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Design of Stirrup-Reinforcement in Concrete Beams

Dimensionnement des armatures de cisaillement dans les poutres en béton armé

Bemessung der Schubarmierung von Eisenbetonbalken

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1. Introduction

Recent experimental research has shown that a concrete member subjected to combined bending and shearing actions can fail in a number of different mechanisms, the mode depending in any particular case upon the section properties, the quantity of web steel reinforcement, the slenderness of the member and the load configuration. It has been further shown that the mode of shear failure most likely to occur in web-reinforced beams of normal design is the so-called "shear-compression" failure.

The sequence of events comprising this failure mechanism has been described by ACI-ASCE Committee 326 as follows [1]:

"As the external load increases after diagonal cracking, the web reinforcement and the compression zone continue to carry shear until the stress in the web reinforcement has reached the yield point. Further increase in external shear must then be resisted by the compression zone alone. Failure occurs when the compression zone is destroyed by the combined compression and shear stresses."

This description is considered by the committee to hold good for most beams covered by ACI design requirements.

A number of analytic studies [2, 3, 8] of the shear-compression failure have been made, based on the "limiting moment" concept, but the resulting ultimate strength theories have so far been quite complicated and not readily adaptable to design. Committee 326 thus concludes that "... although promising, the shear moment approach has not yet been developed sufficiently to provide a basis for a reliable design procedure."

In the proposed revision of the ACI Building Code Requirements [4] a different approach is therefore followed, in which the total shearing resistance, Q_u, is equated to the inclined cracking load of the section, Q_i, plus the shear resistance of the web reinforcement as given by the truss-analogy, Q_s, i.e.

$$Q_u = Q_i + Q_s. \tag{1}$$

Although this approach at present provides a more satisfactory design equation, it is only as reliable as the empirical observation, represented by Eq. (1), on which it is based. Furthermore, a definite value for the inclined cracking load is not always exhibited by a beam. Inclined cracks quite often develop gradually with increasing load, forming initially as vertical flexural cracks, then inclining in-span and extending upward into the concrete compression zone until failure is brought about by crushing of the compression concrete. In such cases it is left entirely to the investigator to choose the value of the inclined cracking load.

The very real advantage of a design equation such as (1) is that it represents an empirical modification of the simple and safe — but often over-conservative — truss analogy theory.

In the present study, an approach is suggested in which shear compression theory is used to estimate the portion of the shearing resistance in excess of the truss analogy value. The resulting equation is similar to (1) except that the inclined cracking load Q_i is replaced by the term Q_c, which represents the shear carried by the concrete compression zone at the instant of failure. This analysis is therefore more appropriate for use in the development of design equations than previous shear-compression studies; on the other hand it is considered to have a more rational basis than the inclined cracking load theory.

2. Development of Basic Equations

In the following analysis we consider a member subjected to transverse loadings which produce a maximum moment M_{max} at some section YY and a moment M and a shear force Q at any typical section XX. The assumed purpose of the stirrups in the region of XX is to prevent premature shear failure there before the full flexural capacity is achieved at the section of maximum moment, YY. The conditions pertaining at XX are examined on the assumptions that an inclined crack has formed, the stirrups crossing the crack have yielded, and the concrete in the compression zone above the inclined crack is in a state of imminent failure.

2.1. Equilibrium

The assumed equilibrium situation is shown in Fig. 1. Considering first the triangular wedge of concrete beneath the inclined crack (Fig. 1c), we see that the

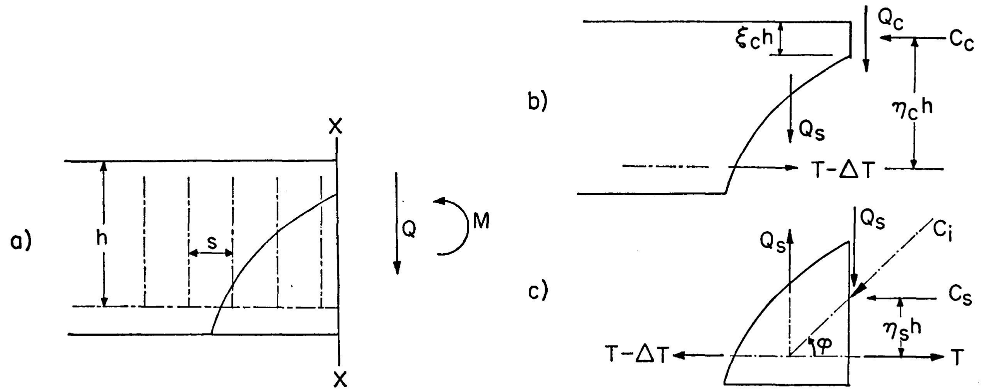

Fig. 1.

stirrup force Q_s is brought into equilibrium by an inclined compressive force C_i in the web concrete and a change ΔT in the horizontal steel force. The inclination of C_i is approximately equal to the angle of the inclined crack φ, so that the horizontal component of C_i is

$$C_s = \Delta T = Q_s \cot\text{an}\,\varphi. \qquad (2)$$

The possibility of other types of failure is also indicated by the equilibrium situation in the concrete below the inclined crack. Large values of Q_s, which may occur when the web is heavily reinforced, must be accompanied by large C_i and ΔT. In beams with thin webs and low concrete strengths, the force C_i can become critical, and produce inclined crushing of the web concrete before the stirrups reach yield. Similarly, an excessive ΔT can produce slip of the longitudinal steel. If either slip or inclined crushing occurs, the force in the stirrups will fall off, and the beam will fail, either by "sagging" or by crushing of the top concrete fibres as a result of the transfer of shear from the stirrups to the concrete compression zone. The present analysis, however, will be restricted to the more common shear-compression failure.

In the region above the inclined crack the state of combined shearing and compressive stress may be represented by a vertical and a horizontal force, Q_c and C_c. According to the simple truss analogy, Q_c would be equal to zero.

