Zeitschrift: IABSE publications = Mémoires AIPC = IVBH Abhandlungen

Band: 27 (1967)

Artikel: Inelastic buckling of plates with residual stresses

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DOI: https://doi.org/10.5169/seals-21550

Inelastic Buckling of Plates with Residual Stresses

Voilement inélastique des âmes comportant des contraintes résiduelles

Unelastisches Ausbeulen von Platten mit Eigenspannungen

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1. Introduction

Welded built-up members are being used more frequently in steel construction due to economy, convenience, and esthetics. The residual stresses produced in the member as a result of the welding play an important role in the buckling strength of the member.

It hade been believed that residual stresses do not affect the elastic buckling of structural members, but this is only true for column buckling of the Euler type.

When a flat plate containing residual stresses is subjected to a thrust, it may buckle in one of three ways according to the magnitude of the thrust, that is, either elastic buckling, elastic-plastic buckling, or plastic buckling.

When a thin plate is subjected to compressive forces, shearing forces, or their combination, the differential equation for the plate in the elastic range takes the form [1]

$$D\left[\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\,\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right]+h\left[\sigma_x\frac{\partial^2 w}{\partial x^2}+2\,\tau_{xy}\frac{\partial^2 w}{\partial x\,\partial y}+\sigma_y\frac{\partial^2 w}{\partial y^2}\right]=0\,, \tag{1}$$

where D = flexural rigidity of the plate $= \frac{E\,h^3}{12\,(1-\nu^2)}$,
w = deflection of plate,
σ_x, σ_y = normal stress components in the cartesian coordinates,
τ_{xy} = shearing stress in the cartesian coordinates,
E = Young's modulus,
h = thickness of plate,
ν = Poisson's ratio.

This differential equation was solved for many cases under different boundary conditions by TIMOSHENKO [1, 2] and many others.

From 1924, attempts to extend the theory of plate stability into the inelastic range were made by F. BLEICH [3], CHWALLA [4], ROS and EICHINGER [5].

In 1941, MADSEN noted and discussed qualitatively the effect of residual stress on plate buckling [6]; his work led directly to the later recognition of residual stress effects in columns.

As the theory of plasticity developed, new theories were presented for plastic buckling. One was based on the deformation theory [7], and another on the flow theory of plasticity [8, 9]. Later, both theories were modified by using the SHANLEY concept [10, 11, 12, 13].

In 1960, OKERBLOM presented a paper [14] concerning the influence of residual stresses on the stability of welded structures and structural members, based on experimental results. His paper showed that there was a possibility of elastic buckling of plate elements of a structure which had been fabricated by welding.

In the same year, YOSHIKI, FUJITA and KAWAI [15) investigated analytically the influence of residual stresses on the elastic buckling of centrally welded plates, and showed that the residual stresses could influence the elastic buckling of a plate.

These two studies are apparently the only papers concerned with elastic buckling of plates with residual stresses. There is no theoretical research other than in the elastic range.

In fact, the method of analysis presented in this paper is believed to be the first approach to the solution of the elastic-plastic and plastic buckling of plates with residual stresses.

This paper is based on a dissertation [16] to which reference may be made for detailed information on the history of the study of plate buckling and for a complete summary of the theories and formulas involved.

2. Analysis of Buckling Strength

1. General Approach

The buckling strength of a plate with residual stresses is evaluated by the energy method in this paper. The behavior of a plate is analyzed by the theory of elasticity and by the two theories of plasticity (the secant modulus deformation theory and the flow theory). These theories are based on relationships between stress and strain which are described below, in Section 2.2.

The theorem of minimum potential energy [17] is valid for an elastic body; it is valid also for a plastic body in which the reversal of strain is not allowed. Expressions for the total potential energy of a plate with residual stresses were derived in this study for the elastic and plastic ranges. In the plastic range, the expression was based on both the deformation theory and on the flow theory, using the Shanley concept. By adopting a suitable stress-strain relationship for each domain of the plate and by substituting the appropriate deflection functions, the potential energy in the plate may be evaluated taking into account the effect of residual stresses in the plate. The minimization of the potential energy leads to the equilibrium condition according to the theorem of minimum potential energy. When the Ritz method is employed to minimize the potential energy, there results a set of simultaneous equations with respect to the coefficients which appear in the assumed deflection function. In the Ritz method, the assumed deflection functions must satisfy the geometric boundary conditions and these functions must be as complete as possible. For buckling problems, the set of simultaneous equations are homogeneous. The non-trivial solution of this set of homogeneous simultaneous equations is possible only if the coefficient is equal to zero

The roots of such a coefficient determinant will give the critical values of buckling strength of the member, these corresponding to the characteristic values, the lowest of which is the critical buckling strength.

2. *Stress-Strain Relationship in the Elastic and Plastic Ranges*

The behavior of the plate was analyzed by using the theories of elasticity and plasticity in the elastic and the plastic parts respectively. For material strained into the plastic range, two theories of plasticity were used, one being the secant modulus deformation theory, and the other the flow theory. In the plane stress problem, those theories are based on the following stress-strain relationship:

a) Elastic range, from the theory of elasticity [18, 19].

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu\,\sigma_y], \qquad \epsilon_y = \frac{1}{E}[\sigma_y - \nu\,\sigma_x], \qquad \gamma_{xy} = \frac{2\,(1+\nu)}{E}\tau_{xy}, \tag{2}$$

where ϵ_x, ϵ_y = normal strain components in cartesian coordinates,
γ_{xy} = shearing strain in cartesian coordinates.

b) Plastic range, from the secant modulus deformation theroy [7].
for loading:

$$\epsilon_x = \frac{1}{E_s}[\sigma_x - \nu\,\sigma_y], \qquad \epsilon_y = \frac{1}{E_s}[\sigma_y - \nu\,\sigma_x], \qquad \gamma_{xy} = \frac{2\,(1+\nu)}{E_s}\tau_{xy}, \tag{3}$$

where
$$E_s = \frac{\sigma_i}{\epsilon_i},$$
$$\sigma_i = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\,\sigma_y + 3\,\tau_{xy}^2} \quad \text{intensity of stress}, \tag{4}$$
$$\epsilon_i = \frac{2}{\sqrt{3}}\sqrt{\epsilon_x^2 + \epsilon_y^2 + \epsilon_x\,\epsilon_y + \frac{\gamma_{xy}}{4}} \quad \text{intensity of strain;}$$

for unloading (the material is assumed to behave completely elastically):

$$d\epsilon_x = \frac{1}{E}[d\sigma_x - \nu\, d\sigma_y], \qquad d\epsilon_y = \frac{1}{E}[d\sigma_y - \nu\, d\sigma_x], \qquad d\gamma_{xy} = \frac{2\,(1+\nu)}{E}\, d\tau_{xy}, \tag{5}$$

where the relationship is given in the form of a variation to eliminate the effect of a permanent set.

c) Plastic range, from the flow theory [20].

for loading:

$$\begin{aligned} \dot{\epsilon}_x &= \frac{1}{E}\left[\lambda\,\dot{\sigma}_x - \left(\nu + \frac{\lambda-1}{2}\right)\dot{\sigma}_y\right], \\ \dot{\epsilon}_y &= \frac{1}{E}\left[-\left(\nu + \frac{\lambda-1}{2}\,\dot{\sigma}_x + \frac{\lambda+3}{4}\,\dot{\sigma}_y\right]\right., \\ \dot{\gamma}_{xy} &= \frac{2\,(1+\nu)}{E}\,\dot{\tau}_{xy}, \end{aligned} \tag{6}$$

where $\dot{\epsilon}_x, \dot{\epsilon}_y$ = rate of change of strain components in the cartesian coordinates,
$\dot{\sigma}_x, \dot{\sigma}_y$ = rate of change of stress components in the cartesian coordinates,
$\lambda = E/E_t = l/\alpha t$ (ratio).

for unloading:

$$\dot{\epsilon}_x = \frac{1}{E}[\dot{\sigma}_x - \nu\,\dot{\sigma}_y], \qquad \dot{\epsilon}_y = \frac{1}{E}[\dot{\sigma}_y - \nu\,\dot{\sigma}_x], \qquad \dot{\gamma}_{xy} = \frac{2\,(1+\nu)}{E}\,\dot{\tau}_{xy}. \tag{7}$$

3. Residual Stress Distribution

Steel structures fabricated by welding contain residual stresses due to the plastic deformations set up by the temperature gradient induced at welding [21, 22, 23].

In general, two residual stress patterns may be regarded as typical for welded plates and for shapes fabricated from plates by welding. One is that due to an edge weld, and the other that due to a center weld [21, 22].

The buckling strength of these plate elements may be investigated on the basis of these two distributions.

It is advantageous to simplify the residual stress distribution for the analysis of the buckling strength of plates [22, 24]. The residual stress pattern used in this study will be of the form shown in Fig. 1, which corresponds to the

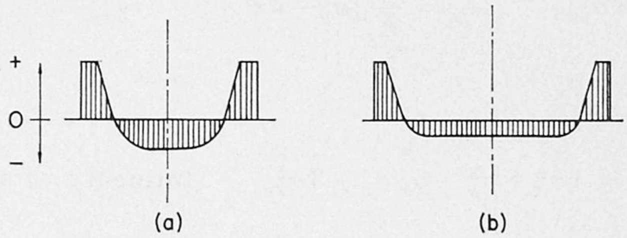

Fig. 1. Usual residual stress distributions in edge welded plates.

pattern obtained in experimental work. For this study the residual stress distribution of Fig. 2a was chosen as a simple approximation of the true pattern. By adjusting the appropriate parameters, this pattern can be reduced readily to other patterns such as those shown in Figs. 2b, 2c and 2d, which correspond to different geometric proportions of plates [16].

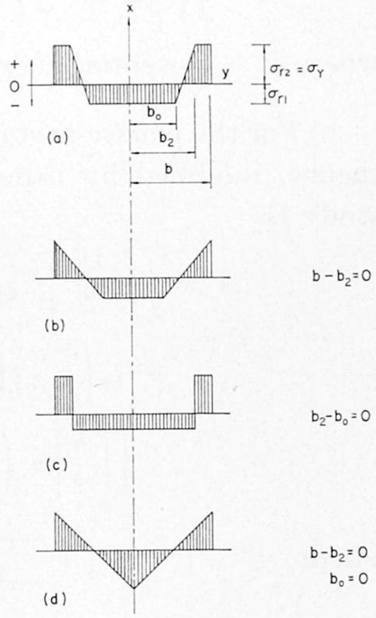

Fig. 2. Simplified residual stress distribution used in the analysis.

4. *Potential Energy of the Plate*

The theorem of minimum potential energy [16, 17] leads to equilibrium differential equations in the elastic and plastic ranges. In this study, the equilibrium differential equations in the plastic range obtained from the theorem of minimum potential energy was shown to be the same as those obtained from consideration of the equilibrium of an element of the body [16]. The characteristic values of these differential equations give the values of the buckling strength.

If the solution of the differential equations is difficult to obtain, the energy method can be used as a powerful tool to solve the problem to sufficient accuracy for engineering purposes.

The total energy of the plate is obtained in the form of a summation of the strain energy stored in the plate and the work done by the external forces acting on the plate, with an additive constant which depends on the reference position. In this study, the reference position was taken as the loaded state prior to buckling.

The potential energy at buckling is shown below for the following cases:

a) In the elastic region of the plate,

b) in the plastic region, using both the secant modulus deformation theory and the flow theory.

a) For the elastic part of the plate, the energy equation may be shown to be [1]:

$$V = \iint \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] dx\,dy$$
$$- \iint \frac{h}{2}\left[\sigma_x\left(\frac{\partial w}{\partial x}\right)^2 + 2\tau_{xy}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) + \sigma_y\left(\frac{\partial w}{\partial y}\right)^2\right] dx\,dy, \quad (8)$$

where V = potential energy stored in the plate.

b) For the plastic part of the plate, based on the secant modulus deformation theory, modified by using Shanley's concept, the expression derived in this study is

$$V = \iint \frac{D_d}{2}\left[C_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2 - C_2\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + C_3'\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right.$$
$$\left. + C_3''\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) - C_4\left(\frac{\partial^2 w}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right) + C_5\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] dx\,dy \quad (9)$$
$$- \iint \frac{h}{2}\left[\sigma_x\left(\frac{\partial w}{\partial x}\right)^2 + 2\tau_{xy}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) + \sigma_y\left(\frac{\partial w}{\partial y}\right)^2\right] dx\,dy,$$

where

$$C_1 = 1 - \frac{\sigma_x^2}{\sigma_i^2}(1-\nu^2)\,\kappa,$$

$$C_2 = \frac{4\,\sigma_x\,\tau_{xy}}{\sigma_i^2}(1-\nu)^2\,\kappa,$$

$$C_3' = 2\left[(1-\nu) - \frac{\tau_{xy}^2}{\sigma_i^2}(1-\nu^2)\,\kappa\right],$$

$$C_3'' = 2\left[\nu - \frac{\sigma_x\,\sigma_y}{\sigma_i^2}(1-\nu^2)\,\kappa\right],$$

$$C_4 = \frac{4\,\sigma_y\,\tau_{xy}}{\sigma_i^2}(1-\nu^2)\,\kappa,$$

$$C_5 = 1 - \frac{\sigma_y^2}{\sigma_i^2}(1-\nu^2)\,\kappa,$$

$$\kappa = 1 - \frac{E_t}{E_s},$$

D_d = flexural rigidity of plate in the plastic range, based on the deformation theory $= \dfrac{E_s h^3}{12(1-\nu^2)}$.

Eq. (9) is similar to Stowells's equation [11, 12], except for the coefficients C_3' and C_3''. In Stowells's equation,

$$C_3' = C_3'' = 1 - \frac{3}{4}\,\frac{\sigma_x\,\sigma_y + 2\,\tau_{xy}^2}{\sigma_i^2}\,\kappa.$$

Based on the flow theory [13, 20] modified by using Shanley's concept, the expression derived in this study is:

$$V = \iint \frac{D_f}{2}\left[C_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + C_3'\left(\frac{\partial^2 w}{\partial x\,\partial y}\right) + C_3''\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + C_5\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] dx\,dy$$
$$- \iint \frac{h}{2}\left[\sigma_x\left(\frac{\partial w}{\partial x}\right)^2 + 2\,\tau_{xy}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) + \sigma_y\left(\frac{\partial w}{\partial y}\right)^2\right] dx\,dy, \qquad (10)$$

where $C_1 = \dfrac{(1-\nu^2)(\lambda+3)}{(5-4\nu)\lambda-(1-2\nu)^2}$,

$C_3' = 2(1-\nu)$,

$C_3'' = \dfrac{4(1-\nu^2)(2\nu+\lambda-1)}{(5-4\nu)\lambda-(1-2\nu)^2}$,

$C_5 = \dfrac{4\lambda(1-\nu^2)}{(5-4\nu)\lambda-(1-2\nu)^2}$,

D_f = flexural rigidity of plate in the plastic range, based on the flow theory $= \dfrac{E h^3}{12(1-\nu^2)}$.

Eq. (9) coincides with Eq. (8), when it is applied to the elastic zone, where $E_t = E_s = E$. Likewise, Eq. (10) reduces to Eq. (8), since $E_t = E$ in the elastic zone.

5. *Residual Stresses and the Plate Equation*

For an elastic isotropic plate of a constant thickness, h, which is subjected to an edge thrust in the direction x, the equilibrium differential equation may be expressed as [1].

$$D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] = -\sigma_x h \frac{\partial^2 w}{\partial x^2}. \qquad (11)$$

When the plate contains residual stress, the stress in the x direction may be expressed in the form of the summation of the stresses, $\sigma_{rx} + \sigma_x$, and Eq. (11) becomes [16]

$$D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] = -(\sigma_{rx} + \sigma_x)\, h \frac{\partial^2 w}{\partial x^2}, \qquad (12)$$

where σ_{rx} is the residual stress in the x direction.

The energy equation, Eq. (8), modified for the effects of residual stress in this case, becomes

$$V = \iint \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] dx\,dy$$
$$- \iint \frac{h}{2}(\sigma_{rx} + \sigma_x)\left(\frac{\partial w}{\partial x}\right)^2 dx\,dy. \qquad (13)$$

Similarly, the components of residual stresses, σ_{rx}, σ_{ry} and τ_{xy}, may be introduced into the expression for the potential energy of the plate [16].

$$
\begin{aligned}
V = & \iint \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] dx\,dy \\
& - \iint \frac{h}{2}\left[(\sigma_{rx}+\sigma_x)\left(\frac{\partial w}{\partial x}\right)^2 + 2(\tau_{rxy}+\tau_{xy})\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) + (\sigma_{ry}+\sigma_y)\left(\frac{\partial w}{\partial y}\right)^2\right] dx\,dy .
\end{aligned}
\tag{14}
$$

where σ_{rx}, σ_{ry}, τ_{rxy} are the components of residual stresses in the cartesian co-ordinates.

The above equations are the elastic domains, but similar equations may be derived for the plastic domain, by taking into account the yield condition.

The relationship between stress and strain, secant and tangent moduli under load are shown in Table 1, and illustrated in Fig. 3. These relationship are based on the assumed residual stress distributions of Figs. 2a and 3a.

