y\frac{\partial^2 w}{\partial y^2} + \frac{\sigma_y}{r}\right] - p = 0. \tag{4.12}$$

In the discussion of the boundary conditions it will be necessary to know u and v in terms of w. It is convenient, therefore, to deduce these relationships here.

From the first of eqs. (4.9) it is found that

$$E\frac{\partial}{\partial x}\nabla^4 u = \frac{\partial^2}{\partial y^2}\nabla^4 F - \mu\frac{\partial^2}{\partial x^2}\nabla^4 F, \tag{4.13}$$

which together with eq. (4.10) yields

$$\frac{\partial}{\partial x}\nabla^4 u = \frac{1}{r}\left[\mu\frac{\partial^4 w}{\partial x^4} - \frac{\partial^4 w}{\partial x^2\,\partial y^2}\right]. \tag{4.14}$$

Similarly, from the second of eqs. (4.9):

$$E\frac{\partial}{\partial y}\nabla^4 v = \frac{\partial^2}{\partial x^2}\nabla^4 F - \mu\frac{\partial^2}{\partial y^2}\nabla^4 F + \frac{E}{r}\nabla^4 w, \tag{4.15}$$

which, together with eq. (4.10) yields

$$\frac{\partial}{\partial y}\nabla^4 v = \frac{1}{r}\left[\mu\frac{\partial^4 w}{\partial x^2\,\partial y^2} - \frac{\partial^4 w}{\partial x^4} + \nabla^4 w\right]. \tag{4.16}$$

By integrating eqs. (4.14) and (4.16) with respect to x and y respectively, the following relationships are found:

$$\begin{aligned} \nabla^4 u &= \frac{1}{r}\left[\mu \frac{\partial^3 w}{\partial x^3} - \frac{\partial^3 w}{\partial x\, \partial y^2}\right], \\ \nabla^4 v &= \frac{1}{r}\left[(2+\mu)\frac{\partial^3 w}{\partial x^2\, \partial y} + \frac{\partial^3 w}{\partial y^3}\right]. \end{aligned} \tag{4.17}$$

5. Derivation of the Differential Equation for a Shell Reinforced by Ring Ribs

In the case of a shell stiffened by ring ribs the expression (4.1) for M_y' must have a correction to account for the effect of the ribs.

For a curved bar with circular axis the following relation exists between moment and radial displacement [15]:

$$\frac{d^2 w}{d y^2} + \frac{w}{r^2} = -\frac{M}{E J}. \tag{5.1}$$

The second term on the left side of eq. (5.1) may be neglected in the present derivation. A corresponding term has been omitted in the fifth one of eqs. (4.1). The effect of this simplification is discussed in Section 12.

Hence one may take into consideration the stiffening effect of the ribs by the following expression:

$$M'_{y\,rib} = -E J \sum_{i=1}^{q} \delta(x - i d) \frac{\partial^2 w}{\partial y^2}, \tag{5.2}$$

where $\sum_{i=1}^{q} \delta(x-id)$ is a delta-function which is zero between the ribs, and equals one along the lines of action of the ribs, and q is the number of ribs.

Combining now eq. (5.2) and expression (4.1) for M_y' one obtains

$$M_y' = -D\left[\left(\mu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \gamma d \sum_{i=1}^{q} \delta(x - i d) \frac{\partial^2 w}{\partial y^2}\right], \tag{5.3}$$

where
$$\gamma = \frac{E J}{D d},$$

d = distance between ribs.

Eq. (5.3) introduced into the last of eqs. (4.6) yields:

$$Q_y' = -D\left[\left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2\, \partial y}\right) + \gamma d \sum_{i=1}^{q} \delta(x - i d) \frac{\partial^3 w}{\partial y^3}\right]. \tag{5.4}$$

The circumferential force N_y just before buckling starts has the following value:

$$N_y = \sigma_y t\left[1 + \alpha d \sum_{i=1}^{q} \delta(x - i d)\right]. \tag{5.5}$$

Introducing now the modified expression (5.4) and N_y from eq. (5.5) into the last of eqs. (4.4), the following differential equation is obtained

$$\begin{aligned} &\nabla^4 w + \gamma d \sum_{i=1}^{q} \delta(x - i d) \frac{\partial^4 w}{\partial y^4} - \frac{t}{D}\left[\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2}\right. \\ &\left. + \alpha d \sigma_y \sum_{i=1}^{q} \delta(x - i d) \frac{\partial^2 w}{\partial y^2} + \frac{\sigma_y}{r}[1 + \alpha d \sum_{i=1}^{q} \delta(x - i d)] + \frac{1}{r} \frac{\partial^2 F}{\partial x^2}\right] - \frac{p}{D} = 0. \end{aligned} \tag{5.6}$$

In the case of constant radial pressure one must have

$$p r = -t \sigma_y [1 + \alpha d \sum_{i=1}^{q} \delta(x - i d)], \tag{5.7}$$

if edge effects are neglected.

Thus eq. (5.6) can be simplified as follows:

$$\begin{aligned} &\nabla^4 w + \frac{12 Z^2}{L^4} \nabla^{-4} \frac{\partial^4 w}{\partial x^4} + \gamma d \sum_{i=1}^{q} \delta(x - i d) \frac{\partial^4 w}{\partial y^4} \\ &- \frac{t}{D}\left[\sigma_x \frac{\partial^2 w}{\partial x^2} + \sigma_y [1 + \alpha d \sum_{i=1}^{q} \delta(x - i d)] \frac{\partial^2 w}{\partial y^2} + 2\tau \frac{\partial^2 w}{\partial x \partial y}\right] = 0, \end{aligned} \tag{5.8}$$

after elimination of F by means of eq. (4.10).

Note that the term taking into account additional axial forces in the ribs due to buckling is neglected. The inclusion of these additional forces would lead to a more complicated differential equation. The importance of this simplification is discussed in Section 10.

In the further discussions it is assumed that $\tau = 0$. Buckling due to pure shear is discussed by STEIN, SANDERS and CRATE [12].

6. Solution of the Differential Equation

It is now assumed that the shape of the buckled surface of the shell can be expressed by a series of the following form

$$w = \sin \frac{n \pi y}{b} \sum_{m=1}^{\infty} a_m \sin \frac{m \pi x}{L} = f(y) \sum_{m=1}^{\infty} a_m g_m(x), \tag{6.1}$$

where b is the circumferential width of the panel under consideration, and n is the number of half waves in the circumferential direction. If the expression (6.1) for w is introduced into eq. (5.8) one cannot find the solution in the general way by considering one term of the deflection function separately. This is due to the delta-functions appearing in the equation.