The equilibrium equations for section XX are as follows:

$$Q = Q_s + Q_c, \qquad (3)$$

$$M = C_c \eta_c h + C_s \eta_s h, \qquad (4)$$

in which $\eta_c h$ and $\eta_s h$ are the lever arms for the forces C_c and C_s about the tension steel. An average, constant value of 45 degrees will be assumed for the angle of the inclined crack φ, so that

$$C_s = Q_s. \qquad (5)$$

Also, equating Q_s to the yield force in the stirrups, we may write with sufficient accuracy

$$Q_s = \sigma_f \rho b h, \qquad (6)$$

in which σ_f is the steel yield stress and ρ is the proportion of web reinforcement, $\frac{F_B}{b\,s}$.

The concrete compression zone above the extremity of the inclined crack is of depth $\zeta_c h$, and the average compressive and shear stresses in this region at failure are

$$\bar{\sigma} = \frac{C_c}{b\,h\,\zeta_c} = \frac{M - Q_s\,\eta_s\,h}{b\,h^2\,\eta_c\,\zeta_c} \tag{7}$$

and

$$\bar{\tau} = \frac{Q_c}{b\,h\,\zeta_c}. \tag{8}$$

2.2. Failure of Concrete in Combined Shear and Compression

The condition of imminent failure in the compression flange lends itself to an analysis using Mohr's failure criterion, which can be adapted for direct application to the case of a small region of concrete subjected to longitudinal compression and complementary shears. WALTHER [2] has derived the following equations to define this state of failure,

$$\left(\frac{2\,\bar{\tau}}{\sigma_u}\right)^2 - \left(\frac{\bar{\sigma}}{\sigma_u}\right) + \left(\frac{\bar{\sigma}}{\sigma_u}\right)^2 = 0, \tag{9a}$$

when

$$\left(\frac{\bar{\sigma}}{\sigma_u}\right) \geqq 0.25 \tag{10a}$$

and

$$\left(\frac{8\,\bar{\tau}}{\sigma_u}\right)^2 - \left(\frac{8\,\bar{\sigma}}{\sigma_u}\right) - 1 = 0, \tag{9b}$$

when

$$\left(\frac{\bar{\sigma}}{\sigma_u}\right) \leqq 0.25. \tag{10b}$$

The term σ_u represents the strength of the concrete under uniaxial compression. The relation between $(\bar{\tau}/\sigma_u)$ and $(\bar{\sigma}/\sigma_u)$ is shown graphically in Fig. 2.

Considering first the range $(\bar{\sigma}/\sigma_u) \geqq 0.25$ and substituting Eqs. (7) and (8) in (9a) we obtain

$$\left[\frac{2\,Q_c}{\sigma_u\,b\,h\,\zeta_c}\right]^2 = \left[\frac{M - Q_s\,\eta_s\,h}{\sigma_u\,b\,h^2\,\eta_c\,\zeta_c}\right] - \left[\frac{M - Q_s\,\eta_s\,h}{\sigma_u\,b\,h^2\,\eta_c\,\zeta_c}\right]^2. \tag{11}$$

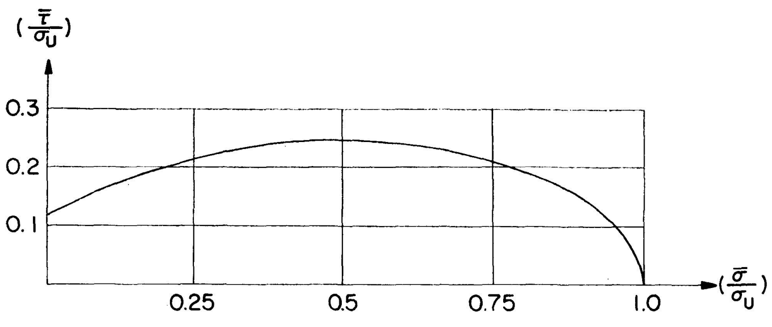

Fig. 2.

The ultimate flexural strength of the section is given by the equation

$$M_u = \sigma_u b h^2 \zeta_u \eta_u,$$

which may be used in a re-arrangement of (11) to give

$$\left[\frac{2 Q_c}{\sigma_u b h \zeta_c}\right]^2 = \frac{\zeta_u \eta_u}{\zeta_c \eta_c} \frac{M}{M_u}\left[1 - \eta_s \frac{Q_s h}{M}\right] - \left(\frac{\zeta_u \eta_u}{\zeta_c \eta_c}\right)^2 \left(\frac{M}{M_u}\right)^2 \left[1 - \eta_s \frac{Q_s h}{M}\right]^2.$$

Defining the term $m = \frac{M}{M_u}$ we obtain

$$Q_c = \sigma_u b h \frac{1}{2} \sqrt{\zeta_c \left(\frac{\eta_u \zeta_u}{\eta_c}\right) m \left(1 - \eta_s \frac{Q_s h}{M}\right) - \left(\frac{\eta_u \zeta_u}{\eta_c}\right)^2 m^2 \left(1 - \eta_s \frac{Q_s h}{M}\right)^2} \quad (12a)$$

for the condition

$$\left(\frac{\zeta_u \eta_u}{\zeta_c \eta_c}\right) m \left(1 - \eta_s \frac{Q_s h}{M}\right) \geqq 0.25. \quad (13a)$$

A similar treatment using Eq. (9b) leads to

$$Q_c = \sigma_u b h \frac{1}{8} \sqrt{\zeta_c^2 + 8 \zeta_c \left(\frac{\eta_u \zeta_u}{\eta_c}\right) m \left(1 - \eta_s \frac{Q_s h}{M}\right)}, \quad (12b)$$

when

$$\left(\frac{\zeta_u \eta_u}{\zeta_c \eta_c}\right) m \left(1 - \eta_s \frac{Q_s h}{M}\right) \leqq 0.25. \quad (13b)$$

2.3. Compatibility

Apart from Q_c and Q_s, which we wish to evaluate, the unknown term in Eqs. (12a) and (12b) is ζ_c, of which the η_s and η_c are both functions. In order to evaluate ζ_c accurately, the deformations in the region of the inclined crack would have to be studied. This has been attempted by several investigators [2, 3, 8] but owing to the complexity of the situation a completely satisfactory formulation of compatibility has yet to be made. Furthermore, the compatibility equations inevitably involve empirical parameters which must be evaluated from beam tests, and although they have been given different physical interpretations, they tend to become, in effect, not physical constants representing actual properties of the beam or the component materials, but open parameters which are evaluated to provide a best fit between predicted and observed beam strengths. The complexity of the resulting equations is also a considerable hinderance to the development of simple design equations.