Table 1. Distribution of Stress and Strain, and Secant and Tangent Moduli in Plates

Original State

Domain	Strain	Stress	E_s	E_t
$0-b_0$	$-\sigma_{r1}/E$	$-\sigma_{r1}$	E	E
t_0-b_2	$-\phi(y)/E$	$-\phi(y)$	E	E
b_2-b	$\sigma_{r2}/E \leqq \epsilon_Y$	$\sigma_{r2} \leqq \sigma_Y$	E	E

Elastic Buckling

$0-b_0$	$-[\sigma_{r1}/E+\epsilon_c]$	$-[\sigma_{r1}+\sigma_c]$	E	E
b_0-b_2	$-[\phi(y)/E+\epsilon_c]$	$-[\phi(y)+\sigma_c]$	E	E
b_2-b	$\sigma_{r2}/E-\epsilon_c$	$\sigma_{r2}-\sigma_c$	E	E

Elastic-Plastic Buckling

$0-b_0$	$-[\sigma_{r1}/E+\epsilon_c]$	$-\sigma_Y$	$E\,\sigma_Y/\sigma_{r2}+\sigma_c$	0
b_0-b_1	$-[\phi(y)/E+\epsilon_c]$	$-\sigma_Y$	$E\,\sigma_Y/\phi(y)+\sigma_c$	0
b_1-b_2	$-[\phi(y)/E+\epsilon_c]$	$-[\phi(y)+\sigma_c]$	E	E
b_2-b	$\sigma_{r2}/E-\epsilon_c$	$\sigma_{r2}-\sigma_c$	E	E

Plastic Buckling

$0-b_0$	$-[\sigma_{r1}/E+\epsilon_c]$	$-\sigma_Y$	$E\,\sigma_Y/\sigma_{r1}+\sigma_c$	0
b_0-b_1	$-[\phi(y)/E+\epsilon_c]$	$-\sigma_Y$	$E\,\sigma_Y/\phi(y)+\sigma_c$	0
b_1-b_2	$-[\phi(y)/E+\epsilon_c]$	$-\sigma_Y$	$E\,\sigma_Y/\phi(y)+\sigma_c$	0
b_2-b	$-[\epsilon_c-\sigma_{r2}/E]$	$-\sigma_Y$	$E\,\sigma_Y/\sigma_c+\sigma_{r2}$	0

where $\phi(y) = \sigma_{r1} - (y-b_0)\dfrac{\sigma_{r1}+\sigma_{r2}}{b_2-b_0} = \dfrac{\sigma_{r1}\,b_2+\sigma_{r2}\,b_0}{b_2-b_0} - \dfrac{\sigma_{r1}+\sigma_{r2}}{b_2-b_0}\,y$, $\sigma_c = E\,\epsilon_c$.

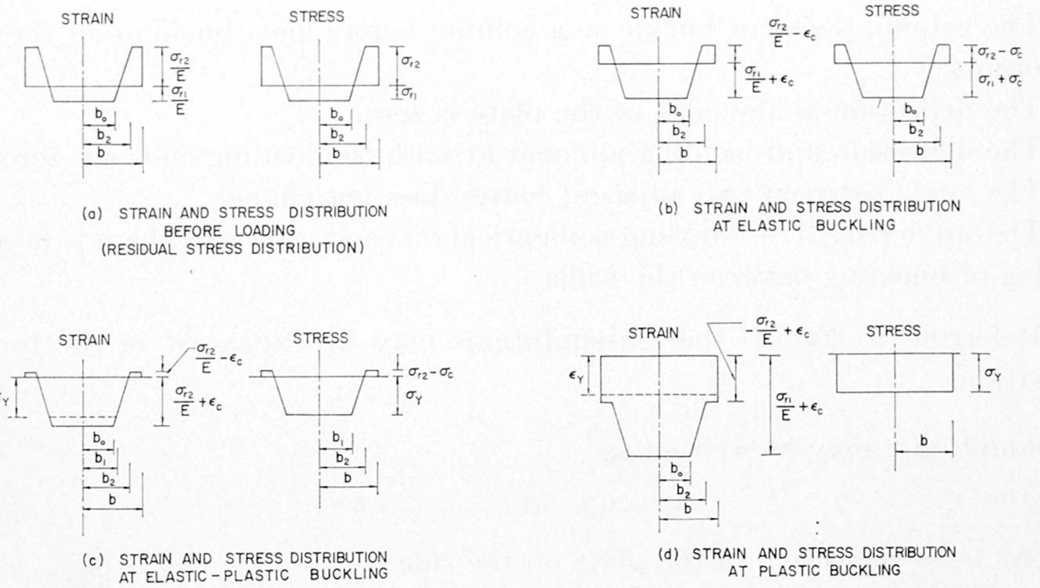

Fig. 3. Relationship between stress and strain in a loaded plate containing residual stresses.

6. *Local Buckling of Built-Up Columns*

Among the problem of the buckling of plates with residual stresses studied in the general investigation [16], special attention was paid to the study of the local buckling of built-up columns.

The method of solution of this kind of problem is quite similar for both closed and open sections except for the boundary conditions at the open plate edges.

In this paper, attention is limited to closed sections.

A closed column section is composed of several walls each of which consists of a flat plate. That is, the study of the local buckling strength of this kind of a column is reduced to a study of the problem of the buckling of plates connected at their edges. The study of the local buckling is considered under the following assumptions [19]:

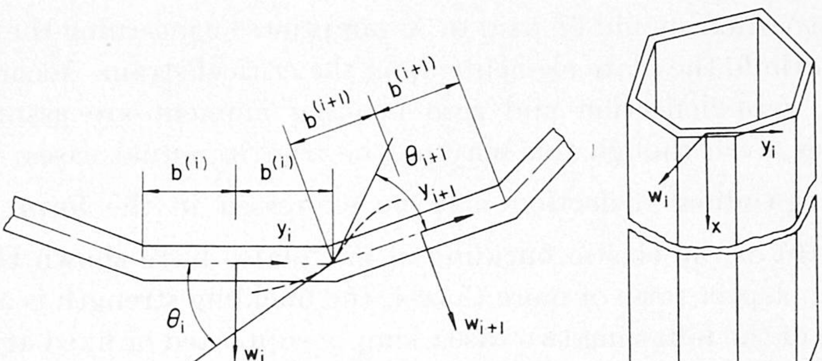

Fig. 4. Coordinate axes for plate elements.

1. The column does not buckle as a column before local buckling of the walls occurs.
2. The deflection at the edge of the plate is zero.
3. The deflection and bending moment at both the loading ends are zero.
4. The angle between two adjacent plates does not change.
5. The wave length of buckling is identical on each wall, and there is no phase lag of buckling between the walls.

Referring to Fig. 4, these assumptions may be expressed in the form of equations.

Assumption 2 may be written as

$$w_i = 0 \quad \text{at} \quad y_i = \pm\, b^{(i)}, \tag{15}$$

where w_i = deflection of the plate on the side i,
y_i = y-axis of the cartesian coordinate of plate on the side i,
$b^{(i)}$ = half width of plate element on side i.

Assumption 4 may be written as

$$\begin{aligned} \theta_i &= \theta_{i+1}, & M_i &= M_{i+1} & \text{at} \quad y_i &= b^{(i)}, \\ \theta_i &= \theta_{i-1}, & M_i &= M_{i-1} & \text{at} \quad y_i &= b^{(i)}, \end{aligned} \tag{16}$$

where θ_i = angle of rotation at edge of plate i,
M_i = bending moment per unit length of section of plate perpendicular to the x axis.

Assumptions 3 and 5 suggest the following equation for the deflection function:

$$w_i = f_i\,(y_i) \sin N \frac{x}{L}, \tag{17}$$

where N = number of half waves in the direction of the x axis,
L = entire length of column,
$f_i\,(y_i)$ = deflection function expressed in the direction of the x axis.

Special attention should be paid to Assumption 3 concerning the influence of the aspect ratio fo the plate elements upon the critical strain. According to this assumption, zero deflection and zero bending moment are assumed at the loading edges, even though this may not be true in actual cases; this implies that the longitudinal deflection may be expressed in the form of $\sin N \frac{x}{L}$. Studies [1, 19] of the elastic buckling of flat plates have shown that, for the plate with an aspect ratio of more than 4, the buckling strength is almost identical in both of the following two cases, simply supported or fixed at the loading edges, regardless of edge conditions. From this, it is quite rational to presume that the walls of a column will buckle at the ratio of L/b which gives the lowest critical value, that is, a plate simply supported at the loading edges, regardless of the conditions of the other edges, The local buckling strength of a built-up column may be predicted from this point of view, since the aspect ratio of plate elements is more than four in most practical cases.

3. Analytical Solutions

Analytical solutions were obtained and are presented for the elastic, elastic-plastic, and plastic buckling of a plate with residual stresses when the plate is simply supported at the loading edges and at the other edges is:

a) elastically restrained,
b) simply supported,
c) fixed.

From this solutions the local buckling strength of a built-up column of rectangular cross section will be obtained. Case a) corresponds to a rectangullar cross section, Case b) to a square cross section, and Case c) to a limiting case. All these cases are illustrated in Fig. 5.

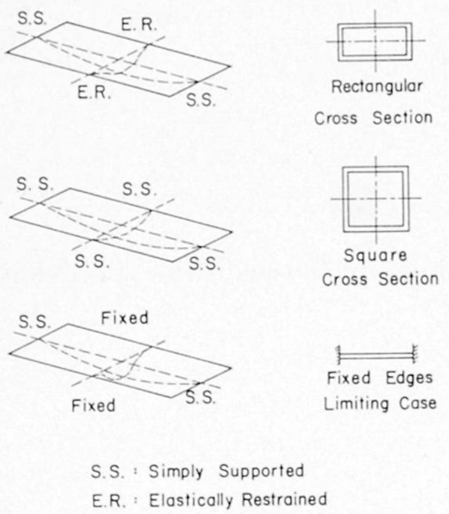

Fig. 5. Boundary conditions of the plates.

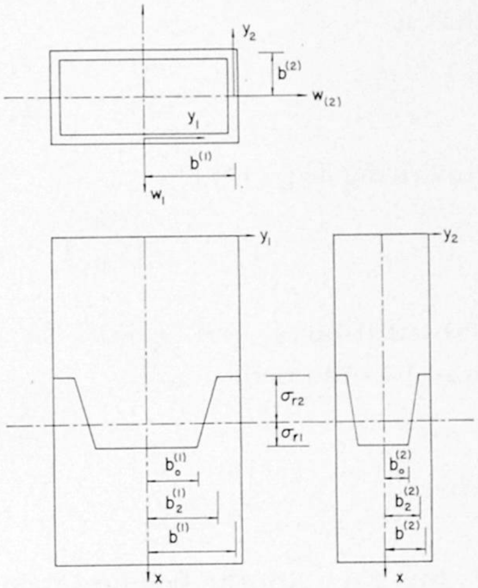

Fig. 6. Coordinate axes for a column of rectangular cross section.

1. Plate Elastically Restrained (Rectangular Cross Section)

The cross section here consists of two different pairs of plates, Fig. 6.

The following additional assumptions [16] are added to those given in Section 2.6.

1. The material properties of all plates are the same: yield point, Young's modulus, Poisson's ratio; both in the elastic and the plastic ranges respectively.

2. Each pair of parallel plates are of the same size.
3. Each pair of parallel plates has the same residual stress distribution.
4. The residual stress distribution in both pairs of plates is similar in shape to each other (Fig. 6).

A combination of two series of sinusoidal functions was chosen [16] as the deflection function of a plate element of a rectangular box column,

$$
\begin{aligned}
w_1 &= \left[\sum a_{1m} \cos\left(\frac{2m-1}{2}\pi\frac{y_1}{b^{(1)}}\right) + \sum C_{1n}\left\{\cos\left(n\pi\frac{y_1}{b^{(1)}}\right) - (-1)^n\right\}\right] \sin N\pi\frac{x}{L},\\
w_2 &= \left[\sum a_{2m} \cos\left(\frac{2m-1}{2}\pi\frac{y_2}{b^{(2)}}\right) + \sum C_{2n}\left\{\cos\left(n\pi\frac{y_2}{b^{(2)}}\right) - (-1)^n\right\}\right] \sin N\pi\frac{x}{L},
\end{aligned}
\tag{18}
$$

where $a_{1m}, a_{2m}, c_{1n}, c_{2n}$ = coefficients of deflection functions,
m, n = positive integers.

These deflection functions were assumed so that the functions satisfy all the boundary conditions mentioned above.

For Assumption 4: the angle between two adjacent plates does not change, that is,

$$\text{a)} \qquad \theta_{1\,y_1=\pm b^{(1)}} = \theta_{2\,y_2=\mp b^{(2)}}, \tag{19}$$

$$\text{b)} \qquad M_{1\,y_1=\pm b^{(1)}} = M_{2\,y_2=\mp b^{(2)}}. \tag{20}$$

Rewriting Eq. (19),

$$\left(\frac{\partial w_1}{\partial y_1}\right)_{y_1=\pm b^{(1)}} = \left(\frac{\partial w_2}{\partial y_2}\right)_{y_2=\mp b^{(2)}}. \tag{21}$$

Substituting w_1 and w_2 into Eq. (21), the relationship between the coefficients may be obtained.

$$\sum a_{2m} = \alpha \sum a_{1m}, \tag{22}$$

where
$$\alpha = -\frac{b^{(2)}}{b^{(1)}}.$$

For Eq. (20), the bending moment in the plate may be expressed [24] in the form of

$$M = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right). \tag{23}$$

At the edge, $w = \frac{\partial w}{\partial x} = \frac{\partial^2 w}{\partial x^2} = 0$ and consequently the bending moment takes the form, $M = -D\left(\frac{\partial^2 w}{\partial y^2}\right)$, and the boundary condition is

$$D_1\left(\frac{\partial^2 w}{\partial y^2}\right)_{y_1=\pm b^{(1)}} = D_2\left(\frac{\partial^2 w}{\partial y^2}\right)_{y_2=\mp b^{(2)}}, \tag{24}$$

where D_1 and D_2 are the respective bending rigidities of the plates,

$$
\begin{aligned}
&\text{in the elastic range} && D = \frac{E\,h^3}{12\,(1-\nu^2)},\\
&\text{in the plastic range (deformation theory) } \nu = 0.5 && D_d = \frac{E_s\,h^3}{12\,(1-\nu^2)} = \frac{E_s\,h^3}{9},\\
&\text{in the plastic range (flow theory) } \nu = 0.5 && D_f = \frac{E\,h^3}{12\,(1-\nu^2)} = \frac{E\,h^3}{9}.
\end{aligned}
\tag{25}
$$

The boundary condition Eq. (24) gives the relationship between the coefficients in the expression for w_1 and w_2 as

$$\sum C_{2n} = \beta \sum C_{1n}, \tag{26}$$

where

$$\beta = \left(\frac{D_1}{D_2}\right)\left(\frac{b^{(2)}}{b^{(1)}}\right)^2.$$

Substituting the relationship of Eqs. (21) and (26), the assumed deflection of Eq. (18) becomes

$$
\begin{aligned}
w_1 &= \left[\sum a_m \cos\left(\frac{2m-1}{2}\,\pi\,\frac{y_1}{b^{(1)}}\right) + \sum C_n\left\{\cos\left(n\,\pi\,\frac{y_1}{b^{(1)}}\right) - (-1)^n\right\}\right] \sin N\,\pi\,\frac{x}{L},\\
w_2 &= \left[\alpha \sum a_m \cos\left(\frac{2m-1}{2}\,\pi\,\frac{y_2}{b^{(2)}}\right) + \beta \sum C_n\left\{\cos\left(n\,\pi\,\frac{y_2}{b^{(2)}}\right) - (-1)^n\right\}\right] \sin N\,\pi\,\frac{x}{L}.
\end{aligned}
\tag{27}
$$

The first term in the brackets of the equation corresponds to the deflection of a plate simply supported at all four edges and the second term is closely associated with that for a plate simply supported at the loading edges and fixed at the other edges.

Sufficiently accurate results were obtained by taking only the first term of each series ($m = 1$, $n = 1$) in Eq. (27) (see Section 4). Then the assumed deflection becomes

$$
\begin{aligned}
w_1 &= \left[a \cos\left(\frac{\pi}{2}\,\frac{y_1}{b^{(1)}}\right) + C\left\{\cos\left(\pi\,\frac{y_1}{b^{(1)}}\right) + 1\right\}\right] \sin N\,\pi\,\frac{x}{L},\\
w_2 &= \left[\alpha\, a \cos\left(\frac{\pi}{2}\,\frac{y_2}{b^{(2)}}\right) + \beta\, C\left\{\cos\left(\pi\,\frac{y_2}{b^{(2)}}\right) + 1\right\}\right] \sin N\,\pi\,\frac{x}{L}.
\end{aligned}
\tag{28}
$$

Introducing the above equations, Eq. (28), into the expressions for the energy integral and carrying out the integration in each part (the elastic parts and the plastic part), and taking into consideration the different stresses and the secant moduli, the total potential energy is obtained; $V = V_1 + V_2$ (where V_1 and V_2 are the potential energy in each plate).