Eq. (5.8) will be solved applying the GALERKIN method. In this method it is assumed that certain weighted averages of the left hand side of eq. (5.8) vanish, instead of the equation itself.

The deflection function (6.1) also acts as a weighing function.

The coefficients a_m should therefore be determined from the following set of simultaneous equations.

$$\sum_{m=1}^{\infty} B_{mp}\, a_m = 0, \qquad p = 1.2\ldots \tag{6.2}$$

where

$$\begin{aligned} B_{mp} &= \int_0^b \int_0^L f(y)\, g_p(x)\, Q\,[f(y), g_m(x)]\, d\,x\, d\,y \\ &= \int_0^b \int_0^L \sin\frac{n\,\pi\,y}{b} \sin\frac{p\,\pi\,x}{L}\, Q\,[f(y), g_m(x)]\, d\,x\, d\,y\,. \end{aligned} \tag{6.3}$$

$Q\,(w)$ denotes the following differential operator (see eq. (5.8)).

$$\left\{ \nabla^4 + \frac{12\,Z^2}{L^4} \nabla^{-4} \frac{\partial^4}{\partial\,x^4} + \gamma\, d \sum_{i=1}^{q} \delta\,(x - i\,d) \frac{\partial^4}{\partial\,y^4} - \frac{t}{D}\left[\sigma_x \frac{\partial^2}{\partial\,x^2} + \sigma_y\,[1 + \alpha\, d \sum_{i=1}^{q} \delta\,(x - i\,d)] \frac{\partial^2}{\partial\,y^2}\right]\right\} w\,.$$

If the integrations (6.3) are carried out, one obtains the following set of homogeneous equations from which a_m should be determined:

$$\left[(p^2+\beta^2)^2 + \frac{12\,Z^2}{\pi^4} \frac{p^4}{(p^2+\beta^2)^2} - p^2\,k_x - \beta^2\,k_y\right] a_p + (\gamma\,\beta^4 - \alpha\,\beta^2\,k_y) \sum_{m=1}^{\infty} a_m\,\delta_{q\,mp} = 0\,, \tag{6.4}$$

where

$$\beta = \frac{L}{\lambda} = n\frac{L}{b}, \tag{6.5}$$

$$k_x = -\frac{\sigma_x\, t\, L^2}{D\,\pi^2}, \qquad k_y = -\frac{\sigma_y\, t\, L^2}{D\,\pi^2}, \tag{6.6}$$

$$\delta_{q\,mp} = \frac{2}{q+1} \sum_{i=1}^{q} \sin\frac{m\,\pi\,i}{q+1} \sin\frac{p\,\pi\,i}{q+1}\,. \tag{6.7}$$

$\delta_{q\,mp}$ takes the following values:

$$\begin{aligned} \delta_{q\,mp} &= +1 \text{ if } p - m \text{ is a multiple of } 2\,(q+1), \\ \delta_{q\,mp} &= -1 \text{ if } p + m \text{ is a multiple of } 2\,(q+1), \\ \delta_{q\,mp} &= 0 \text{ if neither or both are true.} \end{aligned}$$

The following notations are introduced:

$$\begin{aligned} M_p &= (p^2+\beta^2)^2 + \frac{12\,Z^2}{\pi^4} \frac{p^4}{(p^2+\beta^2)^2} - p^2\,k_x - \beta^2\,k_y\,, \\ R &= \gamma\,\beta^4 - \alpha\,\beta^2\,k_y\,. \end{aligned} \tag{6.8}$$

Eq. (6.4) then takes the form:

$$M_p\,a_p + R \sum_{m=1}^{\infty} a_m\,\delta_{q\,mp} = 0\,. \qquad p = 1, 2, \ldots \tag{6.9}$$

A buckled shape of the shell can only exist if the determinant of the coefficients of eq. (6.9) is zero.

This determinant of the coefficients is infinite. The first seven rows and seven columns in the upper left hand corner of the determinants for the two special cases $q=1$ and $q=3$, are shown here:

$q=1$:

$$\begin{vmatrix} M_1+R & 0 & -R & 0 & +R & 0 & -R \\ 0 & M_2 & 0 & 0 & 0 & 0 & 0 \\ -R & 0 & M_3+R & 0 & -R & 0 & +R \\ 0 & 0 & 0 & M_4 & 0 & 0 & 0 \\ +R & 0 & -R & 0 & M_5+R & 0 & -R \\ 0 & 0 & 0 & 0 & 0 & M_6 & 0 \\ -R & 0 & +R & 0 & -R & 0 & M_7+R \end{vmatrix} = 0. \quad (6.10)$$

$q=3$:

$$\begin{vmatrix} M_1+R & 0 & 0 & 0 & 0 & 0 & -R \\ 0 & M_2+R & 0 & 0 & 0 & -R & 0 \\ 0 & 0 & M_3+R & 0 & -R & 0 & 0 \\ 0 & 0 & 0 & M_4 & 0 & 0 & 0 \\ 0 & 0 & -R & 0 & M_5+R & 0 & 0 \\ 0 & -R & 0 & 0 & 0 & M_6+R & 0 \\ -R & 0 & 0 & 0 & 0 & 0 & M_7+R \end{vmatrix} = 0. \quad (6.11)$$

The determinants are extremely simple. The determinant (6.11) for $q=3$ will now be discussed and the significance of the various terms shown.

Firstly, it is observed that in the fourth row and fourth column there is only one term that is different from zero, which can therefore be put outside the determinant, yielding:

$$M_4 \begin{vmatrix} M_1+R & 0 & 0 & 0 & 0 & -R \\ 0 & M_2+R & 0 & 0 & -R & 0 \\ 0 & 0 & M_3+R & -R & 0 & 0 \\ 0 & 0 & -R & M_5+R & 0 & 0 \\ 0 & -R & 0 & 0 & M_6+R & 0 \\ -R & 0 & 0 & 0 & 0 & M_7+R \end{vmatrix} = 0. \quad (6.12)$$

Correspondingly, all terms $M_{(q+1)i}$ can be put outside the determinant for $i=1,2,3\ldots$ These terms correspond to buckling patterns with nodal lines along the ribs, as shown in fig. 2.

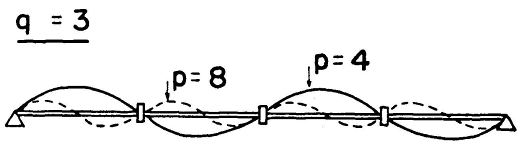

Fig. 2. Buckling Between the Ribs.