For our purposes it will be preferable to make a simple, conservative estimate of ζ_c directly from a consideration of test data. In an earlier study of the strength of reinforced concrete beams failing in shear, LAUPA, SIESS, and NEWMARK [5] adopted a simple approach in which the depth of the neutral axis was considered to be a simple function of the neutral axis depth ζ_0 as obtained from simple reinforced concrete theory for the case of pure, elastic bending. This approach appears to be reasonable when it is considered that both ζ_c and ζ_0 should depend on the same primary variables, namely the percentage of longitudinal steel and the deformation characteristics of the concrete, and has also been adopted here.

Defining the terms

$$\delta = \frac{\zeta_c}{\zeta_0}, \qquad \lambda_0 = \frac{\zeta_u \eta_u}{\zeta_0 \eta_0}, \qquad \tau_s = \frac{Q_s}{b\,h}$$

and rearranging Eqs. (12a) and (12b) we obtain

$$Q_c = \sigma_u b\, h\, \zeta_0 \frac{1}{2} \sqrt{\delta \left(\frac{\eta_0}{\eta_c}\right) \lambda_0 \left[m - \frac{\eta_s}{\zeta_u \eta_u} \left(\frac{\tau_s}{\sigma_u}\right)\right] - \left(\frac{\eta_0}{\eta_c}\right)^2 \lambda_0^2 \left[m - \frac{\eta_s}{\zeta_u \eta_u} \left(\frac{\tau_s}{\sigma_u}\right)\right]^2} \tag{14a}$$

for condition (13a), and for (13b)

$$Q_c = \sigma_u b\, h\, \zeta_0 \frac{1}{8} \sqrt{\delta^2 + 8\,\delta \left(\frac{\eta_0}{\eta_c}\right) \lambda_0 \left[m - \frac{\eta_s}{\zeta_u \eta_u} \left(\frac{\tau_s}{\sigma_u}\right)\right]}. \tag{14b}$$

Although not a necessary step, these equations will be simplified by approximating constant values for η_s and the ratio (η_0/η_c). The term

$$\eta_s = \tfrac{1}{2}(1 - \zeta_c),$$

the maximum possible value of which is 0.5, will be assumed to be equal to 0.4; the ratio (η_0/η_c) will be approximated by the value unity.

The equations may now be written in the form

$$Q_c = \sigma_u b\, h\, \zeta_0 K, \tag{15}$$

in which

$$K = \frac{1}{2} \sqrt{\delta\, \lambda_0 \left[m - \frac{0.4}{\zeta_u \eta_u} \left(\frac{\tau_s}{\sigma_u}\right)\right] - \lambda_0^2 \left[m - \frac{0.4}{\zeta_u \eta_u} \left(\frac{\tau_s}{\sigma_u}\right)\right]^2}, \tag{16a}$$

when

$$\delta\, \lambda_0 \left[m - \frac{0.4}{\zeta_u \eta_u} \left(\frac{\tau_s}{\sigma_u}\right)\right] \geqq 0.25 \tag{17a}$$

and

$$K = \frac{1}{8} \sqrt{\delta^2\, 8\, \lambda_0 \left[m - \frac{0.4}{\zeta_u \eta_u} \left(\frac{\tau_s}{\sigma_u}\right)\right]}, \tag{16b}$$

when

$$\delta\, \lambda_0 \left[m - \frac{0.4}{\zeta_u \eta_u} \left(\frac{\tau_s}{\sigma_u}\right)\right] \leqq 0.25. \tag{17b}$$

2.4. Empirical Evaluation of δ

For the empirical evaluation of δ, the equations are rearranged to

$$K = \frac{Q - \sigma_f \rho\, b\, h}{\sigma_u b\, h\, \zeta_0} \tag{18}$$

with

$$\delta = \frac{\lambda_0^2 \left[m - \frac{0.4}{\zeta_u \eta_u} \left(\frac{\tau_s}{\sigma_u}\right)\right]^2 + 4 K^2}{\lambda_0 \left[m - \frac{0.4}{\zeta_u \eta_u} \left(\frac{\tau_s}{\sigma_u}\right)\right]} \tag{19a}$$

for condition (17a), and for condition (17b)

$$\delta = \frac{\frac{1}{8} + 8\,K^2}{\lambda_0 \left[m - \frac{0.4}{\zeta_u\,\eta_u}\left(\frac{\tau_s}{\sigma_u}\right)\right]}. \tag{19b}$$

In the present study, data have been analyzed for 58 shear tests conducted by Clark [6] and Moretto [7] on reinforced concrete beams of rectangular section containing vertical stirrup reinforcement. Details of the test data and results of the calculations are shown in Tables 1 and 2. The values computed for δ, contained in the last column of the tables, are grouped around the value unity. The computed mean value from tests is 0.99 with a standard deviation of 0.18.

Before a numerical value is adopted for δ, a check must be made on the assumption that δ is dependent only on ζ_0. To investigate the possibility of systematic variation with the prime variables, the computed values of δ have been plotted against (τ_s/σ_u), m, σ_u, and also the shear-to-moment ratio $\frac{Q\,h}{M}$,

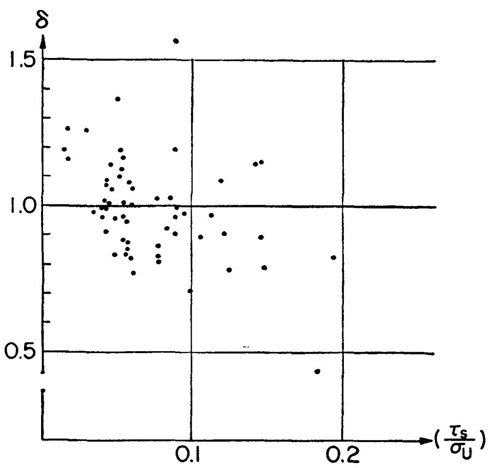

Fig. 3.