Using the Ritz method, and minimizing the potential energy, the partial differentation with respect to the coefficients a and c leads to the following homogeneous equations,

$$\frac{\partial V}{\partial a} = 0 \quad \text{and} \quad \frac{\partial V}{\partial c} = 0 \tag{29}$$

with the result that

$$a F_{11} + c F_{12} = 0, \qquad a F_{21} + c F_{22} = 0. \tag{30}$$

Each component in Eq. (30), F_{11}, F_{22}, F_{12}, F_{21}, is listed in Appendix 1, in the sequence of the cases for elastic, elastic-plastic (deformation theory), elastic-plastic (flow theory), and plastic buckling (deformation theory).

The requirement that the coefficient determinant of Eq. (30) is zero gives the stability condition

$$F_{11} F_{22} - F_{12} F_{21} = 0 \tag{31}$$

from which the critical buckling value can be computed.

2. *Plate Simply Supported (Square Cross Section)*

In this study, it was assumed that the four plates which compose a square column are identical in material properties, size of plates, and the distribution of residual stress.

The symmetry of the structure and of the residual stress distribution render the analysis comparatively simple. The solution for this case was obtained as a limiting case of the previous problem of rectangular cross section. The assumed deflection function must satisfy the same boundary conditions as before. Because of symmetry, only one plate, simply supported at all the edges, need be investigated.

Choosing only the terms associated with the deflection of a simply supported plate in Eq. (27), and taking $m=2$, the following equation is obtained, which satisfies all the boundary conditions.

$$w = \left[a_1 \cos\left(\frac{\pi}{2}\frac{y}{b}\right) + a_2 \cos\left(\frac{3\pi}{2}\frac{y}{b}\right)\right] \sin N\pi\frac{x}{L}. \tag{32}$$

Following exactly the same procedure as in the previous section,

$$a_1 F_{11} + a_2 F_{12} = 0, \qquad a_1 F_{21} + a_2 F_{22} = 0 \tag{33}$$

and

$$F_{11} F_{22} - F_{12} F_{21} = 0, \tag{34}$$

which gives the critical strain. F_{11}, F_{22} and $F_{12}\,(=F_{21})$, are presented in Appendix 2.

The following equation gives the first approximation for the critical buckling strain,

$$F_{11} = 0. \tag{35}$$

This corresponds to the limiting case of a rectangular section, and also to the case where the deflection is assumed as

$$w = a \cos\left(\frac{\pi}{2}\frac{y}{b}\right) \sin N\pi\frac{x}{L}. \tag{36}$$

When there is no residual stress in the plate, this deflection is the exact one [1, 3].

3. Plate Fixed at the Edges

A limiting case of the rectangular section is the one which corresponds to a pair of opposite plates which have infinite bending rigidity, as, for example, stiffeners used in ship structures.

The boundary conditions in the case are fundamentally the same as for the rectangular section, except for the condition that $\theta_1 = \theta_2 = 0$ at the edges. This leads to $a_m = 0$ in the deflection function, Eq. (27).

The second term in the brackets of the deflection equation, Eq. (27), fulfills these boundary conditions. That is,

$$w = \sum_{n=0} C_n \left[\cos\left(n\pi\frac{y}{b}\right) - (-1)^n\right] \sin N\pi\frac{x}{L} \tag{37}$$

and for $n = 2$,

$$F_{11}F_{22} - F_{12}F_{21} = 0\,, \tag{38}$$

which gives the critical strain.

The components of the determinant, F_{11}, F_{22} and $F_{12} = F_{21}$, are shown in Appendix 3.

The first approximation of the deflection,

$$w = C\left[\cos\left(\pi\frac{y}{b}\right) + 1\right] \sin N\pi\frac{x}{L} \tag{39}$$

is identical to that by Cox [26] for elastic buckling without residual stresses.

As above, $F_{11} = 0$, gives the first approximation for the solution, and is the same as for the limiting case of rectangular sections.

4. Numerical Illustration

The analytical solutions were used to obtain the local buckling strength of built-up columns of square cross section.

When the plate sizes and the distribution of residual stress are specified, the critical stress or strain of the elastic, elastic-plastic, and plastic buckling may be obtained from Eqs. (34) and (35).

The numerical calculation was carried out by a digital computer, the L. G. P. at Lehigh University [16].

In the numerical calculations, Poisson's ratio was assumed to be 0.3 in the elastic range, and 0.5 in the plastic range [16].

The results of numerical calculations from Eqs. (34) and (35) were compared, and it was concluded that Eq. (35) (considering only the first term for the shape of the deflection), gives sufficiently accurate answers [16].

1. Elastic Buckling

In this case, the critical stress of buckling obtained from Eq. (35) is a complicated expression, but can be arranged as

$$\frac{\epsilon_c}{\epsilon_Y} = \frac{\sigma_c}{\sigma_Y} = \left(\frac{\sigma_{cr}}{\sigma_Y}\right)_0 - R, \tag{40}$$

where $\left(\frac{\sigma_{cr}}{\sigma_Y}\right)_0$ = ratio of critical stress to yield point, for elastic buckling without residual stresses,

R = reduction of buckling strength due to residual stresses

$= f\left(\frac{\sigma_{r1}}{\sigma_Y}, \frac{\sigma_{r2}}{\sigma_Y}, \frac{b_0}{b}, \frac{b_2}{b}\right).$

Eq. (40) implies that the influence of residual stresses may be evaluated from the residual stress distribution independently of the critical stresses, and that the critical stress, taking account of the residual stresses, may be obtained readily from the critical stress without residual stresses.

Eq. (40) has been plotted in Figs. 7 and 8.

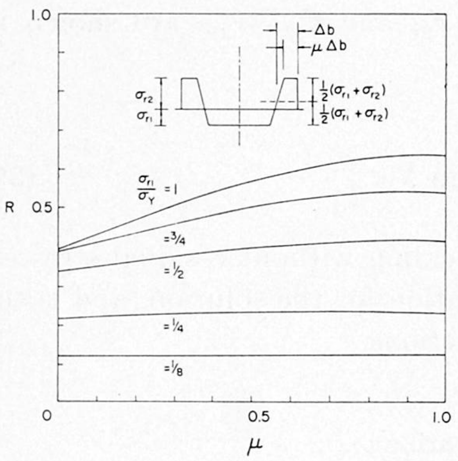

Fig. 7. Relationship between residual stress distribution and reduction in buckling strength (see Eq. 40).

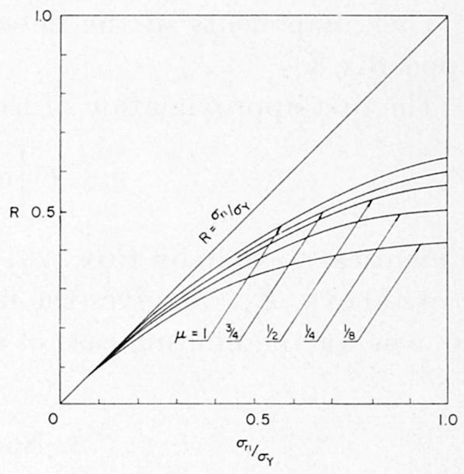

Fig. 8. Relationship between residual stress magnitude and reduction in buckling strength.

According to the results of the numerical calculations, the critical stresses of elastic buckling may be calculated by the following approximate expressions, which have been fitted to the plotted curves of Eqs. (40) (Figs. 7 and 8). The use of them for prediction results in negligible error.

$$\frac{\sigma_c}{\sigma_Y} = \left(\frac{\sigma_{cr}}{\sigma_Y}\right)_0 - \frac{\sigma_{r1}}{\sigma_Y}; \qquad \left(\frac{\sigma_{r1}}{\sigma_Y}\right) < 0.15, \tag{41}$$

$$\frac{\sigma_c}{\sigma_Y} = \left(\frac{\sigma_{cr}}{\sigma_Y}\right)_0 - \left(\frac{\sigma_{r1}}{\sigma_Y}\right) + K\left(\frac{\sigma_{r1}}{\sigma_Y}\right)^2; \qquad \left(\frac{\sigma_{r1}}{\sigma_Y}\right) > 0.15, \tag{42}$$

where σ_{r1} = magnitude of maximum tensile residual stress in the assumed pattern (Fig. 7)

and $K = R$, for $\frac{\sigma_{r1}}{\sigma_Y} = 1.0$.

The factor (l/b) influences the buckling strength of a plate, and at a certain value of (l/b) the minimum critical strain may be obtained. However, in the elastic buckling, the influence of residual stresses on the buckling strength of a plate is independent of the critical load, as shown by Eq. (40). The factor (l/b) is contained only in the first term of Eq. (40), but not in the second term. The first term gives the elastic buckling strength for a plate without residual stresses which is a minimum for $(l/b) = 1.0$ [16].

The results of the numerical calculations are summarized in Figs. 9 and 11, for critical stress critical and strain respectively. Some of the curves for the elastic buckling intersect the abscissa. This interesting fact shows the possibility of the buckling of a plate without any external load and explains the reason why plates can be distorted solely due to the process of welding.

2. *Elastic-Plastic Buckling*

As in the preceding section, a comparison was made of the accuracy obtained by taking the first and the first two terms of the deflection equation. The computation showed that the value for σ_{cr}/σ_Y as obtained by the deflection equations, differed only by 3% in the worst case [16]. It was judged that the

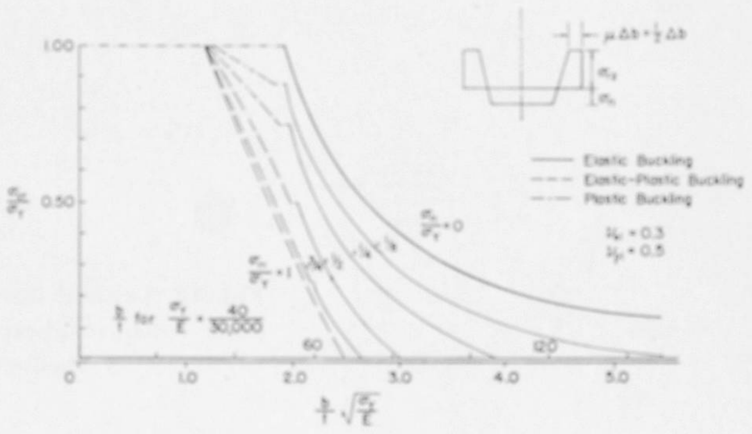

Fig. 9. Buckling strength of plates with residual stresses (deformation theory).

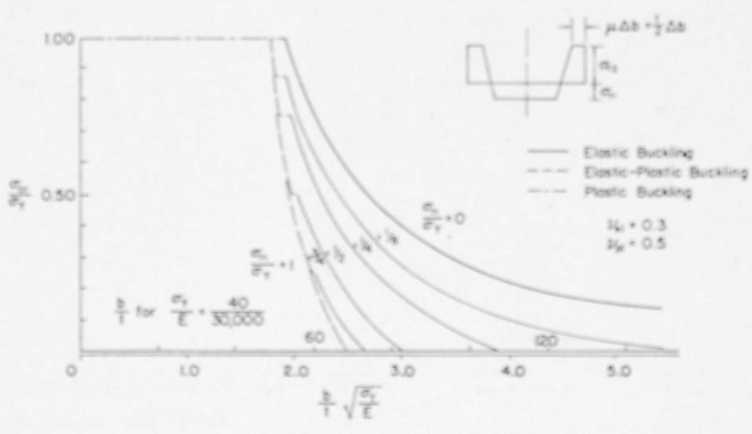

Fig. 10. Buckling strength of plates with residual stresses (flow theory).

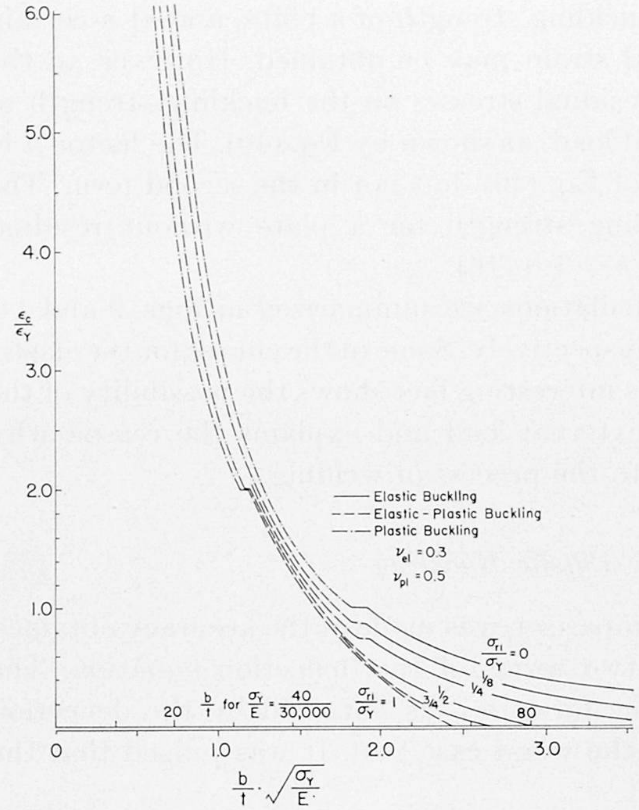

Fig. 11. Critical buckling strain of plate with residual stresses (deformation theory).

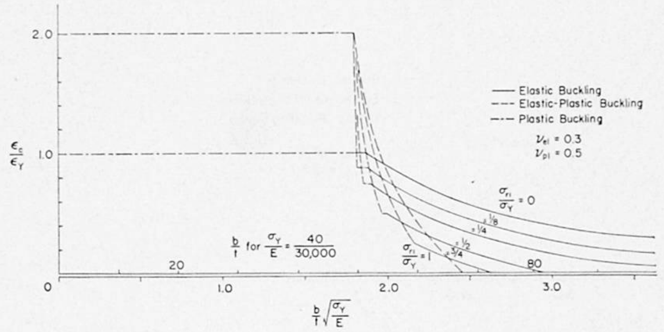

Fig. 12. Critical buckling strain of plate with residual stresses (flow theory).

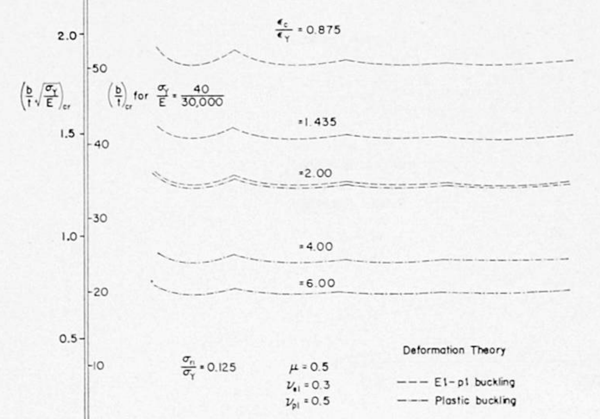

Fig. 13. Relationship between $(b/t)_{cr}$ and (L/B) for $\sigma_{r1}/\sigma_Y = 0.125$.

use of the first term is accurate enough to carry out a comprehensive numerical computation with due consideration to the economy of computer time. (For instance, using only the first term, the running time of the computer was only one fourth of the time spent for the computation using the first two terms.)

Even when only the first term of the deflection equation was used, the computation did not become much simpler, as was the case for elastic buckling. In that case, the influence of residual stresses on the buckling strength of a plate was separated from the original buckling strength of the plate without residual stresses. For elastic-plastic buckling, it is not possible to separate these two factors in the equation.

In contrast with the buckling of a single plate, the local buckling of a box column normally occurs at the critical (l/b) ratio which gives the minimum critical stress.

For each residual stress pattern, various values of l/b were chosen for a given value of critical strain. From the results of the computation, the curves (L/B) vs. critical (b/t) were drawn and the most critical l/b was determined corresponding to the minimum critical strain. This is illustrated in Figs. 13 to 17. For example, in Fig. 15, the lowest critical stress corresponds to an l/b ratio between 0.7 and 0.8 for elastic-plastic buckling of a plate with residual stress.

Figs. 9, 10, 11, and 12 summarize the computation results for the elastic-plastic local buckling of the box column using the deformation theory and the flow theory. Figs. 9 and 10 show the ratio of the average critical stress to the yield point vs. the b/t ratio, and Figs. 11 and 12 show the ratio of the average critical strain to the yield strain vs. the b/t ratio. For elastic buckling, the critical stress is calculated from the critical strain multiplied by Young's

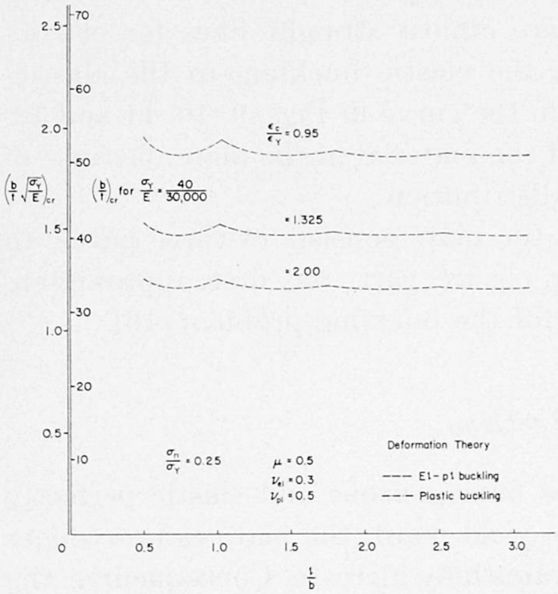

Fig. 14. Relationship between $(b/t)_{cr}$ and (L/B) for $\sigma_{r1}/\sigma_Y = 0.25$.