Next, the reduced determinant is considered.

As the small deflection theory which is fundamental for the present paper, is insufficient for cylindrical shells in axial compression, the following discussion is limited to the case when $\sigma_x = 0$.

7. Buckling under Lateral Pressure

The terms $M_1, M_7, M_{13} \ldots M_{(2iq+1)}$ (where $i = 0, 1, 2 \ldots$) in (6.12) constitute a determinant which is independent of the other terms. Fig. 3 shows the deformations corresponding to M_1 and M_7.

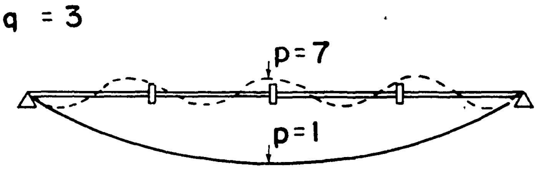

Fig. 3. Buckling Across the Ribs.

In the case of pure lateral pressure it is known that the equations containing M_1 correspond to the lowest eigenvalue of the system. This eigenvalue can therefore be determined from the very simple determinant

$$M_4 \begin{vmatrix} M_1 + R & -R \\ -R & M_7 + R \end{vmatrix} = 0 \tag{7.1}$$

or in general form

$$M_{q+1} \begin{vmatrix} M_1 + R & -R \\ -R & M_{2q+1} + R \end{vmatrix} = 0, \tag{7.2}$$

which reduces to the equation

$$M_{q+1} \left[M_1 + R \left(1 + \frac{M_1}{M_{2q+1}} \right) \right] = 0. \tag{7.3}$$

Also the determinants (7.1) and (7.2) are really infinite, but numerical calculations show that it is quite accurate enough to consider only the first two terms.

In fact, when $q > 1$, also the correcting term $\frac{M_1}{M_{2q+1}}$ is of very little importance, and one arrives at the following equation:

$$M_{q+1} [M_1 + R] = 0 \qquad (\text{when } q > 1). \tag{7.4}$$

The solution

$$M_1 + R = 0,$$

which by introduction of expressions (6.8) reduces to

$$(1 + \alpha) k_y = \frac{(1 + \beta^2)^2}{\beta^2} + \frac{12 Z^2}{\pi^4} \frac{1}{\beta^2 (1 + \beta^2)^2} + \gamma \beta^2, \tag{7.5}$$

determines the buckling load as long as this is below the value which corresponds to $M_{q+1}=0$. The latter solution expresses the condition under which the shell between two consecutive ribs buckles independently of the rest of the structure.

The buckling criteria presented above contain a parameter β which expresses the wave length ratio. When numerical calculations are carried out, the buckling load must be minimized with respect to β. This is done by trial and error.

In fig. 4 curves are given which show the relation between the stress parameter $(1+\alpha)\,k_y$ and the stiffness ratio γ for various values of the curvature parameter Z. The curve marked $Z=0$ applies to the flat plate. The dashed curves will be discussed later on. They apply only when $q=1$,

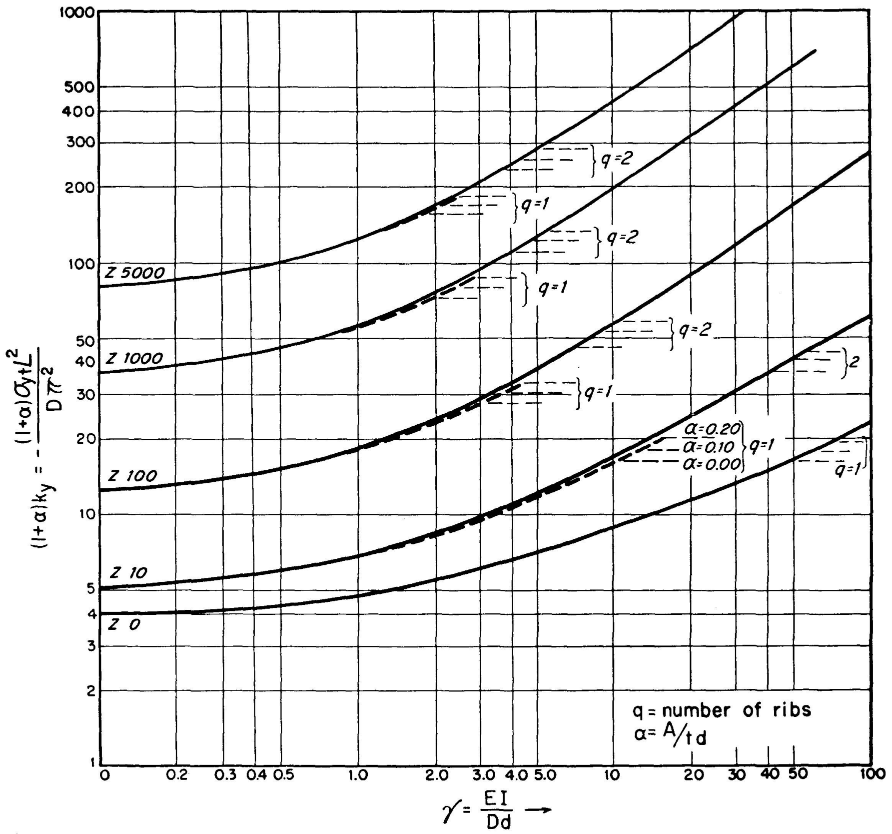

Fig. 4. Influence of Rib Stiffness on the Buckling Stresses of Plates and Shells.

When minimizing the buckling load, β has been assumed to vary continuously. This will give buckling loads which are a little too low when the buckling pattern, which corresponds to the found value of β, does not agree with boundary conditions and dimensions of the shell. However, the discrepancies are in most cases of very little importance and on the safe side. The buckling load varies very little with variations in β in the neighbourhood of the value at which buckling starts. In fig. 5 the variations of β with γ will be found for various values of Z.

The terms in the buckling determinant (6.12) which have not been considered, correspond to higher eigenvalues and will not be discussed further in the present paper. If the shell is also loaded axially, those higher buckling modes may under certain circumstances become important and should not be neglected.

To facilitate the interpolation for various values of Z, a new set of buckling curves is presented in fig. 5. In that figure the straight lines corresponding to the equations $M_{q+1} = 0$ have been omitted.

The panel between two consecutive ribs should therefore be treated as an isotropic shell, and the safety against local buckling of the panel is found by using the curves in fig. 6, which are drawn from eq. (7.5) after putting $\alpha\gamma = 0$. In this case one must of course introduce the distance between two consecutive ribs $\left(\frac{L}{q+1}\right)$ as the length of the shell.