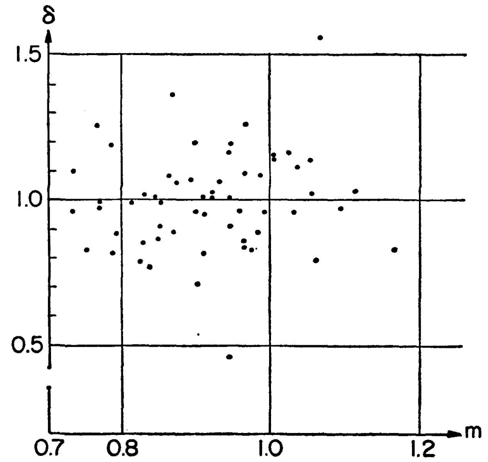

Fig. 4.

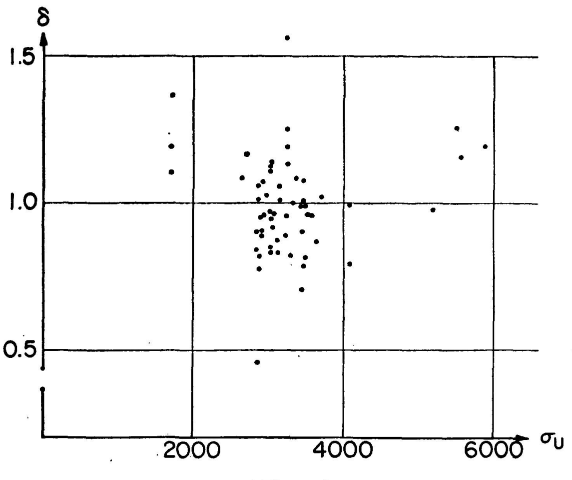

Fig. 5.

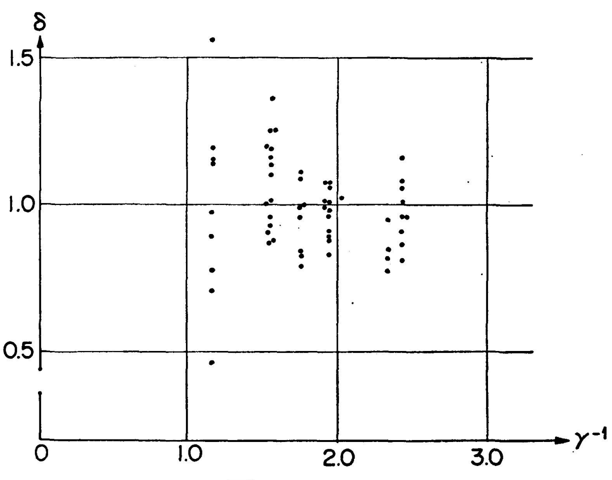

Fig. 6.

Table I. Analysis of Test Data of Clark [6]

Beam No.	σ_u *) lb./in.^{2}	$\frac{M}{Q\,h}$	M in. kips	Q kips	M_{ult} in. kips	m	$\frac{\tau_s}{\sigma_u}$	K	δ
A 11	3040	2.34	1800	50.02	2174	0.83	0.057	0.152	0.85
12	2920	2.34	1692	47.02	2152	0.79	0.059	0.139	0.82
13	2880	2.34	1800	50.02	2145	0.84	0.060	0.158	0.77
14	3050	2.34	1982	55.02	2174	0.91	0.057	0.176	0.95
B 11	2880	1.95	1880	62.68	2145	0.88	0.060	0.228	1.06
12	3130	1.95	1731	57.69	2190	0.79	0.055	0.187	0.88
13	2920	1.95	1921	64.03	2152	0.89	0.059	0.232	1.07
14	2870	1.95	1806	60.28	2143	0.84	0.060	0.214	1.00
15	3030	1.95	1628	54.28	2173	0.75	0.057	0.173	0.84
B 21	2860	1.95	2030	67.69	2140	0.95	0.121	0.138	0.90
22	3250	1.95	2170	72.44	2210	0.98	0.106	0.152	0.90
23	3070	1.95	2258	75.28	2182	1.03	0.113	0.170	0.97
B 61	5190	1.95	2630	85.28	2400	1.10	0.033	0.239	0.98
C 11	3160	1.56	1497	62.44	1579	0.95	0.055	0.245	1.00
12	3250	1.56	1675	69.94	1585	1.06	0.053	0.281	1.13
13	2950	1.56	1326	55.28	1563	0.85	0.058	0.210	0.87
14	3580	1.56	1541	64.28	1607	0.96	0.048	0.234	0.96
C 21	2920	1.56	1563	65.19	1558	1.00	0.089	0.207	0.91
22	3080	1.56	1623	67.69	1573	1.03	0.084	0.214	0.93
23	2980	1.56	1745	72.78	1564	1.12	0.087	0.249	1.03
24	3320	1.56	1553	64.78	1590	0.98	0.074	0.187	0.82
C 31	1730	1.56	1205	50.28	1386	0.87	0.050	0.334	1.37
32	1700	1.56	1087	45.03	1380	0.79	0.051	0.288	1.19
33	1720	1.56	1014	42.28	1383	0.73	0.051	0.266	1.10
C 41	3020	1.56	1668	69.53	2173	0.77	0.029	0.311	1.25
C 62	5580	1.56	2283	95.28	2422	0.94	0.016	0.300	1.15
63	5500	1.56	2344	97.78	2420	0.97	0.016	0.313	1.25
64	5860	1.56	2310	96.36	2440	0.95	0.015	0.300	1.20
D 11	3230	1.17	1218	67.69	1356	0.90	0.089	0.300	1.20
12	3220	1.17	1443	80.19	1356	1.07	0.089	0.376	1.57
13	3020	1.17	1038	57.69	1340	0.77	0.095	0.241	0.97
D 21	2960	1.17	1173	65.19	1343	0.87	0.146	0.224	0.90
22	3415	1.17	1263	70.19	1400	0.90	0.098	0.176	0.71
23	3050	1.17	1353	75.19	1348	1.00	0.141	0.286	1.14
24	3020	1.17	1355	75.28	1347	1.00	0.143	0.287	1.15
D 31	3480	1.17	1598	88.78	1947	0.82	0.124	0.187	0.78
D 41	2850	1.17	1263	70.19	1337	0.94	0.183	0.045	0.45