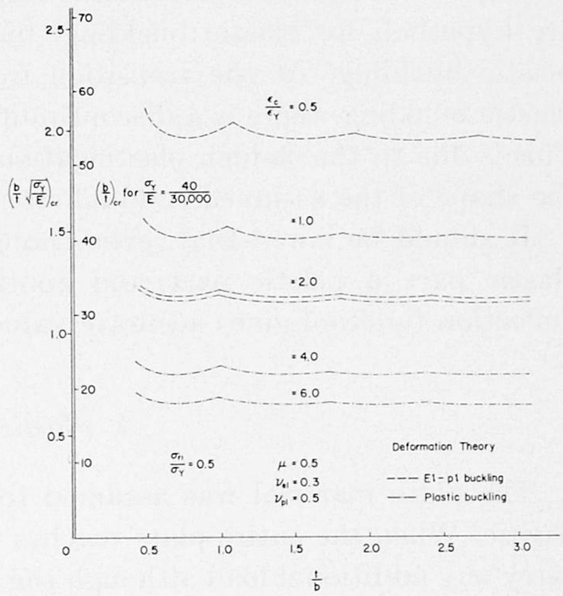

Fig. 15. Relationship between $(b/t)_{cr}$ and (L/B) for $\sigma_{r1}/\sigma_Y = 0.5$.

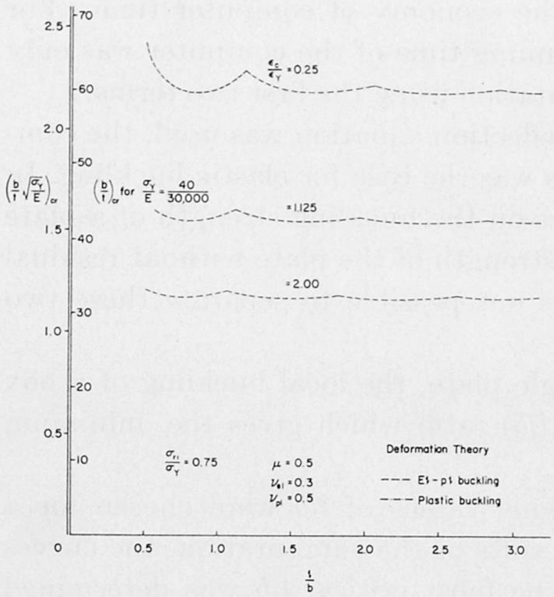

Fig. 16. Relationship between $(b/t)_{cr}$ and (L/B) for $\sigma_{r1}/\sigma_Y = 0.75$.

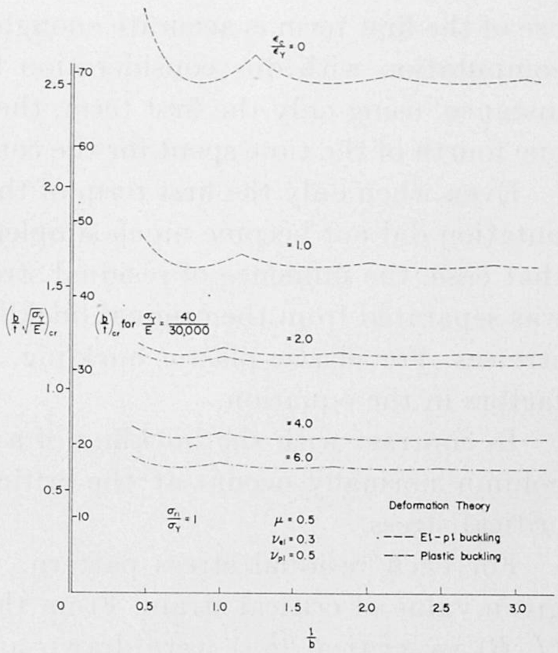

Fig. 17. Relationship between $(b/t)_{cr}$ and (L/B) for $\sigma_{r1}/\sigma_Y = 1.0$.

modulus. But in the elastic-plastic buckling this relationship is no longer applicable and the critical stress must be calculated from a more complicated relationship [16].

As expected [27, 28, 29], the flow theory gives higher critical stresses than does the deformation theory. The elastic-plastic buckling curves based on the flow theory lie very close to the boundary of the elastic and the elastic-plastic buckling regions.

The curves which represent the buckling strength of a plate vs. (b/t) ratio are hyperboli for elastic buckling, but are almost straight lines for elastic-plastic buckling. At the transition from the elastic buckling to the elastic-plastic buckling, there is a discontinuity in the curve in Figs. 9, 10, 11 and 12. This is due to the sudden plastification of the material in the plate, because of the shape of the assumed residual stress distribution.

It should be noted that, even though the plate consists of three parts, an elastic part, a plastic part, and another elastic part, the first approximate deflection function gives accurate values for the buckling problem [16].

3. Plastic Buckling

The plate material was assumed to be homogeneous and elastic perfectly plastic. When the entire plate reaches the yield point, the plate can no longer carry any additional load although the strains may increase. Consequently, the critical strain may be investigated. For this reason the results of the numerical analysis do no appear in Figs. 9 and 10 which are drawn with respect to σ_{cr}/σ_Y

and b/t. On the other hand, the secant modulus is affected by the magnitude of plastic strain.

As far as the flow theory is concerned, the complete plastification of the plate may be delayed by the existence of residual stresses, but after the whole plate has reached the yield point, the plate behaves completely plastically in the same manner as if the plate had not been subjected to any residual stresses before. While the residual stresses do not play any role in the flow theory for the plastic buckling, they do influence the deformation theory because the secant modulus defines the relationship between stress and strain in the plastic range of an elastic perfectly plastic material.

The result of the numerical calculation according to the deformation theory is shown in Fig. 11.

For the plastic buckling, the study of the influence of residual stresses is similar to that for elastic-plastic buckling, that is, no separation of the effect of residual stresses is possible.

When the plate is not subject to any residual stresses, the plastic buckling of the plate occurs at $1/\sqrt{2}$ of the (l/b) ratio which gives the lowest critical strain [12]. For the plate with residual stresses, the corresponding critical value of (l/b) is approximately 0.7, which is approximately the same as $1/\sqrt{2}$. This fact suggests that the existence of residual stresses in the plate affects the critical strain of the plastic buckling of the plate, but not the wave length of buckling. (Figs. 13, 15 and 17.)

5. Experimental Results

Experiments were conducted to verify the theories for the elastic buckling and elastic-plastic buckling of plate elements in built-up square columns. These columns were built up from plates by welding. Both ASTM A 36 steel and A 514 steel (T-1 constructional alloy steel) were used.

Ratios of b/t were chosen so that the plate elements would buckle in definite ranges of elastic, or elastic-plastic buckling, as defined by Figs. 9 and 10.

The experiments consisted of tensile coupon tests, residual stress measurements, and plate buckling tests.

Table 2 lists the dimensions of the test specimens, and their yield loads. Specimens designated by S are structural carbon steel and specimens designated by T are constructional alloy steel. The plate buckling tests were conducted on short columns to simulate the local buckling of columns without the occurrence of column buckling.

The experiments and test results have been described in detail in [30]; only the results will be considered briefly in this paper.

The test results are presented in Table 3, and in Figs. 18, 19, 20 and 21. Figs. 18 and 19 compare the test results with the deformation theory, and

Table 2. Dimensions of Specimens

Specimen No.	Length (in)	B [1]) (in)	h [1]) (in)	Area (in^2)	L/B	b/t	P_Y (kips)
S-1	50	11.4	0.256	11.7	4.35	44.5	455
S-11	83	11.5	0.256	11.8	7.20	45.0	460
S-2	80	16.2	0.253	16.3	4.91	64.0	630
S-21	87	16.3	0.254	16.5	5.34	64.2	635
T-1A	60	11.3	0.256	11.5	5.31	44.0	1340
T-1B	60	11.2	0.255	11.5	5.34	44.0	1340
T-2A	35	6.77	0.258	6.98	5.18	26.2	724
T-2B	35	6.77	0.258	6.98	5.18	26.2	724

[1]) Average value of four faces.

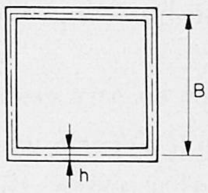

Table 3. Test Results

Specimen No.	Test Results			Theoretical Prediction [3])		Tension Coupon Test
	$\frac{\sigma_{r1}}{\sigma_y}$ [1])	P_{cr} [2]) (kips)	P_u (kips)	P_{cr} (kips) with σ_r	P_{cr} (kips) without σ_r	P_Y (kips)
S-1	0.23	340	357	380	455	455
S-11	0.23	355	366	383	460	460
S-2	0.16	260	337	348	462	630
S-21	0.16	270	342	350	465	635
T-2A	0.15	620	651	650	724	724
T-2B	0.15	640	657	650	724	724
T-1A	0.10	500	700	510	638	1340
T-1B	0.10	490	694	510	638	1340

[1]) Ratio between average compressive residual stress and static yield stress.
[2]) The critical loads were determined by means of the "top-of-the-knee" method [31].
[3]) Prediction was based on the deformation theory of plasticity.

Figs. 20 and 21 also show the comparison with the flow theory. The ultimate strengths of the buckled plates are shown for comparison, although this was not considered in the theoretical study.

The theoretical predictions gave good correlation with the experimental results for the deformation theory, but were quite high for the flow theory.

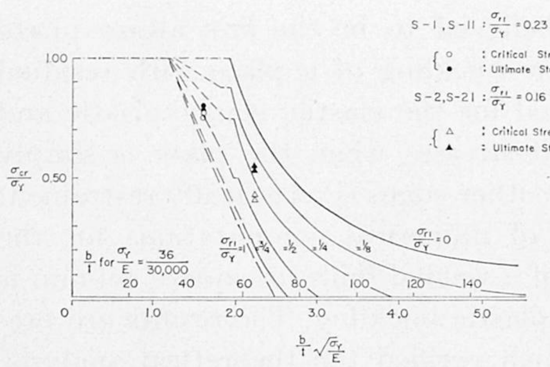

Fig. 18. Plate buckling curve with test points (A-36 steel).

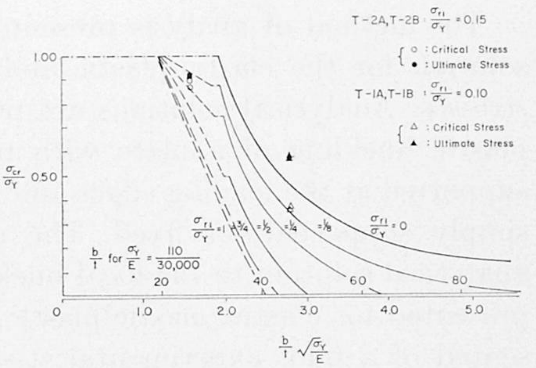

Fig. 19. Plate buckling curve with test points (A-514 steel).

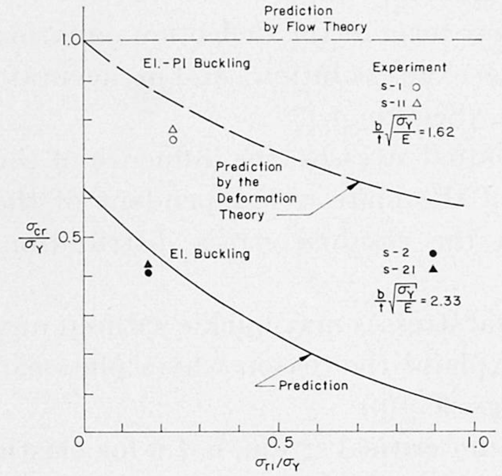

Fig. 20. Critical local buckling strength of columns (A 36 steel).

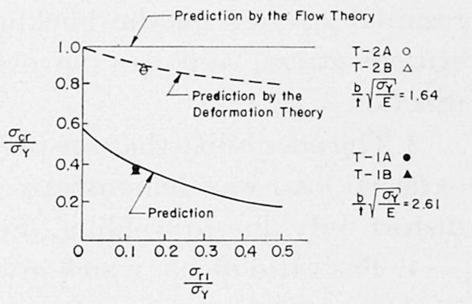

Fig. 21. Critical local buckling strength of columns (A 514 steel).

The result of these pilot tests have shown that considerable post buckling strength may be expected for the elastic buckling of plates, but not for the elastic-plastic buckling.

6. Conclusions

This paper presents the results of an investigation into the elastic, elastic-plastic, and plastic buckling of plates containing residual stresses. Particular attention has been paid to the local buckling of plate elements of built-up columns of box-shaped cross sections. An experimental study was correlated with the theoretical predictions.

In the theoretical analysis, the pattern of the residual stress distribution was simplified and the theorem of minimum potential energy was employed with the restriction that there is no reversal of strain at any point in the plastified material. The plastic part of the plate was analyzed by plastic theories, the secant modulus deformation theory and the flow theory.

The method of analysis presented is believed to be the first approximate solution for the elastic-plastic and plastic buckling of a plate with residual stresses. Analytical solutions are presented for the elastic, elastic-plastic and plastic buckling of a plate with residual stresses when the plate is simply supported at the loading edges and at the other edges is: elastically restrained; simply supported; or fixed. The result of numerical computations for the analytical solution to the local buckling of a welded built-up square column is presented for elastic, elastic-plastic, and plastic buckling. The results are presented of a pilot experimental study which verified the theoretical analysis. The experimental study showed the relationship between the buckling strength and the ultimate strength of a plate element of the column.

The following conclusions may be drawn from the study:

1. The approximation of using only the first term of the deflection equation gives an answer which is very close to the exact solution, and is accurate enough for analysis of the buckling problem. (Section 4.)

2. For elastic buckling of a plate with residual stresses, the influence of the residual stresses on the buckling strength of the plate is independent of the critical stress, and can be evaluated from the residual stress distribution. (Section 4.1.)

3. The possibility that the plate with residual stresses may buckle without any external load was demonstrated. This fact explains the reason why a plate can distort only due to welding. (Section 4.1, Figs. 9, 10.)

4. The ratio of l/b, which gives the minimum critical strain, is 1.0 for elastic buckling, 0.7 to 0.8 for elastic-plastic buckling, and 0.7 for plastic buckling. (Section 4, Figs. 13 to 17.)

5. For elastic-plastic buckling of the plate, the analysis based on the flow theory gives a much higher critical strain than the one based on the deformation theory. (Section 4.2, 4.3, Figs. 9 to 12.)

6. A plate containing residual stresses will not buckle until the critical stress reaches the yield point, if the b/t ratio of the plate is less than

a) $1.17 \sqrt{E/\sigma_Y}$ based on the deformation theory,

b) $1.83 \sqrt{E/\sigma_Y}$ based on the flow theory regardless of the magnitude of the residual stresses, and less than

c) $1.90 \sqrt{E/\sigma_Y}$ for the plate free of residual stresses. (Section 4.2, 4.3, Figs. 9, 10.)

7. The experiments verified the validity of the theoretical analysis for the elastic and elastic-plastic buckling of a plate containing residual stresses. The theory based on the secant modulus deformation theory gave good correlation with the experimental results, but the theory based on the flow theory did not. (Section 5, Figs. 18 to 21.)

8. Although considerable post buckling strength occurred for elastic buckling of the plate, this was not the case for elastic-plastic buckling. (Section 5, Figs. 18, 19.)

7. Acknowledgements

This paper presents a part of a theoretical and experimental investigation made during the course of studies into the influence of residual stress on the strength of welded built-up columns of both structural carbon steel and "T-1" constructional alloy steel.

The investigation was conducted at Fritz Engineering Laboratory in the Department of Civil Engineering of Lehigh University in Bethlehem, Pennsylvania. The United States Steel Corporation and the Pennsylvania Department of Highways, U.S. Department of Commerce — Bureau of Public Roads sponsored the research program. Column Research Council Task Group 1 under the chairmanship of John A. Gilligan provided valuable guidance.

Sincere appreciation is due Fumio Nishino for his criticism, advice, and assistance during the course of this study and the preparation of this paper.

Nomenclature

a_{1m}, a_{2m} coefficients of deflection functions.
B width of plate.
b half width of plate.
$b^{(i)}$ half width of plate element on side i.
c_{1m}, c_{2m} coefficients of deflection functions.
D flexural rigidity of plate $= \frac{E h^3}{12(1-\nu^2)}$.
D_d flexural rigidity of plate in the plastic range, based on deformation theory.
D_f flexural rigidity of plate in the plastic range, based on flow theory.
E modulus of elasticity.
E_s secant modulus.
E_t tangent modulus.
$F_{11}, F_{12}, F_{21}, F_{22}$ component of the coefficient determinant of the stability equation.
h thickness of plate.
L entire length of column or plate.
$2l$ half wave length of buckling of plate.
$(l/b)_{cr}$ (l/b) ratio giving minimum critical strain of buckling of plate.
M_i bending moment per unit length of section of plate about x axis on side i.
m, n positive integers.
N number of half waves in the direction of x axis.
P_{cr} critical load.
P_u ultimate load.
P_Y yield load.