Fig. 7 shows the values of γ at which the safety against buckling for the complete structure equals the safety against individual buckling of the panels.

As previously mentioned, the third term in the series (6.1) for w is of some importance in the case of one single rib ($q = 1$). The buckling pattern in this case will be studied a little more to determine the importance of a_3.

When $q = 1$ the determinant (7.2) corresponds to the following homogeneous set of equations:

$$\begin{aligned} (M_1 + R)\, a_1 - R\, a_3 &= 0, \\ - R\, a_1 + (M_3 + R)\, a_3 &= 0. \end{aligned} \tag{7.6}$$

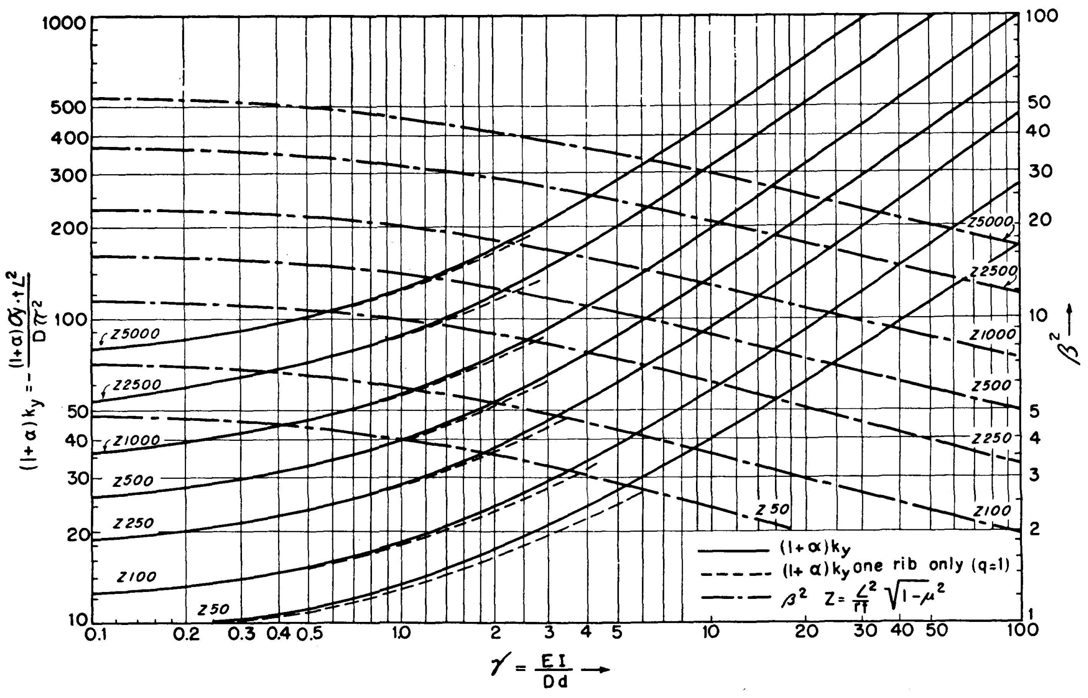

Fig. 5. Buckling Parameters for Ring-stiffened Shells under Lateral Pressure.

Adding the two equations one obtains

$$M_1 a_1 + M_3 a_3 = 0,$$

from which

$$a_3 = -\frac{M_1}{M_3} a_1. \tag{7.7}$$

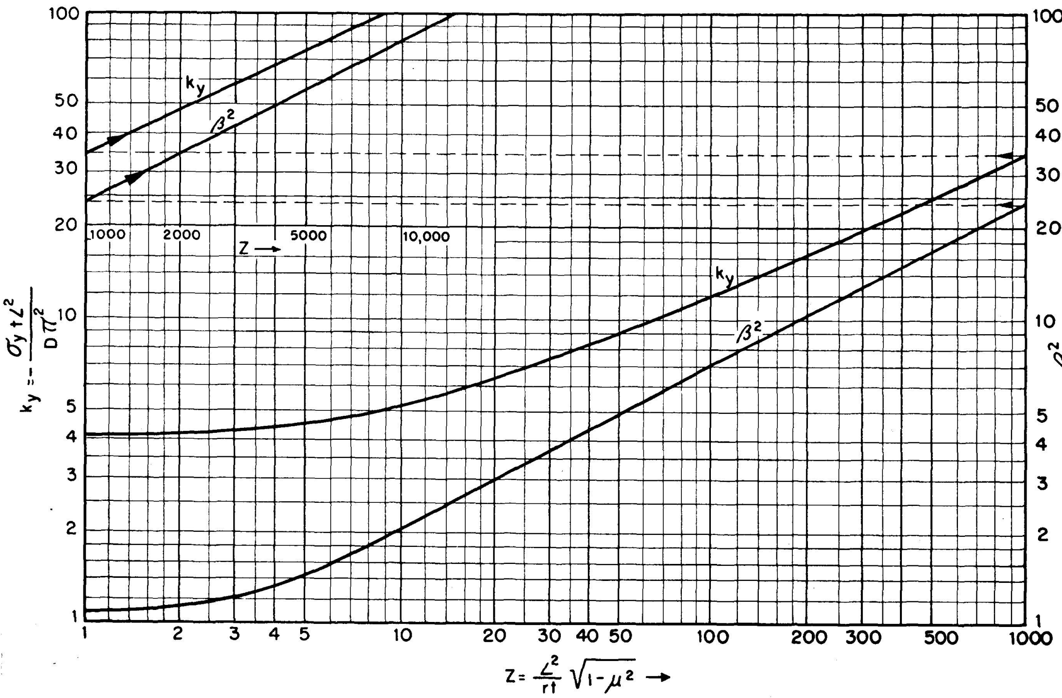

Fig. 6. Buckling Parameters for Isotropic Shells Under Lateral Pressure.