Beam No.	σ_u *) lb./in.^{2}	$\frac{M}{Q\,h}$	M in. kips	Q kips	M_{ult} in. kips	m	$\frac{\tau_s}{\sigma_u}$	K	δ
D 16	3410	1.94	943	39.28	1158	0.81	0.042	0.224	0.99
17	3450	1.94	966	40.28	1162	0.83	0.042	0.230	1.02
18	3420	1.94	1003	41.78	1159	0.87	0.042	0.244	1.08
E 12	3720	2.02	1247	49.86	1182	1.06	0.077	0.212	1.02
D 26	3630	2.43	1137	37.86	1174	0.97	0.079	0.126	0.86
27	3500	2.43	1062	35.36	1163	0.91	0.082	0.109	0.81
28	3220	2.43	1137	37.86	1141	1.00	0.089	0.134	0.96
D 41	3380	2.43	1137	37.86	1154	0.98	0.043	0.214	1.08
42	3160	2.43	1062	35.36	1137	0.93	0.046	0.204	1.06
43	2720	2.43	1113	37.11	1088	1.02	0.053	0.243	1.16
D 51	3420	2.43	987	32.86	1159	0.85	0.042	0.173	0.91
52	3570	2.43	1062	35.36	1170	0.91	0.040	0.187	0.96
53	3340	2.43	1062	35.36	1152	0.92	0.043	0.197	1.01

*) σ_u assumed equal to 0.85 times quoted cylinder strength.

Table II. Analysis of Test Data of Moretto [7]

Beam No.	σ_u *) lb./in.^{2}	$\frac{M}{Q\,h}$	M in. kips	Q kips	M_{ult} in. kips	m	$\frac{\tau_s}{\sigma_u}$	K	δ
1 V $\frac{1}{4}$	2940	1.75	1864	58.30	2540	0.73	0.056	0.217	0.96
2 V $\frac{1}{4}$	4085	1.75	2161	67.60	2807	0.77	0.040	0.236	1.00
1 V $\frac{3}{8}$	2648	1.75	2344	73.40	2438	0.96	0.119	0.217	1.09
2 V $\frac{3}{8}$	3500	1.75	2297	71.80	2697	0.85	0.090	0.217	0.99
1 V $\frac{1}{2}$	3115	1.75	2510	78.50	2600	0.97	0.194	0.100	0.84
2 V $\frac{1}{2}$	4090	1.75	2980	83.20	2810	1.06	0.148	0.102	0.80
1a V $\frac{1}{4}$	3010	1.75	1698	53.10	1637	1.04	0.046	0.224	1.10
1a V $\frac{3}{8}$	2833	1.75	1877	58.70	1622	1.16	0.049	0.182	0.83

*) σ_u assumed equal to 0.85 times quoted cylinder strength.

in Figs. 3 through 6. No systematic variation is to be observed in any of the plots, and the value

$$\delta = 1.0$$

is here adopted. The equations then become

$$K = \frac{1}{2}\sqrt{\lambda_0\left[m - \frac{0.4}{\zeta_u\,\eta_u}\left(\frac{\tau_s}{\sigma_u}\right)\right] - \lambda_0^2\left[m - \frac{0.4}{\zeta_u\,\eta_u}\left(\frac{\tau_s}{\sigma_u}\right)\right]^2} \qquad (20\text{a})$$

for $$\lambda_0 \left[m - \frac{0.4}{\zeta_u \eta_u}\left(\frac{\tau_s}{\sigma_u}\right)\right] \geqq 0.25 \tag{21a}$$

and $$K = \frac{1}{8}\sqrt{1 + 8\lambda_0 \left[m - \frac{0.4}{\zeta_u \eta_u}\left(\frac{\tau_s}{\sigma_u}\right)\right]} \tag{20b}$$

for $$\lambda_0 \left[m - \frac{0.4}{\zeta_u \eta_u}\left(\frac{\tau_s}{\sigma_u}\right)\right] \leqq 0.25. \tag{21b}$$

2.5. Simplification

The use of the above expressions is considerably complicated by the two ranges denoted in the equations by "*a*" and "*b*". An approximation may however be made for K using a simple second order parabola with constant coefficients chosen to provide matching values when $\lambda_0 \left[m \frac{0.4}{\zeta_u \eta_u}\left(\frac{\tau_s}{\sigma_u}\right)\right]$ is equal to zero, 0.5, and unity. Such an equation is

$$K = \frac{1}{8} + \frac{5}{8}\lambda_0 \left[m - \frac{0.4}{\zeta_u \eta_u}\left(\frac{\tau_s}{\sigma_u}\right)\right] - \frac{3}{4}\lambda_0^2 \left[m - \frac{0.4}{\zeta_u \eta_u}\left(\frac{\tau_s}{\sigma_u}\right)\right]^2, \tag{22}$$

which holds for the full range of applicability. Eq. (22) is compared with Eqs. (20a) and (20b) in Fig. 7. Although the approximation is not too close, it is probably within the accuracy of the analysis; use of a higher order approximating parabola would certainly not be justified. It will be seen later that abscissa values greater than 0.5, where the correspondence of the two curves is poorest, can only occur for the unusual case of heavily over-reinforced concrete sections.