R	magnitude of reduction of elastic buckling strength due to residual stresses.
$2t$	thickness of a plate.
V	potential energy of plate.
w	deflection of plate.
w_i	deflection of plate on the side i.
x, y, z	cartesian coordinates.
y_i	y-axis of cartesian coordinate on a plate on the side i.
γ_{xy}	shearing strain in the cartesian coordinates.
$\dot{\gamma}$	rate of change of shearing strain.
ϵ_c	critical normal strain.
ϵ_i	intensity of strain.
ϵ_x, ϵ_y	normal strain components in the cartesian coordinates.
ϵ_Y	yield strain in tension or compression.
$\dot{\epsilon}_x, \dot{\epsilon}_y$	rate of change of strain components in the cartesian coordinates.
θ_i	angle of rotation at edge of plate i.
μ	a parameter, where $\mu \varDelta b$ is the width of the tensile residual stress distribution in the assumed pattern.
ν	Poisson's ratio.
λ	E/E_t.
σ_{cr}	average critical normal stress.
σ_i	intensity of stress.
σ_{rx}, σ_{ry}	normal residual stress components in the cartesian coordinates.
σ_{r1}	magnitude of maximum compressive residual stress in the assumd pattern.
σ_{r2}	magnitude of maximum tensile residual stress in the assumed pattern.
σ_x, σ_y	normal stress components in the cartesian coordinates.
$\dot{\sigma}_x, \dot{\sigma}_y$	rate of change of stress components in the cartesian coordinates.
σ_Y	yield stress in tension or compression.
τ_{rxy}	residual shearing stress in the cartesian coordinates.
τ_{xy}	shearing stress in the cartesian coordinates.

Appendix

Analytical Solutions for Buckling Strength of Plates with Residual Stresses

Analytical solutions are presented for elastic, elastic-plastic and plastic buckling of a plate which is simply supported at the loading edges and at other edges is:

a) elastically restrained,
b) simply supported,
c) fixed.

General Notation

$$\Gamma_0 = \frac{b_0}{b}, \qquad \Gamma_1 = \frac{b_1}{b}, \qquad \Gamma_2 = \frac{b_2}{b}, \qquad \Gamma_3 = \frac{b}{l},$$

$$_j\Gamma_3 = \frac{b^{(j)}}{l}, \qquad \Gamma_4 = \frac{t}{b}, \qquad \Gamma_5 = \frac{h_2}{h_1},$$

$$\Gamma_6 = \frac{\dfrac{\sigma_c}{\sigma_Y} + \dfrac{\dfrac{\sigma_{r1}}{\sigma_Y}\dfrac{b_2}{b} + \dfrac{\sigma_{r2}}{\sigma_Y}\dfrac{b_0}{b}}{\dfrac{b_2}{b} - \dfrac{b_0}{b}}}{\dfrac{\dfrac{\sigma_{r1}}{\sigma_Y} + \dfrac{\sigma_{r2}}{\sigma_Y}}{\dfrac{b_2}{b} - \dfrac{b_0}{b}}},$$

$$k_1 = \frac{\sigma_{r1}}{\sigma_Y} + \frac{\sigma_c}{\sigma_Y},$$

$$k_2 = \frac{\sigma_c}{\sigma_Y} + \frac{\dfrac{\sigma_{r1}}{\sigma_Y}\dfrac{b_2}{b} + \dfrac{\sigma_{r2}}{\sigma_Y}\dfrac{b_0}{b}}{\dfrac{b_2}{b} - \dfrac{b_0}{b}},$$

$$k_3 = \frac{\dfrac{\sigma_{r1}}{\sigma_Y} + \dfrac{\sigma_{r2}}{\sigma_Y}}{\dfrac{b_2}{b} - \dfrac{b_0}{b}},$$

$$k_4 = \frac{\sigma_c}{\sigma_Y} - \frac{\sigma_{r2}}{\sigma_Y},$$

$$u = \frac{1}{12}\left(\frac{1}{N\pi}\right)^2\left(\frac{E}{\sigma_Y}\right)\left(\frac{l}{b}\right)^2\frac{1}{1-\nu^2},$$

$$u_0 = \frac{1}{9}\left(\frac{1}{N\pi}\right)^2\left(\frac{E}{\sigma_Y}\right)\left(\frac{l}{b}\right)^2,$$

$$u_1 = \frac{1}{9}\left(\frac{1}{N\pi}\right)^2\left(\frac{E}{\sigma_Y}\right)\left(\frac{l}{b}\right)^2\frac{1}{\dfrac{\sigma_{r1}}{\sigma_Y} + \dfrac{\sigma_{r2}}{\sigma_Y}},$$

$$u_2 = \frac{1}{9}\left(\frac{1}{N\pi}\right)^2\left(\frac{E}{\sigma_Y}\right)\left(\frac{l}{b}\right)^2\frac{\dfrac{b_2}{b} - \dfrac{b_0}{b}}{\dfrac{\sigma_{r1}}{\sigma_Y} + \dfrac{\sigma_{r2}}{\sigma_Y}},$$

$$u_3 = \frac{1}{9}\left(\frac{1}{N\pi}\right)^2\left(\frac{E}{\sigma_Y}\right)\left(\frac{l}{b}\right)^2\frac{1}{\dfrac{\sigma_c}{\sigma_Y} - \dfrac{\sigma_{r2}}{\sigma_Y}},$$

$$S_m^k = \cos m\pi\Gamma_6 \log \frac{\Gamma_k - \Gamma_6}{\Gamma_0 - \Gamma_6} - (m\pi)\frac{1}{1!}\sin m\pi\Gamma_6\,[(\Gamma_k - \Gamma_6) - (\Gamma_0 - \Gamma_6)]$$
$$- (m\pi)^2 \frac{1}{(2)\,2!}\cos m\pi\Gamma_6\,[(\Gamma_k - \Gamma_6)^2 - (\Gamma_0 - \Gamma_6)^2]$$
$$+ (m\pi)^3 \frac{1}{(3)\,3!}\sin m\pi\Gamma_6\,[(\Gamma_k - \Gamma_6)^3 - (\Gamma_0 - \Gamma_6)^3]$$
$$+ (m\pi)^4 \frac{1}{(4)\,4!}\cos m\pi\Gamma_6\,[(\Gamma_k - \Gamma_6)^4 - (\Gamma_0 - \Gamma_6)^4]$$
$$- \cdots$$

(where $k=1$ or 2, $m=\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, 3 or 4).

For elastic region $i=E$ and $w=1$.
For plastic region $i=P$ and $w=\frac{1}{4}$.
For the side 1 of column $j=1$.
For the side 2 of column $j=2$.

1. *Analytical Solutions for Elastically Restrained Plate*

1.1. *Additional Notation*

$${}_jc_{11}^i = \pi^4[w\,{}_j\Gamma_3^4 + 2\,{}_j\Gamma_3^2 + 1], \qquad {}_jc_{12}^i = \pi^4[w\,{}_j\Gamma_3^4 - 2(1-2\nu)\,{}_j\Gamma_3^2 + 1],$$
$${}_jc_{13}^i = \pi^4[3w\,{}_j\Gamma_3^4 + (4+3\nu)\,{}_j\Gamma_3^2 + 4], \qquad {}_jc_{14}^i = \pi^4[w\,{}_j\Gamma_3^4 - (4-9\nu)\,{}_j\Gamma_3^2 + 4],$$
$${}_jc_{21}^i = {}_jc_{13}^i, \qquad {}_jc_{22}^i = {}_jc_{14}^i,$$
$${}_jc_{23}^i = \pi^4[3w\,{}_j\Gamma_3^4 + 8\,{}_j\Gamma_3^2 + 16], \qquad {}_jc_{24}^i = 4\pi^4[w\,{}_j\Gamma_3^4 + 4\nu\,{}_j\Gamma_3^2],$$
$${}_jc_{25}^i = \pi^4[w\,{}_j\Gamma_3^4 - 8(1-2\nu)\,{}_j\Gamma_3^2 + 16].$$

1.2. *Analytical Solutions*

The analytical solution is given in the form:

$$F_{11}F_{22} - F_{12}F_{21} = 0,$$

where

$$F_{11} = (\Gamma_4)^2(v\,110) + (v\,111),$$
$$F_{12} = F_{21} = (\Gamma_4)^2(v\,120) + (v\,121),$$
$$F_{22} = (\Gamma_4)^2(v\,220) + (v\,221).$$

1.2a. Elastic Buckling

$$(v\,110) = u\,(t\,130),$$
$$(v\,120) = u\,(t\,230),$$
$$(v\,220) = u\,(t\,430),$$
$$(v\,111) = -k_1(t\,140) - k_2(t\,151) + k_3(t\,152) + k_4(t\,160),$$
$$(v\,121) = -k_1(t\,240) - k_2(t\,251) + k_3(t\,252) + k_4(t\,260),$$
$$(v\,221) = -k_1(t\,440) - k_2(t\,451) + k_3(t\,452) + k_4(t\,460).$$

$$(t\,130) = \left[{}_1c_{11}^E - \frac{\alpha}{\beta}\,{}_2c_{11}^E\right],$$

$$(t\,140) = (1-\alpha^3\,\Gamma_5)\left[\Gamma_0 - \frac{1}{\pi}\sin\pi\,\Gamma_0\right],$$

$$(t\,151) = (1-\alpha^3\,\Gamma_5)\left[(\Gamma_2-\Gamma_0) + \frac{1}{\pi}(\sin\pi\,\Gamma_2 - \sin\pi\,\Gamma_0)\right],$$

$$(t\,152) = (1-\alpha^3\,\Gamma_5)\left[\frac{1}{2}(\Gamma_2^2-\Gamma_0^2) + \frac{1}{\pi}(\Gamma_2\sin\pi\,\Gamma_2 - \Gamma_1\sin\pi\,\Gamma_1) + \left(\frac{1}{\pi}\right)^2(\cos\pi\,\Gamma_2 - \cos\pi\,\Gamma_1)\right],$$

$$(t\,160) = (1-\alpha^3\,\Gamma_5)\left[(\Gamma_2-1) + \left(\frac{1}{\pi}\right)\sin\pi\,\Gamma_2\right],$$

$$(t\,230) = [{}_1c_{13}^E - {}_2c_{13}^E]\left(\frac{2}{\pi}\right) - [{}_1c_{14}^E - {}_2c_{14}^E]\left(\frac{2}{3\,\pi}\right),$$

$$(t\,240) = (1-\alpha^2\beta\,\Gamma_5)\left[+\frac{6}{\pi}\sin\pi\frac{\Gamma_0}{2} + \frac{2}{3\,\pi}\sin\frac{3\,\pi}{2}\,\Gamma_0\right],$$

$$(t\,251) = (1-\alpha^2\beta\,\Gamma_5)\left[\frac{6}{\pi}\left(\sin\frac{\pi}{2}\,\Gamma_2 - \sin\frac{\pi}{2}\,\Gamma_0\right) + \frac{2}{3\,\pi}\left(\sin\frac{3\,\pi}{2}\,\Gamma_2 - \sin\frac{3\,\pi}{2}\,\Gamma_0\right)\right],$$

$$(t\,252) = (1-\alpha^2\beta\,\Gamma_5)\left[\frac{2}{\pi}\left(\Gamma_2\sin\frac{\pi}{2}\,\Gamma_2 - \Gamma_0\sin\frac{\pi}{2}\,\Gamma_0\right) + 3\left(\frac{2}{\pi}\right)^2\left(\cos\frac{\pi}{2}\,\Gamma_2 - \cos\frac{\pi}{2}\,\Gamma_0\right) + \frac{2}{3\,\pi}\left(\Gamma_2\sin\frac{3\,\pi}{2}\,\Gamma_2 - \Gamma_0\sin\frac{3\,\pi}{2}\,\Gamma_0\right) + \left(\frac{2}{3\,\pi}\right)^2\left(\cos\frac{3\,\pi}{2}\,\Gamma_2 - \cos\frac{3\,\pi}{2}\,\Gamma_0\right)\right],$$

$$(t\,260) = (1-\alpha^2\beta\,\Gamma_5)\left[\frac{6}{\pi}\left(-1+\sin\frac{\pi}{2}\,\Gamma_2\right) + \frac{2}{3\,\pi}\left(1+\sin\frac{3\,\pi}{2}\,\Gamma_2\right)\right],$$

$$(t\,430) = \left[{}_1c_{23}^E + \frac{\beta^3}{\alpha^4}\,{}_2c_{23}^E\right],$$

$$(t\,440) = (1-\alpha\,\beta^2\,\Gamma_5)\left[3\,\Gamma_0 + \frac{4}{\pi}\sin\pi\,\Gamma_0 + \frac{1}{2\,\pi}\sin 2\,\pi\,\Gamma_0\right],$$

$$(t\,451) = (1-\alpha\,\beta^2\,\Gamma_5)\left[3\,(\Gamma_2-\Gamma_0) + \frac{4}{\pi}(\sin\pi\,\Gamma_2 - \sin\pi\,\Gamma_0) + \frac{1}{2\,\pi}(\sin 2\,\pi\,\Gamma_2 - \sin 2\,\pi\,\Gamma_0)\right],$$

$$(t\,452) = (1-\alpha\,\beta^2\,\Gamma_5)\left[\frac{3}{2}(\Gamma_2^2-\Gamma_0^2) + \frac{4}{\pi}(\Gamma_2\sin\pi\,\Gamma_2 - \Gamma_0\sin\pi\,\Gamma_0) + \left(\frac{2}{\pi}\right)^2(\cos\pi\,\Gamma_2 - \cos\pi\,\Gamma_0) + \frac{1}{2\,\pi}(\Gamma_2\sin 2\,\pi\,\Gamma_2 - \Gamma_0\sin 2\,\pi\,\Gamma_0) + \left(\frac{1}{2\,\pi}\right)^2(\cos 2\,\pi\,\Gamma_2 - \cos 2\,\pi\,\Gamma_0)\right],$$

$$(t\,460) = (1-\alpha\,\beta^2\,\Gamma_5)\left[3\,(\Gamma_2-1) + \frac{4}{\pi}\sin\pi\,\Gamma_2 + \frac{1}{2\,\pi}\sin 2\,\pi\,\Gamma_2\right].$$

1.2b. Elastic-plastic Buckling
(Based on the deformation theory)

$$(v\,110) = u_1\,(t\,110) - u_2\,(t\,120) + u\,(t\,130)\,,$$

$$(v\,120) = u_1\,(t\,210) - u_2\,(t\,220) + u\,(t\,230)\,,$$

$$(v\,220) = u_1\,(t\,410) - u_2\,(t\,420) + u\,(t\,430)\,,$$

$$(v\,111) = -\,(t\,140) - k_2\,(t\,151) + k_3\,(t\,152) + k_4\,(t\,160)\,,$$

$$(v\,121) = -\,(t\,240) - k_2\,(t\,251) + k_3\,(t\,252) + k_4\,(t\,260)\,,$$

$$(v\,221) = -\,(t\,440) - k_2\,(t\,451) + k_3\,(t\,452) + k_4\,(t\,460)\,.$$

$$(t\,110) = \left[{}_1c_{11}^P - \frac{\alpha}{\beta}\,{}_2c_{11}^P\right] \Gamma_0 + \left[{}_1c_{12}^P - \frac{\alpha}{\beta}\,{}_2c_{12}^P\right]\left(\frac{1}{\pi}\sin\pi\,\Gamma_0\right),$$

$$(t\,120) = \left[{}_1c_{11}^P - \frac{\alpha}{\beta}\,{}_2c_{11}^P\right] \log\frac{\Gamma_1 - \Gamma_6}{\Gamma_0 - \Gamma_6} + \left[{}_1c_{12}^P - \frac{\alpha}{\beta}\,{}_2c_{12}^P\right] S_1^1,$$

$$(t\,130) = \left[{}_1c_{11}^E - \frac{\alpha}{\beta}\,{}_2c_{11}^E\right](1 - \Gamma_0) - \left[{}_1c_{12}^E - \frac{\alpha}{\beta}\,{}_2c_{12}^E\right]\frac{1}{\pi}\sin\pi\,\Gamma_1\,,$$

$$(t\,140) = (1 - \alpha^3\,\Gamma_5)\left(\Gamma_1 - \frac{1}{\pi}\sin\pi\,\Gamma_1\right),$$

$$(t\,151) = (1 - \alpha^3\,\Gamma_5)\left[(\Gamma_2 - \Gamma_1) + \frac{1}{\pi}(\sin\pi\,\Gamma_2 - \sin\pi\,\Gamma_1)\right],$$

$$(t\,152) = (1 - \alpha^3\,\Gamma_5)\left[\frac{1}{2}(\Gamma_2^2 - \Gamma_1^2) + \frac{1}{\pi}(\Gamma_2\sin\pi\,\Gamma_2 - \Gamma_1\sin\pi\,\Gamma_1) + \left(\frac{1}{\pi}\right)^2(\cos\pi\,\Gamma_2 - \cos\pi\,\Gamma_1)\right],$$

$$(t\,160) = (-\,\alpha^3\,\Gamma_5)\left[(\Gamma_2 - 1) + \frac{1}{\pi}\sin\pi\,\Gamma_2\right],$$

$$(t\,210) = [c_{13}^P - {}_2c_{13}^P]\,\frac{2}{\pi}\sin\frac{\pi}{2}\,\Gamma_0 + [{}_1c_{14}^P - {}_2c_{14}^P]\,\frac{2}{3\,\pi}\sin\frac{3\,\pi}{2}\,\Gamma_0\,,$$

$$(t\,220) = [{}_1c_{13}^P - {}_2c_{13}^P]\,S_{1/2}^1 + [{}_1c_{13}^P - {}_2c_{13}^P]\,S_{3/2}^1\,,$$