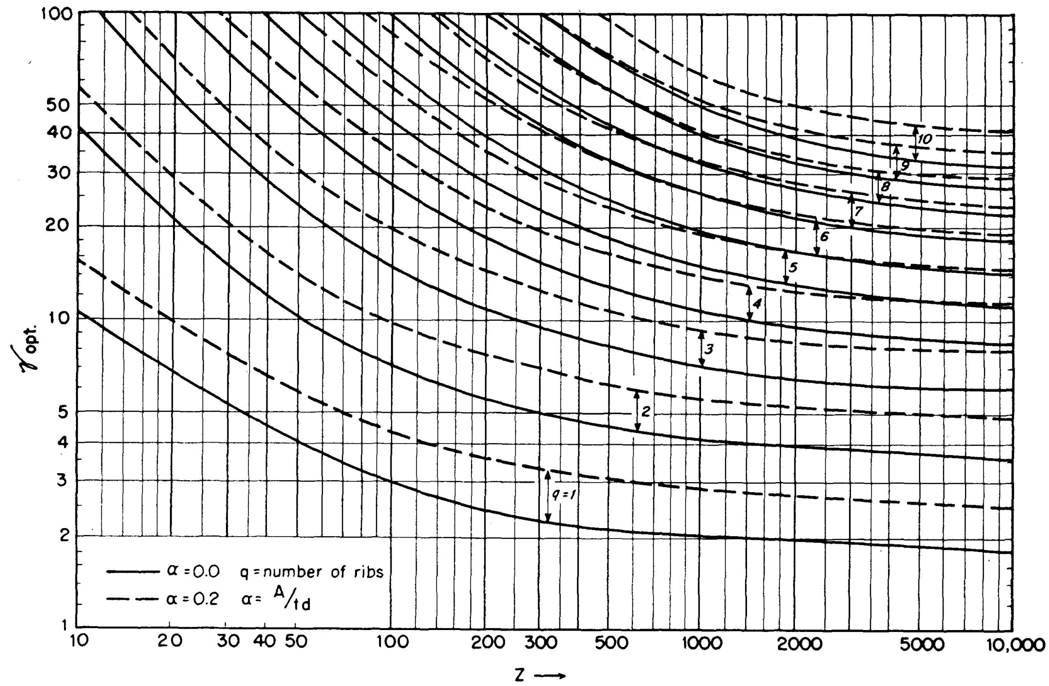

Fig. 7. Optimum Rib Stiffness.

Numerical calculations show that the value of a_3 is zero for $\gamma = 0$ and increases to $0{,}05\,a_1 - 0{,}20\,a_1$, depending on the shell parameters, when γ reaches the value at which one panel buckles separately. The influence of the rigidity of the ribs on the buckling pattern is shown in fig. 8. In fig. 8a there is no rib ($\gamma = 0$) and the shell buckles longitudinally into one half sine wave.

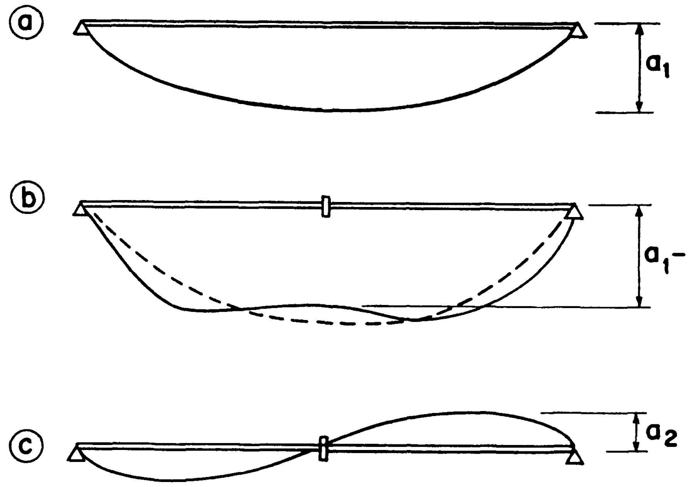

Fig. 8. Effect of Variations in the Rib Stiffness on the Buckling Pattern ($q = 1$).

When $\gamma > 0$ the buckling pattern will deviate from the sinecurve, as demonstrated in fig. 8b. The deviation, taking its maximum value just when γ reaches the value at which the two panels buckle separately (fig. 8c), will usually be 10—20% of a_1.

The dashed lines in fig. 5 applies when $q = 1$. In this case the buckling load is slightly reduced on account of the modifications of the buckling pattern just discussed.

It is quite natural that such modifications will not influence the buckling load when $q > 1$. Considering once again fig. 3, it is clear that the buckling pattern corresponding to the value $p = 7$ represents a very high buckling load. The cylinder will surely buckle in a half-sine wave without noticeable modification due to the influence of the term $\frac{M_1}{M_{2q+1}}$ in eq. (7.3).

8. Buckling of the Flat Plate Reinforced by Ribs

If Z equals zero, one obtains the buckling condition for a flat plate reinforced by longitudinal ribs as shown in fig. 9.

The general buckling conditions in this case take the following form:

$$(1+\alpha)\,k_y = \frac{(1+\beta^2)^2 + \gamma\,\beta^4}{\beta^2}, \tag{8.1}$$

$$k_y = \frac{[(q+1)^2 + \beta^2]^2}{\beta^2}. \tag{8.2}$$

Eq. (8.1) corresponds to the condition $M_1 + R = 0$. Eq. (8.2) corresponds to $M_{q+1} = 0$, and the buckling load is the minimum value found from the two equations. In fig. 4 the variation of k_y with γ is shown.

Eq. (8.1) is exactly the same equation as the one given by TIMOSHENKO [15] when due regard is taken to the different notations used. It is only necessary to introduce

$$\beta_T = \frac{1}{\beta}, \qquad 2\gamma_T = \gamma, \qquad 2\delta_T = \alpha \quad \text{and} \quad b_T = L, \tag{8.3}$$

where the index T denotes symbols used by Timoshenko for $q = 1$.

Correspondingly, one may also find from eq. (6.4) the buckling load for a flat plate loaded in the direction perpendicular to the stiffening ribs as shown in fig. 10.

In this case

$$k_x = \frac{(p^2 + \beta^2)^2 + \gamma \beta^4 \delta_{qpp}}{p^2}, \tag{8.4}$$

where again only one term of the deflection function (6.1) has been considered.

In the case of one rib ($q = 1$), a better approximation is obtained by taking into account the combined influence of a_1 and a_3 as discussed in Section 7.

If p is chosen equal to the number of panels $(q + 1)$, then δ_{qpp} is zero and one obtains

$$k_x = \frac{[(q+1)^2 + \beta^2]^2}{(q+1)^2}. \tag{8.5}$$

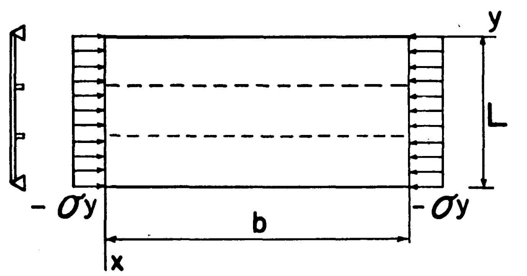

Fig. 9.