The equations derived in this section for the stirrup steel requirements in the vicinity of section XX may thus be written as

$$\rho = \frac{Q - Q_c}{\sigma_f b h}, \tag{23}$$

$$Q_c = \sigma_u \zeta_0 K b h. \tag{15}$$

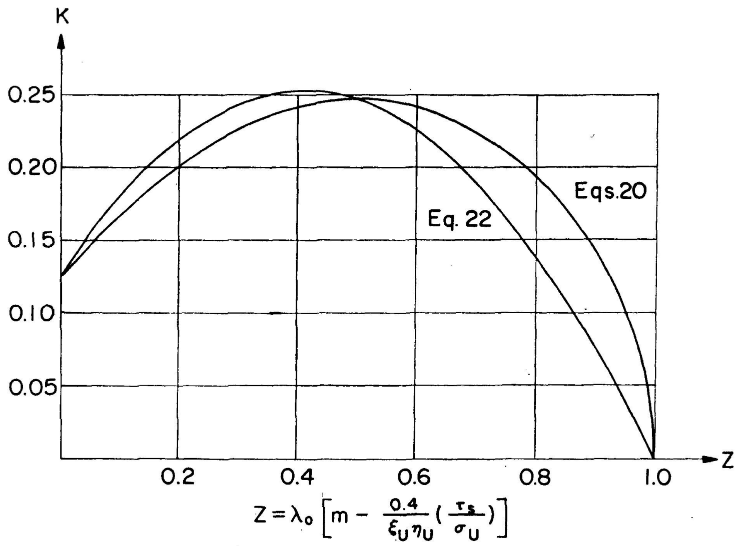

Fig. 7.

The dimensionless term K, evaluated in Eq. (22), can vary between zero and 0.25. The manner in which its value is affected by the prime variables will now be considered briefly.

3. Variation of K

Since the value of Q_c depends directly upon K, as shown in Eq. (15), the effect of the different variables on Q_c is reflected in their influence on the value of K.

In order to indicate more clearly the prime variables in Eq. (22) a rearrangement may be made as follows,

$$\lambda_0 \left[m - \frac{0.4}{\zeta_u \eta_u}\left(\frac{\tau_s}{\sigma_u}\right)\right] = \lambda_0 \left[m - 0.4 \frac{Q_s}{Q} \frac{Q\,h}{M} \frac{M}{M_u}\right] = \lambda_0 m (1 - 0.4 q_s \gamma),$$

where $$q_s = \frac{Q_s}{Q}, \quad \text{and} \quad \gamma = \frac{Q\,h}{M}.$$

Eq. (22) may now be rewritten as

$$K = \frac{1}{8} + \frac{5}{8}\lambda_0 m (1 - 0.4 q_s \gamma) - \frac{3}{4}\lambda_0^2 m^2 (1 - 0.4 q_s \gamma)^2, \tag{22a}$$

which shows directly the effect of the shear-to-moment ratio γ and the proportion of web reinforcement as represented by $q_s = Q_s/Q$.

The manner in which K varies with the terms on the right hand side of (22a) is shown graphically in Fig. 8. Considering initially the extreme condition $q_s \gamma = 0$, we see that the relation between K and $\lambda_0 m$ is similar to the original $(\bar{\tau}/\sigma_u) - (\bar{\sigma}/\sigma_u)$ relation. The term λ_0 in effect expresses the degree of flexural reinforcement in the member: In the extreme case of a heavily over-reinforced beam λ_0 can approach unity; for normal designs the value will be much smaller, in the order of 0.3. The variation between K and m is shown in Fig. 8b for λ_0 equal to 1.0 and 0.3. In the first case the peak K value of 0.25 occurs at $m = 0.4$, whereas for $\lambda_0 = 0.3$ the maximum of $K = 0.24$ is found at $m = 1.0$. A reduction in λ_0 is thus seen to move the peak K value to the right in the direction of increasing m.

The introduction of non-zero $q_s \gamma$ values into Eq. (22a) has an effect similar to a further reduction of λ_0, as is seen in Fig. 8c for $\lambda_0 = 1.0$, and in Fig. 8d for $\lambda_0 = 0.3$. The effect of an increase in $q_s \gamma$ or a decrease in λ_0 is therefore most easily pictured qualitatively as a movement to the left along the curve of Fig. 8a.

A lower limit to the value of K in under-reinforced beams is thus seen to be 0.125. Referring to Eq. (22a) it can be seen that this value is obtained either when

$$m = 0$$

or $$(1 - 0.4\, q_s \gamma) = 0,$$

the latter condition implies that the shear-to-moment ratio γ be greater than 2.5. These conditions in general will only exist in the immediate region of a support or a point of contraflexure.

3.1. Extreme Value $m = 1.0$; ($M = M_u$)

If any shearing action is present in the concrete compression zone, failure of the concrete will occur before the maximum longitudinal compressive stresses, corresponding to flexure failure, are reached (The reduction in $(\bar{\sigma}/\sigma_u)$ with increasing $(\bar{\tau}/\sigma_u)$ has been shown in Fig. 2.) From a theoretical viewpoint, therefore, it is not possible for full flexural strength to be achieved in a section subjected to combined bending and shear, since compatibility considerations will always ensure that a portion of the shear Q is in fact carried by the concrete.

A very slight reduction in m from 1.0 is however sufficient to allow the section to carry considerable shearing action. This may be deduced from the fact that the $(\bar{\sigma}/\sigma_u) - (\bar{\tau}/\sigma_u)$ curve is vertical at $(\bar{\sigma}/\sigma_u) = 1.0$. In practice, furthermore, it is not possible to predict M_u precisely, and the reduction in M theoretically required to allow a shear Q in the beam often cannot be detected