$$(t\,230) = [{}_1c_{13}^E - {}_2c_{13}^E]\,\frac{2}{\pi}\left(1 - \sin\frac{\pi}{2}\,\Gamma_1\right) - [{}_1c_{14}^E - {}_2c_{14}^E]\,\frac{2}{3\,\pi}\left(1 + \sin\frac{3\,\pi}{2}\,\Gamma_1\right),$$

$$(t\,240 = (1 - \alpha^2\beta\,\Gamma_5)\left[\frac{6}{\pi}\sin\frac{\pi}{2}\,\Gamma_1 + \frac{2}{3\,\pi}\sin\frac{3\,\pi}{2}\,\Gamma_1\right],$$

$$(t\,251) = (1 - \alpha^2\beta\,\Gamma_5)\left[\frac{6}{\pi}\left(\sin\frac{\pi}{2}\,\Gamma_2 - \sin\frac{\pi}{2}\,\Gamma_1\right) + \frac{2}{3\,\pi}\left(\sin\frac{3\,\pi}{2}\,\Gamma_2 - \sin\frac{3\,\pi}{2}\,\Gamma_1\right)\right],$$

$$(t\,252) = (1 - \alpha^2\beta\,\Gamma_5)\left[\frac{6}{\pi}\left(\Gamma_2\sin\frac{\pi}{2}\,\Gamma_2 - \Gamma_1\sin\frac{\pi}{2}\,\Gamma_1\right) + 3\left(\frac{2}{\pi}\right)^2\left(\cos\frac{\pi}{2}\,\Gamma_2 - \cos\frac{\pi}{2}\,\Gamma_1\right) + \frac{2}{3\,\pi}\left(\Gamma_2\sin\frac{3\,\pi}{2}\,\Gamma_2 - \Gamma_1\sin\frac{3\,\pi}{2}\,\Gamma_1\right) + \left(\frac{2}{3\,\pi}\right)\left(\cos\frac{3\,\pi}{2}\,\Gamma_2 - \cos\frac{3\,\pi}{2}\,\Gamma_1\right)\right],$$

$$(t\,260) = (1-\alpha^2\beta\,\Gamma_5)\left[\frac{6}{\pi}\left(-1+\sin\frac{\pi}{2}\,\Gamma_2\right)+\frac{2}{3\,\pi}\left(1+\sin\frac{3\,\pi}{2}\,\Gamma_2\right)\right],$$

$$(t\,410) = \left[{}_1c_{23}^P+\frac{\beta^3}{\alpha^4}\,{}_2c_{23}^P\right]\Gamma_0+\left[{}_1c_{24}^P+\frac{\beta^3}{\alpha^4}\,{}_2c_{24}^P\right]\frac{1}{\pi}\sin\pi\,\Gamma_0+\left[{}_1c_{25}^P+\frac{\beta^3}{\alpha^4}\,{}_2c_{25}^P\right]\frac{1}{2\,\pi}\sin 2\,\pi\,\Gamma_0,$$

$$(t\,420) = \left[{}_1c_{23}^P+\frac{\beta^3}{\alpha^4}\,{}_2c_{23}^P\right]\log\frac{\Gamma_1-\Gamma_6}{\Gamma_0-\Gamma_6}+\left[{}_1c_{24}^P+\frac{\beta_3}{\alpha^4}\,{}_2c_{24}^P\right]S_1^1+\left[{}_1c_{25}^P+\frac{\beta^3}{\alpha^4}\,{}_2c_{25}^P\right]S_2^1,$$

$$(t\,430) = \left[{}_1c_{23}^E+\frac{\beta^3}{\alpha^4}\,{}_2c_{23}^E\right](1-\Gamma_1)-\left[{}_1c_{24}^E+\frac{\beta^3}{\alpha^4}\,{}_2c_{24}^E\right]\frac{1}{\pi}\sin\pi\,\Gamma_1$$
$$-\left[{}_1c_{25}^E+\frac{\beta^3}{\alpha^4}\,{}_2c_{25}^E\right]\frac{1}{2\,\pi}\sin 2\,\pi\,\Gamma_1,$$

$$(t\,440) = (1-\alpha\beta^2\,\Gamma_5)\left[3\,\Gamma_1+\frac{4}{\pi}\sin\pi\,\Gamma_1+\frac{1}{2\,\pi}\sin 2\,\pi\,\Gamma_1\right],$$

$$(t\,451) = (1-\alpha\beta^2\,\Gamma_5)\left[3\,(\Gamma_2-\Gamma_1)+\frac{4}{\pi}(\sin\pi\,\Gamma_2-\sin\pi\,\Gamma_1)\right.$$
$$\left.+\frac{1}{2\,\pi}(\sin 2\,\pi\,\Gamma_2-\sin 2\,\pi\,\Gamma_1)\right],$$

$$(t\,452) = (1-\alpha\beta^2\,\Gamma_5)\left[\frac{3}{2}(\Gamma_2^2-\Gamma_1^2)+\frac{4}{\pi}(\Gamma_2\sin\pi\,\Gamma_2-\Gamma_1\sin\pi\,\Gamma_1)\right.$$
$$+\left(\frac{2}{\pi}\right)^2(\cos\pi\,\Gamma_2-\cos\pi\,\Gamma_1)+\frac{1}{2\,\pi}(\Gamma_2\sin 2\,\pi\,\Gamma_2-\Gamma_1\sin 2\,\pi\,\Gamma_1)$$
$$\left.+\left(\frac{1}{2\,\pi}\right)^2(\cos 2\,\pi\,\Gamma_2-\cos 2\,\pi\,\Gamma_1)\right],$$

$$(t\,460) = (1-\alpha\beta^2\,\Gamma_5)\left[3\,(\Gamma_2-1)+\frac{4}{\pi}\sin\pi\,\Gamma_2+\frac{1}{2\,\pi}\sin 2\,\pi\,\Gamma_2\right].$$

1.2c. Elastic-plastic Buckling (Based on the flow theory)

$$(v\,110) = u_0\,(t\,110)+u\,(t\,130),$$
$$(v\,120) = u_0\,(t\,210)+u\,(t\,230),$$
$$(v\,220) = u_0\,(t\,410)+u\,(t\,430),$$
$$(v\,111) = -(t\,140)-k_2\,(t\,151)+k_3\,(t\,152)+k_4\,(t\,160),$$
$$(v\,121) = -(t\,240)-k_2\,(t\,251)+k_3\,(t\,252)+k_4\,(t\,260),$$
$$(v\,221) = -(t\,440)-k_2\,(t\,451)+k_3\,(t\,452)+k_4\,(t\,460).$$

$$(t\,110) = \left[{}_1c_{11}^P-\frac{\alpha}{\beta}\,{}_2c_{11}^P\right]\Gamma_1+\left[{}_1c_{12}^P-\frac{\alpha}{\beta}\,{}_2c_{12}^P\right]\left(\frac{1}{\pi}\sin\pi\,\Gamma_1\right),$$

$$(t\,130) = \left[{}_1c_{11}^E-\frac{\alpha}{\beta}\,{}_2c_{11}^E\right](1-\Gamma_1)-\left[{}_1c_{12}^E-\frac{\alpha}{\beta}\,{}_2c_{12}^E\right]\frac{1}{\pi}\sin\pi\,\Gamma_1,$$

$$(t\,210) = [{}_1c_{13}^P-{}_2c_{13}^P]\,\frac{2}{\pi}\sin\frac{\pi}{2}\,\Gamma_1+[{}_1c_{14}^P-{}_2c_{14}^P]\,\frac{2}{3\,\pi}\sin\frac{3\,\pi}{2}\,\Gamma_1,$$

$$(t\,230) = [{}_1c^E_{13} - {}_2c^E_{13}]\frac{2}{\pi}\left(1-\sin\frac{\pi}{2}\Gamma_1\right) - [{}_1c^E_{14} - {}_2c^E_{14}]\frac{2}{3\pi}\left(1+\sin\frac{3\pi}{2}\Gamma_1\right),$$

$$(t\,410) = \left[{}_1c^P_{23}+\frac{\beta^3}{\alpha^4}{}_2c^P_{23}\right]\Gamma_1 + \left[{}_1c^P_{24}+\frac{\beta^3}{\alpha^4}{}_2c^P_{24}\right]\sin\pi\Gamma_1 + \left[{}_1c^P_{25}+\frac{\beta^3}{\alpha^4}{}_2c^P_{25}\right]\sin 2\pi\Gamma_1,$$

$$(t\,430) = \left[{}_1c^E_{23}+\frac{\beta^3}{\alpha^4}{}_2c^E_{23}\right](1-\Gamma_1) - \left[{}_1c^E_{24}+\frac{\beta^3}{\alpha^4}{}_2c^E_{23}\right]\frac{1}{\pi}\sin\pi\Gamma_1$$
$$- \left[{}_1c^E_{25}+\frac{\beta^3}{\alpha^4}{}_2c^E_{23}\right]\frac{1}{2\pi}\sin 2\pi\Gamma_1,$$

$(t\,140)$, $(t\,151)$, $(t\,152)$, $(t\,160)$, $(t\,240)$, $(t\,251)$, $(t\,252)$, $(t\,260)$, $(t\,440)$, $(t\,451)$, $(t\,452)$ and $(t\,460)$ are the same as in Section 1.2b.

1.2d. Plastic Buckling
(Based on the deformation theory)

$$(v\,110) = u_1(t\,110) - u_2(t\,120) + u_3(t\,130),$$
$$(v\,120) = u_1(t\,210) - u_2(t\,220) + u_3(t\,130),$$
$$(v\,220) = u_1(t\,410) - u_2(t\,420) + u_3(t\,430),$$
$$(v\,111) = (t\,140),$$
$$(v\,121) = (t\,240),$$
$$(v\,222) = (t\,440).$$

$$(t\,110) = \left[{}_1c^P_{11}-\frac{\alpha}{\beta}{}_2c^P_{11}\right]\Gamma_0 + \left[{}_1c^P_{12}-\frac{\alpha}{\beta}{}_2c^P_{12}\right]\frac{1}{\pi}\sin\pi\Gamma_0,$$

$$(t\,120) = \left[{}_1c^P_{11}-\frac{\alpha}{\beta}{}_2c^P_{11}\right]\log\frac{\Gamma_2-\Gamma_6}{\Gamma_0-\Gamma_6} + \left[{}_1c^P_{12}-\frac{\alpha}{\beta}{}_2c^P_{12}\right]S_1^2,$$

$$(t\,130) = \left[{}_1c^P_{11}+\frac{\alpha}{\beta}{}_2c^P_{11}\right](1-\Gamma_2) - \left[{}_1c^P_{12}-\frac{\alpha}{\beta}{}_2c^P_{12}\right]\frac{1}{\pi}\sin\pi\Gamma_2,$$

$$(t\,140) = (1-\alpha^3\Gamma_5),$$

$$(t\,210) = [{}_1c^P_{13}-{}_2c^P_{13}]\frac{2}{\pi}\sin\frac{\pi}{2}\Gamma_0 + [{}_1c^P_{14}-{}_2c^P_{14}]\frac{2}{3\pi}\sin\frac{3\pi}{2}\Gamma_0,$$

$$(t\,220) = [{}_1c^P_{13}-{}_2c^P_{13}]S^2_{1/2} + [{}_1c^P_{14}-{}_2c^P_{14}]S^2_{1/2},$$

$$(t\,230) = [{}_1c^P_{13}-{}_2c^P_{13}]\frac{2}{\pi}\left(1-\sin\frac{\pi}{2}\Gamma_2\right) - [{}_1c^P_{14}-{}_2c^P_{14}]\frac{2}{3\pi}\left(1+\sin\frac{3\pi}{2}\Gamma_2\right),$$

$$(t\,240) = (1-\alpha^2\beta\Gamma_5),$$

$$(t\,410) = \left[{}_1c^P_{23}+\frac{\beta^3}{\alpha^4}{}_2c^P_{23}\right]\Gamma_0 + \left[{}_1c^P_{24}+\frac{\beta^3}{\alpha^4}{}_2c^P_{24}\right]\frac{1}{\pi}\sin\pi\Gamma_0 + \left[{}_1c^P_{25}+\frac{\beta^3}{\alpha^4}{}_2c^P_{25}\right]\frac{1}{2\pi}\sin 2\pi\Gamma_0,$$

$$(t\,420) = \left[{}_1c^P_{23}+\frac{\beta^3}{\alpha^4}{}_2c^P_{23}\right]\log\frac{\Gamma_2-\Gamma_6}{\Gamma_0-\Gamma_6} + \left[{}_1c^P_{24}+\frac{\beta^3}{\alpha^4}{}_2c^P_{24}\right]S_1^2 + \left[{}_1c^P_{25}+\frac{\beta^3}{\alpha^4}{}_2c^P_{25}\right]S_2^2,$$

$$(t\,430) = \left[{}_1c_{23}^P + \frac{\beta^3}{\alpha^4}\,{}_2c_{23}^P\right](1-\Gamma_2) - \left[{}_1c_{24}^P + \frac{\beta^3}{\alpha^4}\,{}_2c_{24}^P\right]\frac{1}{\pi}\sin\pi\,\Gamma_2$$

$$- \left[{}_1c_{25}^P + \frac{\beta^3}{\alpha^4}\,{}_2c_{25}^P\right]\frac{1}{2\pi}\sin 2\pi\,\Gamma_2\,,$$

$$(t\,440) = -3\,(1-\alpha\beta_2\Gamma_5)\,,$$

2. *Analytical Solutions for Simply Supported Plates*

2.1. *Additional Notation*

$$c_{11}^i = \pi^4[w\,\Gamma_3^4 + 2\,\Gamma_3^2 + 1]\,, \qquad c_{12}^i = \pi^4[w\,\Gamma_3^4 - 2\,(1-2\,\nu)\,\Gamma_3^2 + 1]\,,$$

$$c_{13}^i = \pi^4[w\,\Gamma_3^4 + (6+4\,\nu)\,\Gamma_3^2 + 9]\,, \qquad c_{14}^i = \pi^4[w\,\Gamma_3^4 - 2\,(3-8\,\nu)\,\Gamma_3^2 + 9]\,,$$

$$c_{15}^i = \pi^4[w\,\Gamma_3^4 + 18\,\Gamma_3^2 + 81]\,, \qquad c_{16}^i = \pi^4[w\,\Gamma_3^4 + 18\,\nu\,\Gamma_3^2 + 81]\,.$$

2.2. *Analytical Solutions*

The analytical solution is given in the form:

$$F_{11} = 0$$

or

$$F_{11}F_{22} - F_{12}F_{21} = 0\,,$$

where

$$F_{11} = (\Gamma_4)^2\,(v\,110) + (v\,111)\,,$$

$$F_{12} = F_{21} = (\Gamma_4)^2\,(v\,120) + (v\,121)\,,$$

$$F_{22} = (\Gamma_4)^2\,(v\,220) + (v\,221)\,.$$

2.2a. Elastic Buckling

$$(v\,110) = u\,(t\,130)\,,$$

$$(v\,120) = 0\,,$$

$$(v\,220) = u\,(t\,430)\,,$$

$$(v\,111) = -k_1\,(t\,140) - k_2\,(t\,151) + k_3\,(t\,152) + k_4\,(t\,160)\,,$$

$$(v\,121) = -k_1\,(t\,240) - k_2\,(t\,251) + k_3\,(t\,252) + k_4\,(t\,260)\,,$$

$$(v\,221) = -k_1\,(t\,440) - k_2\,(t\,451) + k_3\,(t\,452) + k_4\,(t\,460)\,.$$

$$(t\,130) = c_{11}^E\,,$$

$$(t\,140) = \Gamma_0 + \frac{1}{\pi}\sin\pi\,\Gamma_0\,,$$

$$(t\,151) = \Gamma_2 - \Gamma_0 + \frac{1}{\pi}\,[\sin\pi\,\Gamma_2 - \sin\pi\,\Gamma_0]\,,$$

$$(t\,152) = \frac{1}{2}[\Gamma_2^2 - \Gamma_0^2) + \frac{1}{\pi}\,[\Gamma_2\sin\pi\,\Gamma_2 - \Gamma_0\sin\pi\,\Gamma_0] + \left(\frac{1}{\pi}\right)^2[\cos\pi\,\Gamma_2 - \cos\pi\,\Gamma_0]\,,$$

$$(t\,160) = \Gamma_2 - 1 + \frac{1}{\pi}\sin\pi\,\Gamma_2\,,$$

$$(t\,240) = \frac{1}{\pi}\sin\pi\Gamma_0 + \frac{1}{2\pi}\sin 2\pi\Gamma_0,$$

$$(t\,251) = \frac{1}{\pi}[\sin\pi\Gamma_2 - \sin\pi\Gamma_0] + \frac{1}{2\pi}[\sin 2\pi\Gamma_2 - \sin 2\pi\Gamma_0],$$

$$(t\,252) = \frac{1}{\pi}[\Gamma_2\sin\pi\Gamma_2 - \Gamma_0\sin\Gamma_0] + \frac{1}{2\pi}[\Gamma_2\sin 2\pi\Gamma_2 - \Gamma_0\sin 2\pi\Gamma_0]$$
$$+ \frac{1}{\pi^2}[\cos\pi\Gamma_2 - \cos\pi\Gamma_0] + \frac{1}{(2\pi)^2}[\cos 2\pi\Gamma_2 - \cos 2\pi\Gamma_0],$$