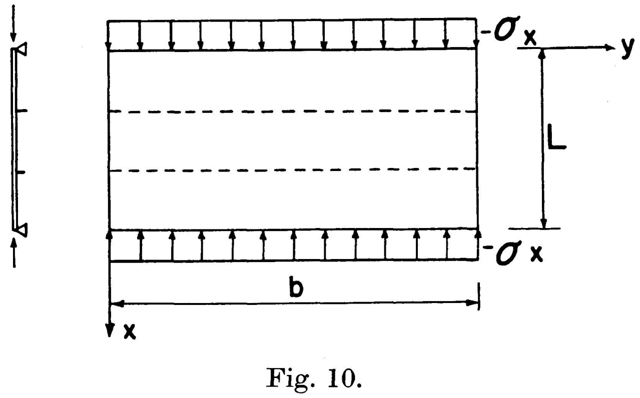

Fig. 10.

If p is not a multiple of $(q+1)$, $\delta_{qpp}=1$ and

$$k_x = \frac{(p^2+\beta^2)+\gamma\beta^4}{p^2}. \tag{8.6}$$

The minimum value of k_x as found from eqs. (8.5) and (8.6) yields the buckling load.

Eq. (8.6) is in every respect in accordance with eq. (288) in [15].

This is easily seen when remembering that in this case:

$$\gamma_T = \frac{EJ}{bD} = \frac{EJ}{LD}\frac{L}{b} = \gamma(q+1)\beta. \tag{8.7}$$

9. Boundary Conditions

The assumed expression (6.1) for the deflected surface corresponds to zero deflection and zero moments along the edges.

Introducing w from eq. (6.1) into expressions (4.17) and integrating, yields

$$u = \sin\frac{n\pi y}{b}\sum_{m=1}^{\infty} C_m \cos\frac{m\pi x}{L}, \qquad v = \cos\frac{n\pi y}{b}\sum_{m=1}^{\infty} D_m \sin\frac{m\pi x}{L}, \tag{9.1}$$

where C_m and D_m are functions of a_m, m and the shell dimensions only.

From eqs. (9.1) it is seen that at the curved edges where $x=0$ or $x=L$

$$\frac{\partial u}{\partial x} = 0, \qquad v = 0, \qquad u \neq 0. \tag{9.2}$$

In the case of a curved panel one correspondingly finds that at the straight edges where $y=0$ or $y=b$

$$\frac{\partial v}{\partial y} = 0, \qquad u = 0, \qquad v \neq 0. \tag{9.3}$$

Hence, the deflection form (6.1) is based upon the following assumptions concerning boundary conditions:

1. The edges are simply supported.
2. No displacements of the shell parallel to the edges.
3. The shell is free to move perpendicular to the edge in the plane of the median surface of the shell.

In the case of a single cylindrical shell roof the edge conditions along the curved edges are generally in good agreement with the conditions of eqs. (9.2).

If the edge beams along the straight edges have little stiffness horizontally, eqs. (9.3) are in most cases fairly well satisfied.

Eqs. (9.2) and (9.3) are, however, not satisfied in the internal valleys between parallel shells, nor at the internal supports of a continuous shell.

BATDORF [10] states that the presence of normal stresses at the curved edges will probably not affect the buckling load appreciably. Stresses normal to the straight edges are, however, believed to be of some importance.

10. Investigation of the Errors Made when Disregarding Extensional Rigidity of the Ribs

As stated before, the extensional rigidity of the ribs was neglected when establishing eq. (5.8).

An idea of the errors involved by that assumption is obtained by comparing the elastic strain energy which is stored in the ribs due to extensional forces, to the energy absorbed by the ribs due to bending stresses. Only the latter part is taken into consideration in eq. (5.8).

The strain energy due to the extensional forces is:

$$U_e = \frac{A}{2}\sum_{i=1}^{q}\int_0^L\int_0^b \sigma'_y\,\epsilon_y\,\delta\,(x-i\,d)\,d\,x\,d\,y = \frac{A}{2\,E}\sum_{i=1}^{q}\int_0^L\int_0^b (\sigma'_y)^2\,\delta\,(x-i\,d)\,d\,x\,d\,y, \tag{10.1}$$

where σ'_y is the additional stress produced during buckling, given by the expression:

$$\sigma'_y = \frac{\partial^2 F}{\partial x^2}. \tag{10.2}$$

From eq. (4.10)

$$\nabla^4 F = -\frac{E}{r}\frac{\partial^2 w}{\partial x^2} = \frac{E}{r}\left(\frac{\pi}{L}\right)^2 a_1 \sin\frac{\pi x}{L}\sin\frac{\beta\,\pi\,y}{L}, \tag{10.3}$$

if the first term only of eq. (6.1) is considered.

By integration one obtains

$$F = \frac{E\left(\frac{\pi}{L}\right)^2}{r\left[\left(\frac{\pi}{L}\right)^2 + \left(\frac{\beta\,\pi}{L}\right)^2\right]^2} a_1 \sin\frac{\pi x}{L}\sin\frac{\beta\,\pi\,y}{L}. \tag{10.4}$$

As σ_y must be periodical, the constants of integration of equation (10.4) equals zero.

Hence

$$\sigma'_y = \frac{\partial^2 F}{\partial x^2} = -\frac{1}{(1+\beta^2)^2}\frac{E\,a_1}{r}\sin\frac{\pi x}{L}\sin\frac{\beta\,\pi\,y}{L}, \tag{10.5}$$

which by introduction in eq. (10.1) yields:

$$U_e = \frac{1}{(1+\beta^2)^4}\frac{E\,A\,b}{4}\left(\frac{a_1}{r}\right)^2\sum_{i=1}^{q}\left(\sin\frac{\pi\,i\,d}{L}\right)^2. \tag{10.6}$$

The strain energy due to bending is

$$U_b = \frac{1}{2}\sum_{i=1}^{q}\int_0^L\int_0^b M'_y \kappa_y \delta(x - i d)\, dx\, dy = \frac{E J}{2}\sum_{i=1}^{q}\int_0^L\int_0^b \left(\frac{\partial^2 w}{\partial y^2}\right)^2 \delta(x - i d)\, dx\, dy. \quad (10.7)$$

Introducing now $\frac{\partial^2 w}{\partial y^2}$ from eq. (6.1)

$$U_b = \frac{E J}{2}\left(\frac{\beta \pi}{L}\right)^4 a_1^2 \sum_{i=1}^{q}\int_0^L\int_0^b \left(\sin\frac{\pi x}{L}\sin\frac{\beta \pi y}{L}\right)^2 \delta(x - i d)\, dx\, dy, \quad (10.8)$$

which by integration yields:

$$U_b = \frac{E J \beta^4 \pi^4 b}{4 L^4} a_1^2 \sum_{i=1}^{q}\left(\sin\frac{\pi i d}{L}\right)^2. \quad (10.9)$$

As a measure of the relative importance of the terms neglected, take the ratio

$$\eta = \frac{U_e}{U_b} = \frac{A L^4}{J r^2}\frac{1}{\pi^4 \beta^4 (1+\beta^2)^4}, \quad (10.10)$$

which in the case of symmetrical cross section (fig. 1) equals

$$\eta = \frac{12 Z^2}{\pi^4 (1-\mu^2)}\left(\frac{t}{h}\right)^2 \frac{1}{\beta^4 (1+\beta^2)^4}. \quad (10.11)$$

From this equation it is possible to calculate in each particular case the part of the stiffening effect of the ribs which has been neglected.