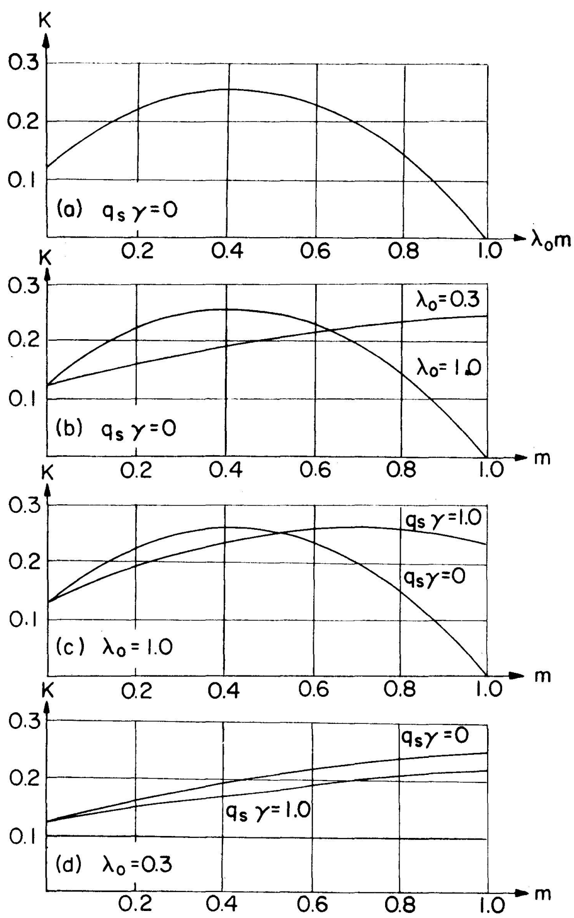

Fig. 8.

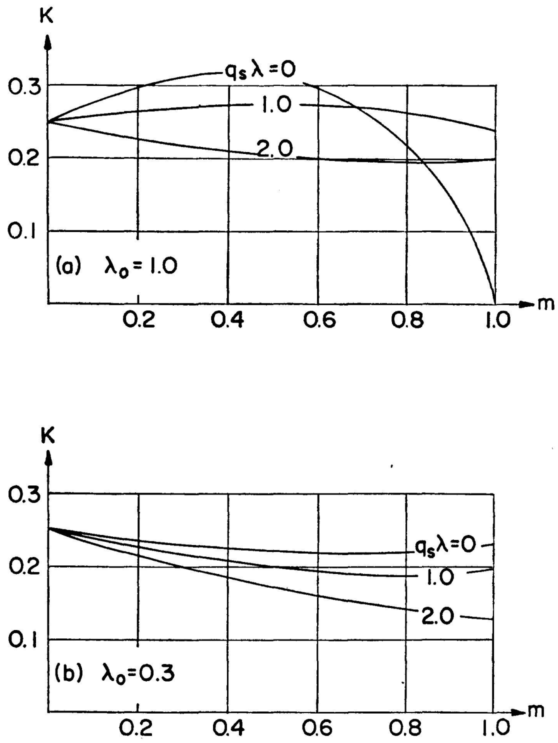

Fig. 9.

experimentally. It is common in laboratory test results to find failures classified as "shear" occurring at moments even greater than the computed flexural capacity. Approximately 30 percent of the test results recorded in Tables 1 and 2 fall into this category.

It may therefore be concluded that although the above analysis will not strictly hold in regions of extremely high moment, the error involved in extrapolating shear design-equations up to $m = 1.0$ is not likely to be serious.

From a theoretical standpoint, allowing the section of maximum moment to carry shearing action amounts to a slight reduction of the ultimate capacity of the member below M_u.

3.2. Small Values of m; M → 0

Of more importance is the situation existing when m becomes small in value. It is to be noted that the quantity δ has been evaluated empirically only from beam shear tests in which the failure moments were greater than 0.73 of M_u. While high values of M_u are almost invariably the case in shear tests of web-reinforced beams, it is an important fact that such test data must be extrapolated to values $0 \leqq m \leqq 0.5$ when the shear reinforcement in all sections of a member is to be proportioned.

Although the equilibrium condition of Fig. 1 will always hold — except for sections closer than h to the supports — it may well be that the height of the inclined crack, represented by

$$\zeta_c h = \delta \zeta_0 h,$$

will not remain constant along the length of the beam. Indeed, it is reasonable to expect that the relatively high moment in the region of failure in the test beams has been in large part responsible for the upward development of the inclined cracks.

This situation is especially emphasized since it can be seen in Eq. (22) that m is a prime variable influencing Q_c and hence the web steel requirements. In previous studies it has always been assumed that the shear-to-moment ratio, Qh/M, alone is sufficient to express the influence of moment on web steel requirements. It has therefore been usual in shear tests to vary only the load configuration to obtain failure at different Qh/M values; the resulting data therefore, almost without exception, lies in the region $m > 0.75$.

That the value $\delta = 1.0$ cannot apply for small moments can be shown by the following reasoning: Since the value of K, and hence Q_c, is seen from Eq. (22) to decrease as m becomes small, the use of $\delta = 1.0$, independent of m, implies that the shear resistance of sections close to the support should be less than for those in the central region where the moment is large. This is contradicted by test results which show that the failure section lies in a region of high shear near the maximum moment.

Since Fig. 4 showed no systematic variation of δ from unity in the range $0.73 \leqq m$, it may be expected that δ will begin to increase sharply only when m becomes relatively small. A simple expression representing this form of variation is

$$\delta = A_1 - A_2 \sin \frac{\pi m}{2}. \tag{23}$$

Retaining the value $\delta = 1.0$ when $m = 1.0$, but stipulating, for example, that $\delta = 2.0$ when $m = 0$, the expression becomes

$$\delta = 2\left(1 - \sin \frac{\pi m}{2}\right). \tag{23a}$$

Use of Eq. (23a) together with (20a) and (20b) produces values for K as shown in Fig. 9; for beams of normal design, the computed strength of the end sections, where m is small, is now at least that of the regions of maximum moment. Before an equation of this type can be adopted, experimental data must be obtained for shear failures at small moments. Such data, which would have to be obtained by over-reinforcing the in-span regions and hence forcing the failure into the outer regions of the member, would provide new information on the design of shear reinforcement in the sections adjacent to supports.

4. Discussion

A number of limitations exist to the above analysis. First, the equations have been derived on the assumption that shear-compression is the governing failure mechanism. It has been seen that inclined-compression failure in the web concrete and bond failure can also occur if the applied shear is high and the web steel percentage is large.

A second limitation is imposed by the empirical evaluation of the term δ. A semi-empirical approach could have been made by introducing a compatibility condition. This may have reduced the scatter exhibited by δ in Tables 1 and 2, and hence improved the reliability of the analysis, but it would have complicated considerably the form of the equations.