$$(t\,260) = \frac{1}{\pi}\sin\pi\Gamma_2 + \frac{1}{2\pi}\sin 2\pi\Gamma_2,$$

$$(t\,430) = c_{15}^E,$$

$$(t\,440) = \Gamma_0 + \frac{1}{3\pi}\sin 3\pi\Gamma_0,$$

$$(t\,451) = \Gamma_2 - \Gamma_0 + \frac{1}{3\pi}[\sin 3\pi\Gamma_2 - \sin 3\pi\Gamma_0],$$

$$(t\,452) = \frac{1}{2}[\Gamma_2^2 - \Gamma_0^2] + \frac{1}{3\pi}[\Gamma_2\sin 3\pi\Gamma_2 - \Gamma_0\sin 3\pi\Gamma_0]$$
$$+ \left(\frac{1}{3\pi}\right)^2[\cos 3\pi\Gamma_2 - \cos 3\pi\Gamma_0],$$

$$(t\,460) = \Gamma_2 - 1 + \frac{1}{3\pi}\sin 3\pi\Gamma_2,$$

2.2b. Elastic-plastic Buckling
(Based on the deformation theory)

$$(v\,110) = u_1(t\,110) - u_2(t\,120) + u(t\,130),$$
$$(v\,120) = u_1(t\,210) - u_2(t\,220) + u(t\,230),$$
$$(v\,220) = u_1(t\,410) - u_2(t\,420) + u(t\,430),$$
$$(v\,111) = -(t\,140) - k_2(t\,151) + k_3(t\,152) + k_4(t\,160),$$
$$(v\,121) = -(t\,240) - k_2(t\,251) + k_3(t\,252) + k_4(t\,260),$$
$$(v\,221) = -(t\,440) - k_2(t\,451) + k_3(t\,452) + k_4(t\,460).$$

$$(t\,110) = c_{11}^P\Gamma_0 + \frac{1}{\pi}c_{12}^P\sin\pi\Gamma_0,$$

$$(t\,120) = c_{11}^P\log\left[\frac{\Gamma_1 - \Gamma_6}{\Gamma_0 - \Gamma_6}\right] + c_{12}^P S_1^1,$$

$$(t\,130) = c_{12}^E(1 - \Gamma_1) - \frac{1}{\pi}c_{12}^E\sin\pi\Gamma_1,$$

$$(t\,140) = \Gamma_1 + \frac{1}{\pi}\sin\pi\Gamma_1,$$

$$(t\,151) = (\Gamma_2 - \Gamma_1) + \frac{1}{\pi}[\sin \pi \Gamma_2 - \sin \pi \Gamma_1],$$

$$(t\,152) = \frac{1}{2}(\Gamma_2^2 - \Gamma_1^2) + \frac{1}{\pi}[\Gamma_2 \sin \pi \Gamma_2 - \Gamma_1 \sin \pi \Gamma_1] + \left(\frac{1}{\pi}\right)^2 [\cos \pi \Gamma_2 - \cos \pi \Gamma_1],$$

$$(t\,160) = \Gamma_2 - 1 + \frac{1}{\pi} \sin \pi \Gamma_2,$$

$$(t\,210) = \frac{1}{\pi} c_{13}^P \sin \pi \Gamma_0 + \frac{1}{2\pi} c_{14}^P \sin 2\pi \Gamma_0,$$

$$(t\,220) = c_{13}^P S_{13}^1 + c_{14}^P S_{14}^1,$$

$$(t\,230) = -\frac{1}{\pi} c_{13}^E \sin \pi \Gamma_1 - \frac{1}{2\pi} c_{14}^E \sin \pi \Gamma_1,$$

$$(t\,240) = \frac{1}{\pi} \sin \pi \Gamma_1 + \frac{1}{2\pi} \sin 2\pi \Gamma_1,$$

$$(t\,251) = \frac{1}{\pi}[\sin \pi \Gamma_2 - \sin \pi \Gamma_1] + \frac{1}{2\pi}[\sin 2\pi \Gamma_2 - \sin 2\pi \Gamma_1],$$

$$(t\,252) = \frac{1}{\pi}[\Gamma_2 \sin \pi \Gamma_2 - \Gamma_1 \sin \pi \Gamma_1] + \frac{1}{2\pi}[\Gamma_2 \sin 2\pi \Gamma_2 - \Gamma_1 \sin 2\pi \Gamma_1]$$
$$+ \frac{1}{\pi^2}[\cos \pi \Gamma_2 - \cos \pi \Gamma_0] + \frac{1}{(2\pi)^2}[\cos 2\pi \Gamma_2 - \cos 2\pi \Gamma_0],$$

$$(t\,260) = \frac{1}{\pi} \sin \pi \Gamma_2 + \frac{1}{2\pi} \sin 2\pi \Gamma_2,$$

$$(t\,410) = c_{15}^P + \frac{1}{3\pi} c_{16}^P \sin 3\pi \Gamma_0,$$

$$(t\,420) = c_{15}^P \log \frac{\Gamma_1 - \Gamma_6}{\Gamma_0 - \Gamma_6} + c_{16}^P S_3^1,$$

$$(t\,430) = c_{15}^E (1 - \Gamma_1) - \frac{1}{3\pi} c_{16}^E \sin 3\pi \Gamma_1,$$

$$(t\,440) = \Gamma_1 + \frac{1}{3\pi} \sin 3\pi \Gamma_1,$$

$$(t\,451) = \Gamma_2 - \Gamma_1 + \frac{1}{3\pi}[\sin 3\pi \Gamma_2 - \sin 3\pi \Gamma_1],$$

$$(t\,452) = \frac{1}{2}(\Gamma_2^2 - \Gamma_1^2) + \frac{1}{3\pi}[\Gamma_2 \sin 3\pi \Gamma_2 - \Gamma_1 \sin 3\pi \Gamma_1]$$
$$+ \left(\frac{1}{3\pi}\right)^2 [\cos 3\pi \Gamma_2 - \cos 3\pi \Gamma_1],$$

$$(t\,460) = \Gamma_2 - 1 + \frac{1}{3\pi} \sin 3\pi \Gamma_2,$$

2.2c. Elastic-plastic Buckling
(Based on the flow theory)

$$(v\,110) = u_0\,(t\,110) + u\,(t\,130)\,,$$
$$(v\,120) = u_0\,(t\,210) + u\,(t\,230)\,,$$
$$(v\,220) = u_0\,(t\,410) + u\,(t\,430)\,,$$
$$(v\,111) = -(t\,140) - k_2\,(t\,151) + k_3\,(t\,152) + k_4\,(t\,160)\,,$$
$$(v\,121) = -(t\,240) - k_2\,(t\,251) + k_3\,(t\,252) + k_4\,(t\,260)\,,$$
$$(v\,221) = -(t\,440) - k_2\,(t\,451) + k_3\,(t\,452) + k_4\,(t\,460)\,.$$

$$(t\,110) = c_{11}^P\,\Gamma_1 + \frac{1}{\pi}\,c_{12}^P \sin\pi\,\Gamma_1\,,$$
$$(t\,130) = c_{11}^E\,(1-\Gamma_1) - \frac{1}{\pi}\,c_{12}^E \sin\pi\,\Gamma_1\,,$$
$$(t\,210) = \frac{1}{\pi}\,c_{13}^P \sin\pi\,\Gamma_1 + \frac{1}{2\,\pi}\,c_{14}^P \sin 2\,\pi\,\Gamma_1\,,$$
$$(t\,230) = -\frac{1}{\pi}\,c_{13}^E \sin\pi\,\Gamma_1 + \frac{1}{2\,\pi}\,c_{14}^E \sin 2\,\pi\,\Gamma_1\,,$$
$$(t\,410) = c_{15}^P\,\Gamma_1 + \frac{1}{3\,\pi}\,c_{16}^P \sin 3\,\pi\,\Gamma_1\,,$$
$$(t\,430) = c_{15}^E\,(1-\Gamma_1) - \frac{1}{3\,\pi}\,c_{16}^E \sin 3\,\pi\,\Gamma_1\,,$$

$(t\,140)$, $(t\,151)$, $(t\,152)$, $(t\,160)$, $(t\,240)$, $(t\,251)$, $(t\,252)$, $(t\,260)$, $(t\,440)$, $(t\,451)$, $(t\,452)$ and $(t\,460)$ are the same as in Section 2.2b.

2.2d. Plastic Buckling
(Based on the deformation theory)

$$(v\,110) = u_1\,(t\,110) - u_2\,(t\,120) + u_3\,(t\,130)\,,$$
$$(v\,120) = u_1\,(t\,210) - u_2\,(t\,220) + u_3\,(t\,230)\,,$$
$$(v\,220) = u_1\,(t\,410) - u_2\,(t\,420) + u_3\,(t\,430)\,,$$
$$(v\,111) = -1\,,$$
$$(v\,121) = 0\,,$$
$$(v\,221) = -1\,,$$

$$(t\,110) = c_{11}^P\,\Gamma_0 + \frac{1}{\pi}\,c_{12}^P \sin\pi\,\Gamma_0\,,$$
$$(t\,120) = c_{11}^P \log\frac{\Gamma_2-\Gamma_6}{\Gamma_0-\Gamma_6} + c_{12}^P\,S_1^2\,,$$
$$(t\,130) = c_{11}^P\,(1-\Gamma_2) - \frac{1}{\pi}\,c_{12}^P \sin\pi\,\Gamma_2\,,$$

$$(t\,210) = \frac{1}{\pi} c_{13}^P \sin \pi \Gamma_0 + \frac{1}{2\pi} c_{14}^P \sin 2\pi \Gamma_0,$$

$$(t\,220) = c_{13}^P S_1^2 + c_{14}^P S_2^2,$$

$$(t\,230) = \frac{1}{\pi} c_{13}^P \sin \pi \Gamma_2 + \frac{1}{2\pi} c_{14}^P \sin 2\pi \Gamma_2,$$

$$(t\,410) = c_{15}^P \Gamma_0 + \frac{1}{3\pi} c_{16}^P \sin 3\pi \Gamma_0,$$

$$(t\,420) = c_{15}^P \log \frac{\Gamma_2 - \Gamma_6}{\Gamma_0 - \Gamma_6} + c_{16}^P S_3^2,$$

$$(t\,430) = c_{15}^P (1 - \Gamma_2) - \frac{1}{3\pi} c_{16}^P \sin 3\pi \Gamma_2,$$

3. *Analytical Solution for Fixed Plates*

3.1. *Additional Notation*

$c_{11}^i = \pi^4 [3 w \Gamma_3^4 + 8 \Gamma_3^2 + 16]$, $c_{12}^i = 4\pi^4 [w \Gamma_3^4 + 4\nu \Gamma_3^2]$,

$c_{13}^i = \pi^4 [w \Gamma_3^4 - 8(1 - 2\nu) \Gamma_3^2 + 16]$, $c_{14}^i = -2\pi^4 w \Gamma_3^4$,

$c_{15}^i = \pi^4 [-w \Gamma_3^4 + 4(4 - \nu) \Gamma_3^2 + 64]$, $c_{16}^i = 2\pi^4 [w \Gamma_3^4 + 16\nu \Gamma_3^2]$,

$c_{17}^i = \pi^4 [w \Gamma_3^4 - 4(4 - 9\nu) \Gamma_3^2 + 64]$, $c_{25}^i = \pi^4 [3 w \Gamma_3^4 + 32 \Gamma_3^2 + 64]$,

$c_{26}^i = -\pi^4 [w \Gamma_3^4 + 16(1 - \nu) \Gamma_3^2]$, $c_{27}^i = \pi^4 [w \Gamma_3^4 - 32(1 - 2\nu) \Gamma_3^2 + 64]$.

3.2. *Analytical Solutions*

The analytical solution is given in the form:

$$F_{11} = 0$$

or

$$F_{11} F_{22} - F_{12} F_{21} = 0,$$

where

$$F_{11} = (\Gamma_4)^2 (v\,110) + (v\,111),$$
$$F_{12} = F_{21} = (\Gamma_4)^2 (v\,120) + (v\,121),$$
$$F_{22} = (\Gamma_4)^2 (v\,220) + (v\,221).$$

3.2a. Elastic Buckling

$$(v\,110) = u\,(t\,130),$$

$$(v\,120) = u\,(t\,230),$$

$$(v\,220) = u\,(t\,430),$$

$$(v\,111) = -k_1 (t\,140) - k_2 (t\,151) + k_3 (t\,152) + k_4 (t\,160),$$

$$(v\,121) = -k_1 (t\,240) - k_2 (t\,251) + k_3 (t\,252) + k_4 (t\,260),$$

$$(v\,221) = -k_1 (t\,440) - k_2 (t\,451) + k_3 (t\,452) + k_4 (t\,460),$$

$$(t\,130) = c^E_{11},$$

$$(t\,140) = 3\,\Gamma_0 + \frac{4}{\pi}\sin\pi\,\Gamma_0 + \frac{1}{2\,\pi}\sin 2\,\pi\,\Gamma_0,$$

$$(t\,151) = 3\,(\Gamma_2 - \Gamma_0) + \frac{4}{\pi}[\sin\pi\,\Gamma_2 - \sin\pi\,\Gamma_0] + \frac{1}{2\,\pi}[\sin 2\,\pi\,\Gamma_2 - \sin 2\,\pi\,\Gamma_0],$$

$$(t\,152) = \frac{3}{2}(\Gamma_2^2 - \Gamma_0^2) + \frac{4}{\pi}[\Gamma_2\sin\pi\,\Gamma_2 - \Gamma_0\sin\pi\,\Gamma_0] + \left(\frac{2}{\pi}\right)^2[\cos\pi\,\Gamma_2 - \cos\pi\,\Gamma_0]$$
$$+ \left(\frac{1}{2\,\pi}\right)[\Gamma_2\sin 2\,\pi\,\Gamma_2 - \Gamma_0\sin 2\,\pi\,\Gamma_0] + \left(\frac{1}{2\,\pi}\right)^2[\cos 2\,\pi\,\Gamma_2 - \cos 2\,\pi\,\Gamma_0],$$

$$(t\,160) = 3\,(\Gamma_2 - 1) + \left(\frac{4}{\pi}\right)\sin\pi\,\Gamma_2 + \frac{1}{2\,\pi}\sin 2\,\pi\,\Gamma_2,$$

$$(t\,230) = c^E_{14},$$

$$(t\,240) = -2\,\Gamma_0 - \frac{1}{\pi}\sin\pi\,\Gamma_0 + \frac{1}{\pi}\sin 2\,\pi\,\Gamma_0 + \frac{1}{3\,\pi}\sin 3\,\pi\,\Gamma_0,$$

$$(t\,251) = -2\,(\Gamma_2 - \Gamma_0) - \frac{1}{\pi}[\sin\pi\,\Gamma_2 - \sin\pi\,\Gamma_0] + \frac{1}{\pi}[\sin 2\,\pi\,\Gamma_2 - \sin 2\,\pi\,\Gamma_0]$$
$$+ \frac{1}{3\,\pi}[\sin 3\,\pi\,\Gamma_2 - \sin 3\,\pi\,\Gamma_0],$$

$$(t\,252) = -(\Gamma_2^2 - \Gamma_0^2) - \frac{1}{\pi}[\Gamma_2\sin\pi\,\Gamma_2 - \Gamma_0\sin\pi\,\Gamma_0] - \left(\frac{1}{\pi}\right)^2[\cos\pi\,\Gamma_2 - \cos\pi\,\Gamma_0]$$
$$+ \left(\frac{1}{\pi}\right)[\Gamma_2\sin 2\,\pi\,\Gamma_2 - \Gamma_0\sin 2\,\pi\,\Gamma_0] + 2\left(\frac{1}{2\,\pi}\right)^2[\cos 2\,\pi\,\Gamma_2 - \cos 2\,\pi\,\Gamma_0]$$
$$+ \left(\frac{1}{3\,\pi}\right)[\Gamma_2\sin 3\,\pi\,\Gamma_2 - \Gamma_0\sin 3\,\pi\,\Gamma_0] + \left(\frac{1}{3\,\pi}\right)^2[\cos 3\,\pi\,\Gamma_2 - \cos 3\,\pi\,\Gamma_0],$$

$$(t\,260) = -2\,(\Gamma_2 - 1) - \frac{1}{\pi}\sin\pi\,\Gamma_2 + \frac{1}{\pi}\sin\pi\,\Gamma_2 + \frac{1}{3\,\pi}\sin 3\,\pi\,\Gamma_2,$$

$$(t\,430) = c^E_{25},$$

$$(t\,440) = 3\,\Gamma_0 - \frac{2}{\pi}\sin 2\,\pi\,\Gamma_0 + \frac{1}{4\,\pi}\sin 4\,\pi\,\Gamma_0,$$

$$(t\,451) = 3\,(\Gamma_2 - \Gamma_0) - \frac{2}{\pi}[\sin 2\,\pi\,\Gamma_2 - \sin 2\,\pi\,\Gamma_0] + \frac{1}{4\,\pi}[\sin 4\,\pi\,\Gamma_2 - \sin 4\,\pi\,\Gamma_0],$$

$$(t\,452) = \frac{3}{2}(\Gamma_2^2 - \Gamma_0^2) - \frac{2}{\pi}[\Gamma_2\sin 2\,\pi\,\Gamma_2 - \Gamma_0\sin 2\,\pi\,\Gamma_0]$$
$$- \left(\frac{1}{\pi}\right)^2[\cos 2\,\pi\,\Gamma_2 - \cos 2\,\pi\,\Gamma_0] + \left(\frac{1}{4\,\pi}\right)[\Gamma_2\sin 4\,\pi\,\Gamma_2 - \Gamma_0\sin 4\,\pi\,\Gamma_0]$$
$$+ \left(\frac{1}{4\,\pi}\right)^2[\cos 4\,\pi\,\Gamma_2 - \cos 4\,\pi\,\Gamma_0],$$

$$(t\,460) = 4\,(\Gamma_2 - 1) - \frac{2}{\pi}\sin 2\,\pi\,\Gamma_2 + \frac{1}{4\,\pi}\sin 4\,\pi\,\Gamma_2,$$