For reinforced concrete shell roofs Z varies between 10 and 10^4, and stiffening ribs are probably only used for values of Z above 100. In such cases η will always be less than 0,05 if $\frac{c}{d} \leqq 0{,}10$ and $\gamma < 100$. Poisson's ratio is assumed to be 0,2.

11. The Effect of Longitudinal Ribs

If the shell also is provided with axial ribs, the previous equations will be slightly altered.

Defining γ_x, d_x and q_x as being analogous to the corresponding terms without subscripts, but now referring to the ribs in the axial direction, the fourth of eqs. (4.1) must be replaced by the following:

$$M'_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} + \gamma_x d_x \sum_{i=1}^{q_x} \delta(y - i d_x)\frac{\partial^2 w}{\partial x^2}\right] \quad (11.1)$$

and correspondingly

$$Q'_x = -D\left[\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x\, \partial y^2} + \gamma_x d_x \sum_{i=1}^{q_x} \delta(y - i d_x)\frac{\partial^3 w}{\partial x^3}\right]. \quad (11.2)$$

Eq. (11.2) is now, together with eqs. (5.4) and (5.5), introduced into the last of eqs. (4.4) thus yielding:

$$\begin{aligned}\nabla^4 w + \frac{12\,Z^2}{L^4}\nabla^{-4}\frac{\partial^4 w}{\partial x^4} + \gamma\, d \sum_{i=1}^{q} \delta\,(x - i\,d)\frac{\partial^4 w}{\partial y^4} + \gamma_x d_x \sum_{i=1}^{q_x} \delta\,(y - i\,d_x)\frac{\partial^4 w}{\partial x^4} \\ - \frac{t}{D}\left[\sigma_x \frac{\partial^2 w}{\partial x^2} + \sigma_y\left[1 + \alpha d \sum_{i=1}^{q} \delta\,(x - i\,d)\right]\frac{\partial^2 w}{\partial y^2} + 2\,\tau \frac{\partial^2 w}{\partial x\, \partial y}\right] = 0.\end{aligned} \tag{11.3}$$

Eq. (11.3) corresponds to the previously derived eq. (5.8).

In the case of pure lateral load the effect of longitudinal ribs is comparatively small. For simplicity eq. (6.1) for the deflection function is applied, though it is possible that a series expansion in the y-direction would give a better approximation to the buckling pattern. The stiffening effect of the longitudinal ribs will therefore probably be slightly overestimated in the present derivations.

Solving eq. (11.3) by the Galerkin method, the following set of homogeneous equations expressing the buckling condition is found:

$$\begin{aligned}\left[(p^2+\beta^2)^2 + \frac{12\,Z^2}{\pi^4}\frac{p^4}{(p^2+\beta^2)^2} + \gamma_x\, p^4\, \delta_{q_x nn} - p^2 k_x - \beta^2 k_y\right] a_p \\ + (\gamma\,\beta^4 - \alpha\,\beta^2 k_y) \sum_{m=1}^{\infty} a_m\, \delta_{qmp} = 0,\end{aligned} \tag{11.4}$$

where

$$\delta_{q_x nn} = \frac{2}{q_x+1} \sum_{i=1}^{q_x} \left(\sin \frac{n\,\pi\, i}{q_x+1}\right)^2.$$

$\delta_{q_x nn}$ takes the following values:

$$\delta_{q_x nn} = +1 \qquad \text{for } n \neq (q_x+1)\,i \qquad i = 1, 2, 3\,\ldots$$

$$\delta_{q_x nn} = 0 \qquad \text{for } n = (q_x+1)\,i\,.$$

Eq. (11.4) may also be written as follows:

$$(M_p)_x\, a_p + R \sum_{m=1}^{\infty} a_m\, \delta_{qmp} = 0, \tag{11.5}$$

where

$$(M_p)_x = (p^2+\beta^2)^2 + \frac{12\,Z^2}{\pi^4}\frac{p^4}{(p^2+\beta^2)^2} + \gamma_x\, p^4\, \delta_{q_x nn} - p^2 k_x - \beta^2 k_y\,. \tag{11.6}$$

As will be seen from eq. (11.6) the influence of γ_x will be the one of reducing the importance of higher buckling modes in the axial direction, since $(M_p)_x \geqq M_p$.

The equality applies only when the buckling pattern involves nodal lines along the axial stiffeners $[n = (q_x+1)\,i]$.

Thus, it is clear that one may, also in this case, calculate the buckling load from the equation

$$M_{q+1}[M_1 + R] = 0. \tag{11.7}$$

The expression in the parenthesis yields, for $\sigma_x = 0$, the following equation

$$(1+\alpha)\,k_y = \frac{1+\beta^2}{\beta^2} + \frac{12\,Z^2}{\pi^4}\,\frac{1}{\beta^2\,(1+\beta^2)^2} + \gamma\,\beta^2 + \frac{\gamma_x}{\beta^2}\,\delta_{q_x nn}. \tag{11.8}$$

The effect of longitudinal ribs for various values of γ and Z is shown in table 1. It is assumed that the circumferential ribs are so closely spaced that local buckling of the panel between two consecutive ribs is prevented. The effect of the circumferential ribs alone is also shown in the table. It will be seen that it is poor economy to provide the shell with axial ribs under the

Table 1. Effect of Longitudinal Ribs

Z		$\gamma=0$	$\gamma=1.0$		$\gamma=10$		$\gamma=100$	
		$\gamma_x=0$	$\gamma_x=0$	$\gamma_x=1$	$\gamma_x=0$	$\gamma_x=10$	$\gamma_x=0$	$\gamma_x=100$
100	k_y	11.90	18.34	18.50	57.60	60.62	271.1	322.0
	Effect of ribs		54.0%	55.4%	384%	408%	2180%	2600%
1000	k_y	34.26	55.95	56.00	192.9	193.7	982.5	996.3
	Effect of ribs		63.0%	63.0%	462%	462%	2760%	2800%

present loading conditions. The strengthening effect of the axial ribs is somewhat overestimated in the table, for the following two reasons:

1. The assumed buckling pattern is probably not very close to the correct one.

2. When establishing table 1 the fact is disregarded that very often the buckling pattern will, due to the effect of the longitudinal ribs, be slightly altered in such a way that $n = (q_x + 1)\,i$ and therefore $\delta_{q_x nn} = 0$. This will imply a lower effect of the axial ribs than the one which is read from the table. When calculating a special shell construction with a given number of ribs this may easily be taken into account.