It was previously stated that the empirical data was restricted to the region $m > 0.73$. The present work is further restricted to rectangular, simply supported beams under concentrated loading.

An important assumption made was that the angle of the inclined crack is 45 degrees. In practice, cracks often form at a more shallow inclination. This implies an increase in the actual Q_s force, therefore a possible over-estimation of the force Q_c form the test data. However, since the 45 degree assumption is also made in the design equations, this probably does not lead to unconservative results.

With the above limitations pointed out, it must be emphasized that the results of this study are presented not as a final design method, but as a line of approach to the problem which reconciles to some extent the theoretical shear-compression theory and the more practical truss analogy analysis. It has been shown that the shear contribution of the concrete compression zone, Q_c, is dependent not only on the section properties, concrete strength, and proportion of longitudinal reinforcement, but also on the moment ratio, M/M_u, and the amount of shear reinforcement present. This implies that the design of web reinforcement, properly carried out, requires a trial and error procedure. A simplified superposition method,

$$Q \leqq Q_c + Q_s,$$

may however be adopted if lower bound values for K, hence for Q_c, are chosen to replace the more complicated Eq. (23). The contribution of the concrete compression zone, Q_c, has been found to lie in the range

$$0.125 \leqq \frac{Q_c}{\sigma_u \zeta_0 b h} \leqq 0.25.$$

Thus, a very simple, conservative — at times up to 50 percent over-conservative — value for Q_c would be

$$Q_c = \tfrac{1}{8}\sigma_u \zeta_0 b h.$$

Notation

Stresses

σ_u = concrete strength in compression zone under uni-axial compression; assumed in calculations to be 0.85 of the cylinder strength f'_c.

σ_f = yield stress of steel reinforcement.

$\bar{\tau}$ = average shear stress in concrete compression zone above the inclined crack.

$\bar{\sigma}$ = average compressive stress in concrete compressure zone above the inclined crack.

$\tau_s = \frac{Q_s}{b h}$.

$\tau_c = \frac{Q_c}{b h}$.

$\tau = \frac{Q}{b h}$.

Forces and Moments

C_c = longitudinal compressive force above the inclined crack.

C_i = inclined compressive force below the inclined crack.

C_s = horizontal component of C_i; horizontal compressive force below the inclined crack.

Q_c = shear force in concrete compression zone above the inclined crack.

Q_s = shear force in stirrup reinforcement crossing the inclined crack.

Q = total shear force at section XX.

T = tensile force in longitudinal steel reinforcement.

M = moment at section XX.

M_u = flexural capacity of section.

Length Terms and Areas

b = width of member.

h = effective depth of member.

s = horizontal spacing of stirrup reinforcement.

F = area of longitudinal reinforcement.

F_B = area of one stirrup unit.

Dimensionless Parameters

$n = \frac{E_s}{E_c}$; modular ratio.

$m = \frac{M}{M_u}$; moment ratio.

$q_s = \frac{Q_s}{Q}$.

$\gamma = \frac{Q\,h}{M}$; shear to moment ratio.

$\delta = \frac{\zeta_c}{\zeta_0}$.

η_c = ratio of distance between internal horizontal forces T and C_c and effective beam depth.

η_0 = $(1 - \frac{1}{3}\zeta_0)$; internal lever arm for elastic bending as a ratio of the effective beam depth.

η_s = ratio of distance between internal horizontal forces T and C_s and effective beam depth.

η_u = internal lever arm at flexure failure as a proportion of the effective beam depth.

$\lambda_0 = \frac{\zeta_u\,\eta_u}{\zeta_0\,\eta_0}$.

$\mu = \frac{F}{b\,h}$; proportion of longitudinal steel reinforcement.

ζ_c = ratio of depth of concrete compression zone above inclined crack to effective beam depth.

ζ_0 = $\sqrt{(n\mu)^2+2n\mu}-n\mu$; ratio of depth of elastically calculated neutral axis to effective beam depth for simple bending.

ζ_u = ratio of depth of neutral axis at flexure failure to effective beam depth.

ρ = $\frac{F_B}{b\,s}$; proportion of stirrup reinforcement.

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Summary

An approach to the problem of web-reinforcement design is suggested in which shear-compression theory is adapted to estimate the portion of the shearing resistance, Q_c, in excess of the truss analogy value Q_s. The analysis indicates that Q_c depends upon the concrete compressive strength, size of section, amount of longitudinal reinforcement, amount of shear reinforcement, and moment ratio M/M_u. The resulting equations are simpler and more readily adaptable to design than those obtained from the usual limiting moment hypothesis.

Résumé

Les auteurs présentent une contribution à l'étude des armatures de cisaillement des poutres en béton armé. La théorie connue de la rupture par cisaillement et compression combinés est utilisée pour le calcul de la fraction de la résistance au cisaillement Q_c dépassant la valeur Q_s, obtenue par l'analogie des poutres en treillis. Il est montré que Q_c dépend de la résistance à la compression du béton, de la section transversale, du pourcentage d'armatures longitudinales ainsi que de celui d'armatures de cisaillement et du rapport des moments M/M_u. Les équations ainsi obtenues sont plus simples et plus faciles à utiliser que les équations obtenues au moyen de l'hypothèse habituelle des moments critiques.

Zusammenfassung

Die Bemessung der Schubarmierung in Stahlbetonbalken wird untersucht. Unter Anwendung der bekannten Biegeschubbruchtheorie wird der Anteil Q_c des Schubwiderstandes berechnet, der zusätzlich zum Anteil Q_s nach der Fachwerk-Analogie auftritt. Es wird gezeigt, daß Q_c von der Betonfestigkeit σ_u, von dem Querschnitt, dem Längsarmierungsgehalt μ, dem Schubarmierungsgehalt ρ und dem Momentenverhältnisse M/M_u abhängig ist. Die hergeleiteten Gleichungen sind für die Anwendung einfacher als die aus der gewöhnlichen Schub-Momenten-Hypothese (Limiting Moment) resultierenden Gleichungen.