3.2b. Elastic-plastic Buckling
(Based on the deformation theory)

$$(v\,110) = u_1\,(t\,110) - u_2\,(t\,120) + u\,(t\,130)\,,$$

$$(v\,120) = u_1\,(t\,210) - u_2\,(t\,220) + u\,(t\,230)\,,$$

$$(v\,220) = u_1\,(t\,420) - u_2\,(t\,420) + u\,(t\,430)\,,$$

$$(v\,111) = -(t\,140) - k_2\,(t\,151) + k_3\,(t\,152) + k_4\,(t\,160)\,,$$

$$(v\,121) = -(t\,240) - k_2\,(t\,251) + k_3\,(t\,252) + k_4\,(t\,260)\,,$$

$$(v\,221) = -(t\,440) - k_2\,(t\,451) + k_3\,(t\,452) + k_4\,(t\,460)\,.$$

$$(t\,110) = c_{11}^P\,\Gamma_0 + \frac{1}{\pi}\,c_{12}^P \sin \pi\,\Gamma_0 + \frac{1}{2\,\pi}\,c_{13}^P \sin 2\,\pi\,\Gamma_0\,,$$

$$(t\,120) = c_{11}^P \log \frac{\Gamma_1 - \Gamma_6}{\Gamma_0 - \Gamma_6} + c_{12}^P\,S_1^1\,,$$

$$(t\,130) = c_{11}^E\,(1 - \Gamma_1) - \frac{1}{\pi}\,c_{12}^E \sin \pi\,\Gamma_1 - \frac{1}{2\,\pi}\,c_{13}^E \sin 2\,\pi\,\Gamma_1\,,$$

$$(t\,140) = 3\,\Gamma_1 + \frac{4}{\pi} \sin \pi\,\Gamma_1 + \frac{1}{2\,\pi} \sin 2\,\pi\,\Gamma_1\,,$$

$$(t\,151) = 3\,(\Gamma_2 - \Gamma_1) + \frac{4}{\pi}\,[\sin \pi\,\Gamma_2 - \sin \pi\,\Gamma_1] + \frac{1}{2\,\pi}\,[\sin 2\,\pi\,\Gamma_2 - \sin 2\,\pi\,\Gamma_1]\,,$$

$$(t\,152) = \frac{3}{2}(\Gamma_2^2 - \Gamma_1^2) + \frac{4}{\pi}\,[\Gamma_2 \sin \pi\,\Gamma_2 - \Gamma_1 \sin \pi\,\Gamma_1] + \left(\frac{2}{\pi}\right)^2 [\cos \pi\,\Gamma_2 - \cos \pi\,\Gamma_1]$$
$$+ \frac{1}{2\,\pi}\,[\Gamma_2 \sin 2\,\pi\,\Gamma_2 - \Gamma_1 \sin 2\,\pi\,\Gamma_1] + \left(\frac{1}{2\,\pi}\right)^2 [\cos 2\,\pi\,\Gamma_2 - \cos 2\,\pi\,\Gamma_1]\,,$$

$$(t\,160) = 3\,(\Gamma_2 - 1) + \frac{4}{\pi} \sin \pi\,\Gamma_2 + \frac{1}{2\,\pi} \sin 2\,\pi\,\Gamma_2\,,$$

$$(t\,210) = c_{14}^P\,\Gamma_0 + \frac{1}{\pi}\,c_{15}^P \sin \pi\,\Gamma_0 + \frac{1}{2\,\pi}\,c_{16}^P \sin 2\,\pi\,\Gamma_0 + \frac{1}{3\,\pi}\,c_{17}^P \sin 3\,\pi\,\Gamma_0\,,$$

$$(t\,220) = c_{14}^P \log \frac{\Gamma_1 - \Gamma_6}{\Gamma_0 - \Gamma_6} + c_{15}^P\,S_1^1 + c_{16}^P\,S_2^1 + c_{17}^P\,S_3^1\,,$$

$$(t\,230) = c_{14}^E\,(1 - \Gamma_1) - \frac{1}{\pi}\,c_{15}^E \sin \pi\,\Gamma_1 - \frac{1}{2\,\pi}\,c_{16}^E \sin 2\,\pi\,\Gamma_1 - \frac{1}{3\,\pi}\,c_{17}^E \sin 3\,\pi\,\Gamma_1\,,$$

$$(t\,240) = -2\,\Gamma_1 - \frac{1}{\pi} \sin \pi\,\Gamma_1 + \frac{1}{\pi} \sin 2\,\pi\,\Gamma_1 + \frac{1}{3\,\pi} \sin 3\,\pi\,\Gamma_1\,,$$

$$(t\,251) = -2\,(\Gamma_2 - \Gamma_1) - \frac{1}{\pi}\,[\sin \pi\,\Gamma_2 - \sin \pi\,\Gamma_1] + \frac{1}{\pi}\,[\sin 2\,\pi\,\Gamma_2 - \sin 2\,\pi\,\Gamma_1]$$
$$+ \frac{1}{3\,\pi}\,[\sin 3\,\pi\,\Gamma_2 - \sin 3\,\pi\,\Gamma_1]\,,$$

$$(t\,252) = -(\Gamma_2^2-\Gamma_1^2)-\frac{1}{\pi}[\Gamma_2\sin\pi\Gamma_2-\Gamma_1\sin\pi\Gamma_1]-\left(\frac{1}{\pi}\right)^2[\cos\pi\Gamma_2-\cos\pi\Gamma_1]$$
$$+\frac{1}{\pi}[\Gamma_2\sin 2\pi\Gamma_2-\Gamma_1\sin 2\pi\Gamma_1]+2\left(\frac{1}{2\pi}\right)^2[\cos 2\pi\Gamma_2-\cos 2\pi\Gamma_1]$$
$$+\left(\frac{1}{3\pi}\right)[\Gamma_2\sin 3\pi\Gamma_2-\Gamma_1\sin 3\pi\Gamma_1]+\left(\frac{1}{3\pi}\right)^2[\cos 3\pi\Gamma_2-\cos 3\pi\Gamma_1],$$

$$(t\,260) = -2(\Gamma_2-1)+\frac{1}{\pi}\sin\pi\Gamma_2+\frac{1}{\pi}\sin 2\pi\Gamma_2+\frac{1}{3\pi}\sin 3\pi\Gamma_2,$$

$$(t\,410) = c_{25}^P\Gamma_0+\left(\frac{1}{2\pi}\right)c_{26}^P\sin 2\pi\Gamma_0+\left(\frac{1}{4\pi}\right)c_{27}^P\sin 4\pi\Gamma_0,$$

$$(t\,420) = c_{25}^P\log\frac{\Gamma_1-\Gamma_6}{\Gamma_0-\Gamma_6}+c_{26}^P S_2^1+c_{27}^P S_4^1,$$

$$(t\,430) = c_{25}^E(1-\Gamma_1)-\frac{1}{2\pi}c_{26}^E\sin 2\pi\Gamma_1-\frac{1}{4\pi}c_{27}^E\sin 4\pi,$$

$$(t\,440) = 3\Gamma_1-\frac{2}{\pi}\sin 2\pi\Gamma_1+\frac{1}{4\pi}\sin 4\pi\Gamma_1,$$

$$(t\,451) = 3(\Gamma_1-\Gamma_0)-\frac{2}{\pi}[\sin 2\pi\Gamma_2-\sin 2\pi\Gamma_1]+\frac{1}{4\pi}[\sin 4\pi\Gamma_2-\sin 4\pi\Gamma_1],$$

$$(t\,452) = \frac{3}{2}(\Gamma_1^2-\Gamma_0^2)-\frac{2}{\pi}[\Gamma_2\sin 2\pi\Gamma_2-\Gamma_1\sin 2\pi\Gamma_1]-\left(\frac{1}{\pi}\right)^2[\cos 2\pi\Gamma_2-\cos 2\pi\Gamma_1]$$
$$+\frac{1}{4\pi}[\Gamma_2\sin 4\pi\Gamma_2-\Gamma_1\sin 4\pi\Gamma_1]-\left(\frac{1}{4\pi}\right)^2[\cos 4\pi\Gamma_2-\cos 4\pi\Gamma_1],$$

$$(t\,460) = 3(\Gamma_2-1)-\frac{2}{\pi}\sin 2\pi\Gamma_2+\frac{1}{4\pi}\sin 4\pi\Gamma_2.$$

3.2c. Elastic-plastic Buckling
(Based on the flow theory)

$$(v\,110) = u_0(t\,110)+u(t\,130),$$
$$(v\,120) = u_0(t\,210)+u(t\,230),$$
$$(v\,220) = u_0(t\,410)+u(t\,430),$$
$$(v\,111) = -(t\,140)-k_2(t\,151)+k_3(t\,152)+k_4(t\,160),$$
$$(v\,121) = -(t\,240)-k_2(t\,251)+k_3(t\,252)+k_4(t\,260),$$
$$(v\,221) = -(t\,440)-k_2(t\,451)+k_3(t\,452)+k_4(t\,460).$$

$$(t\,110) = c_{11}^P\Gamma_1+\left(\frac{1}{\pi}\right)c_{12}^P\sin\pi\Gamma_1+\left(\frac{1}{2\pi}\right)c_{13}^P\sin 2\pi\Gamma_1,$$

$$(t\,130) = c_{11}^E(1-\Gamma_1)-\frac{1}{\pi}c_{12}^E\sin\pi\Gamma_1-\frac{1}{2\pi}c_{13}^E\sin 2\pi\Gamma_1,$$

$$(t\,210) = c_{14}^P\Gamma_1+\frac{1}{\pi}c_{15}^P\sin\pi\Gamma_1+\frac{1}{2\pi}c_{16}^P\sin 2\pi\Gamma_1+\frac{1}{3\pi}c_{17}^P\sin 3\pi\Gamma_1,$$

$$(t\,230) = c_{14}^E (1-\Gamma_1) - \frac{1}{\pi} c_{15}^E \sin \pi \Gamma_1 - \frac{1}{2\pi} c_{16}^E \sin 2\pi \Gamma_1 - \frac{1}{3\pi} c_{17}^E \sin 3\pi \Gamma_1,$$

$$(t\,410) = c_{25}^P \Gamma_1 + \frac{1}{2\pi} c_{26}^P \sin 2\pi \Gamma_1 + \frac{1}{4\pi} c_{27}^P \sin 4\pi \Gamma_1,$$

$$(t\,430) = c_{25}^E (1-\Gamma_1) - \frac{1}{2\pi} c_{26}^E \sin 2\pi \Gamma_1 - \frac{1}{4\pi} c_{27}^E \sin 4\pi \Gamma_1.$$

$(t\,140)$, $(t\,151)$, $(t\,152)$, $(t\,160)$, $(t\,240)$, $(t\,251)$, $(t\,252)$, $(t\,260)$, $(t\,440)$, $(t\,451)$, $(t\,452)$ and $(t\,460)$ are the same as in Section 3.2b.

3.2d. Plastic Buckling
(Based on the deformation theory)

$$(v\,110) = u_1 (t\,110) - u_2 (t\,120) + u_3 (t\,130),$$
$$(v\,120) = u_1 (t\,210) - u_2 (t\,220) + u_3 (t\,230),$$
$$(v\,220) = u_1 (t\,410) - u_2 (t\,420) + u_3 (t\,430),$$
$$(v\,111) = -3,$$
$$(v\,121) = 2,$$
$$(v\,221) = -3.$$

$$(t\,110) = c_{11}^P \Gamma_0 + \frac{1}{\pi} c_{12}^P \sin \pi \Gamma_0 + \frac{1}{2\pi} c_{13}^P \sin 2\pi \Gamma_0,$$

$$(t\,120) = c_{11}^P \log \frac{\Gamma_2 - \Gamma_6}{\Gamma_0 - \Gamma_6} + c_{12}^P S_1^2 + c_{13}^P S_1^2,$$

$$(t\,130) = 1 - \Gamma_2 - \frac{1}{\pi} \sin \pi \Gamma_2 - \frac{1}{2\pi} \sin 2\pi \Gamma_2,$$

$$(t\,210) = c_{14}^P \Gamma_0 + \frac{1}{\pi} c_{15}^P \sin \pi \Gamma_0 + \frac{1}{2\pi} c_{16}^P \sin 2\pi \Gamma_0 + \frac{1}{3\pi} c_{17}^P \sin 3\pi \Gamma_0,$$

$$(t\,220) = c_{14}^P \log \frac{\Gamma_2 - \Gamma_6}{\Gamma_0 - \Gamma_6} + c_{15}^P S_1^2 + c_{16}^P S_2^2 + c_{17}^P S_3^2,$$

$$(t\,230) = c_{14}^P (1-\Gamma_2) - \frac{1}{\pi} c_{15}^P \sin \pi \Gamma_2 - \frac{1}{2\pi} c_{16}^P \sin 2\pi \Gamma_2 - \frac{1}{3\pi} c_{17}^P \sin 3\pi \Gamma_2,$$

$$(t\,410) = c_{25}^P \Gamma_0 + \frac{1}{2\pi} c_{26}^P \sin 2\pi \Gamma_0 + \frac{1}{4\pi} c_{27}^P \sin 4\pi \Gamma_0,$$

$$(t\,420) = c_{25}^P \log \frac{\Gamma_2 - \Gamma_6}{\Gamma_0 - \Gamma_6} + c_{26}^P S_2^2 + c_{27}^P S_4^2,$$

$$(t\,430) = c_{25}^P (1-\Gamma_2) - \frac{1}{2\pi} \sin 2\pi \Gamma_2 - \frac{1}{4\pi} \sin 4\pi \Gamma_2.$$

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Summary

This paper presents the results of a theoretical investigation into the elastic, elastic-plastic, and plastic buckling of steel plates containing residual stresses and simply supported at the loading edges with the other edges:

a) elastically restrained,
b) simply supported,
c) fixed.

Numerical illustrations are presented for the analytical solution of the strength of square built-up columns which fail by local buckling. This study showed that the first term of the series of the assumed deflection function, $w = a\cos\frac{\pi}{2}\frac{y}{b}\sin N\frac{x}{L}$ was sufficient to investigate the elastic, elastic-plastic, and plastic buckling of the plate with residual stresses.

The theoretical predictions were correlated with experimental results obtained from pilot tests on square welded columns of ASTM A 36 steel and of ASTM A 514 steel (T-1 constructional alloy steel).

Résumé

Les auteurs communiquent les résultats d'une étude théorique ayant pour objet le voilement élastique, élasto-plastique et plastique d'âmes métalliques comportant des contraintes résiduelles et simplement appuyées aux bords chargés tandis que les autres bords sont:

a) élastiquement encastrés,
b) simplement appuyés,
c) encastrés.

Des exemples numériques illustrent le calcul de la résistance de poteaux composés, de section carrée, périssant par voilement. La présente étude a fait ressortir que, pour analyser le voilement élastique, élasto-plastique et plastique des âmes comportant des contraintes résiduelles, il était suffisant de ne retenir que le premier terme du développement en série de la fonction admise pour représenter la déformation, $w = a \cos \frac{\pi}{2} \frac{y}{b} \sin N \frac{x}{L}$.

On a déterminé la corrélation existant entre les prévisions données par le calcul et les résultats d'essais pilote exécutés sur des poteaux soudés de section carrée en acier ASTM A 36 et en acier ASTM A 514 (acier de construction allié T-1).

Zusammenfassung

Dieser Aufsatz berichtet über die Ergebnisse einer theoretischen Untersuchung des elastischen, elastisch-plastischen und plastischen Ausbeulens von Stahlplatten mit Eigenspannungen. In allen Fällen sind die Belastungskanten gestützt, die anderen Kanten

a) elastisch gehalten,
b) gestützt,
c) eingespannt.

Die analytische Lösung für die Festigkeit quadratisch zusammengebauter Knickstäbe, die durch örtliches Ausbeulen versagen, wird durch Zahlenbeispiele veranschaulicht. Aus dieser Untersuchung ergab sich, daß das erste Glied des Reihenausdrucks für die angenommene Auslenkungsfunktion $w = a \cos \frac{\pi}{2} \frac{y}{b} \sin N \frac{x}{L}$ zur Untersuchung des elastischen, elastisch-plastischen und plastischen Ausbeulens der Platte mit Eigenspannungen ausreichte.

Die theoretischen Resultate wurden mit den in praktischen Versuchen mit geschweißten quadratischen Säulen aus Stählen ASTM A 36 und ASTM A 514 (Legierungs-Baustahl T-1) gewonnenen verglichen.