If the shell is provided with axial ribs only, the buckling load is found from the formula

$$k_y = \frac{(1+\beta^2)^2}{\beta^2} + \frac{12\,Z^2}{\pi^4}\,\frac{1}{\beta^2\,(1+\beta^2)^2} + \frac{\gamma_x}{\beta^2}\,\delta_{q_x nn}, \tag{11.9}$$

which in the case of a flat plate reduces to

$$k_y = \frac{(1+\beta^2)^2 + \gamma_x\,\delta_{q_x nn}}{\beta^2}. \tag{11.10}$$

This equation of course leads to the same results as eq. (8.4).

12. Limitations on the Formulas

Donnell [9] has demonstrated that the small terms neglected when establishing eqs. (4.1) are of the order of magnitude $\left(\frac{t}{r}\right)^2$ or $\left(\frac{2}{n}\right)^2$ as compared to the remaining terms.

Returning now to eq. (5.1) it is remembered that the second term on the left side of the equation was neglected.

Introducing expression (6.1) for w, one finds that

$$\frac{d^2 w}{d y^2} + \frac{w}{r^2} = \left[\left(\frac{n\pi}{b}\right)^2 + \frac{1}{r^2}\right] w = \left[\left(\frac{n}{2}\right)^2 + 1\right] \frac{w}{r^2}, \tag{12.1}$$

since $b = 2\pi r$ for the complete tube. Also the simplification of eq. (5.1), therefore, implies the neglection of a term with the relative importance $\left(\frac{2}{n}\right)^2$.

13. References

1. Lorenz, R., "Buckling of Thin-walled Cylinders". Phys. Zeitschr., Vol. 12, pp. 241 to 260, 1911.
2. Southwell, R. V., "Collapse of Tubes by External Pressure". Phil. Mag., Vol. 25, pp. 687—698, 1913; Vol. 26, pp. 502—511, 1913; Vol. 29, pp. 67—77, 1915.
3. Southwell, R. V., "On the General Theory of Elastic Stability". Phil. Trans. Royal Soc. London. Vol. 213, Series A, pp. 187—244, 1913.
4. v. Mises, R., "Der kritische Außendruck zylindrischer Rohre". V.D.I. Zeitschr., Vol. 58, pp. 750—755, 1914.
5. Sanden and Tølke, "Stabilitätsprobleme dünner, kreiszylindrischer Schalen". Ing. Arch., Vol. 3, pp. 24—66, 1932.
6. Flügge, W., "Die Stabilität der Kreiszylinderschale". Ing. Arch., Vol. 3, pp. 463 to 506, 1932.
7. Windenburg, D. F. and Trilling, C., "Collapse by Instability of Thin Cylindrical Shells under External Pressure". ASME Vol. 56, pp. 819—825, 1934.
8. Sturm, R. G., "A Study on the Collapsing Pressure on Thin-walled Cylinders". Univ. Illinois, Engr. Exp. Station. Bull. No. 329, 1941.
9. Donnell, L. H., "Stability of Thin-walled Tubes under Torsion". NACA Rep. No. 479, 1934.
10. Batdorf, S. B., "A Simplified Method of Elastic-Stability Analysis for Thin Cylindrical Shells". NACA Rep. No. 874, 1947.
11. Lundgren, H., "Stability Tests on Cylindrical Reinforced Concrete Shells". Ingeniørvidenskapelige Skrifter No. 5, Copenhagen. (In Danish), 1945.
12. Stein, M., Sanders, J. L., Crate, H., "Critical Stress of Ring-stiffened Cylinders in Torsion". NACA Rep. No. 989, 1951.
13. Jenkins, R. S., "Theory and Design of Cylindrical Shell Structures". London, 1947.
14. Holand, I., "Design of Circular Cylindrical Shells". Oslo University Press, 1957.
15. Timoshenko, S., "Theory of Elastic Stability". McGraw-Hill, New York, 1936.
16. Moe, J., "On the Theory of Cylindrical Shells". Intern. Assn. Bridge and Struct. Eng. Publications. Vol. 13, pp. 283—296, 1953.

Summary

In the present paper formulas are derived and graphs presented which estimate the stiffening effect of ring ribs on a complete thin-walled cylinder loaded with external lateral pressure. As a complete cylinder generally will buckle into a great number of circumferential waves, the formulas are also believed to apply in predicting the safety against buckling near the top of a rib reinforced shell roof construction. Additional account must be taken of the influence of axial stresses on the stability of the shell.

Résumé

Les relations et les courbes qui figurent dans ce mémoire sont destinées au calcul de l'influence raidissante des anneaux de renforcement sur un cylindre fermé à paroi mince soumis à une pression extérieure latérale.

Etant donné que généralement le voilement d'un cylindre fermé s'effectue selon de nombreuses ondulations disposées sur ce périmètre, les formules doivent trouver également application à la détermination de la sécutité vis-à-vis du voilement au voisinage de l'arête supérieure des ouvrages en voûte mince raidis par des nervures. En outre, il y a lieu de tenir compte de l'influence des contraintes axiales sur la stabilité du voile.

Zusammenfassung

Die in der vorliegenden Abhandlung abgeleiteten Beziehungen und die gegebenen Kurven dienen der Berechnung des versteifenden Einflusses von Verstärkungsringen auf einen geschlossenen dünnwandigen Zylinder unter einer äußeren seitlichen Pressung.

Da ein geschlossener Zylinder allgemein in einer großen Zahl von, dem Kreisumfang folgenden Wellen ausbeult, sollten die Formeln auch Anwendung finden in der Bestimmung der Sicherheit gegen Beulen in der Nähe des Scheitels von rippenversteiften Schalendachkonstruktionen. Zusätzlich muß der Einfluß von Axialspannungen auf die Stabilität der Schale berücksichtigt werden.