Semi-infinite plate subjected to the concentrated load acting at its internal point

Semi-Infinite Plate Subjected to the Concentrated Load Acting at its Internal Point

Plaque semi-infinie soumise à une charge concentrée agissant en un point intérieur

Halbebenen-Platte unter Einzellast im Innern

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In this paper, the complex variable method of N. I. MUSKHELISHVILI is applied to the solution of semi-infinite plate subjected to the concentrated load acting at its internal point. The coefficients of the two functions of complex variable, whose combination gives the Airy Stress function, are taken as unknowns. Then they are determined by using the boundary conditions and considering the limits of these two functions while the loaded point approaches to the boundary or tends to the infinity. Two examples are given in order to visualize the stress pattern in the semi-infinite plate.

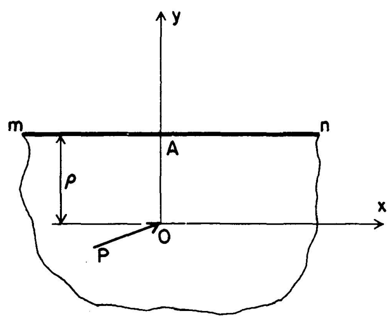

Fig. 1.

Consider the region S occupied by the semi-infinite plate as the "lower" half plane (Fig. 1) bounded by the horizontal line mn at a distance ρ from the origin 0. Let the horizontal and vertical components of the load, P, applied at the origin 0, be respectively X, Y and let the two independent functions of complex variable be:

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$$\begin{aligned} \tau(z) &= -(A+iB)\log z-(C+iD)\log(z-2i\rho)+\frac{E+iF}{z-2i\rho}, \\ \psi(z) &= (C-iD)\log z+(A-iB)\log(z-2i\rho)+\frac{E+iF}{z-2i\rho}, \end{aligned} \tag{1}$$

where the constants A, B, C, D, E, F, are the unknowns to be determined by the boundary conditions and from the limiting conditions corresponding to ρ, approaching zero or infinity.

The boundary condition is given by MUSKHELISHVILI[1])

$$\tau(t)+t\overline{\tau'(t)}+\overline{\psi(t)} = \text{constant}, \qquad t = x+i\rho. \tag{2}$$

Substituting the expressions (1) into the boundary condition (2) one obtains the following relations between the constants

$$E = 2\rho B, \qquad F = 2\rho A.$$

Then $\tau(z)$ and $\psi(z)$ become:

$$\begin{aligned} \tau(z) &= -(A+iB)\log z-(C+iD)\log(z-2i\rho)+2i\rho\frac{A-iB}{z-2i\rho}, \\ \psi(z) &= (C-iD)\log z+(A-iB)\log(z-i\rho)+2i\rho\frac{A-iB}{z-2i\rho}. \end{aligned} \tag{3}$$

The first derivatives of the functions $\tau(z)$ and $\psi(z)$ are:

$$\begin{aligned} \Phi(z) &= -\frac{A+iB}{z}-\frac{C+iD}{z-2i\rho}-2i\rho\frac{A-iB}{(z-2i\rho)^2}, \\ \Psi(z) &= \frac{C-iD}{z}+\frac{A-iB}{z-2i\rho}-2i\rho\frac{A-iB}{(z-2i\rho)^2}, \end{aligned} \tag{4}$$

where, $\Phi(z) = \tau'(z), \qquad \Psi(z) = \psi'(z).$

Now, consider the case when ρ approaches zero. Putting $\rho=0$ in the relations (4) one obtains

$$\Phi(z) = -\frac{A+iB}{z}-\frac{C+iD}{z}, \qquad \Psi(z) = \frac{C-iD}{z}+\frac{A-iB}{z}. \tag{5}$$

Similarly, when ρ increases to infinity the relations (4) become

$$\Phi(z) = -\frac{A+iB}{z}, \qquad \Psi(z) = \frac{C-iD}{z}. \tag{6}$$

But the solution of a semi-infinite plate under the concentrated load applied to its boundary is known.

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[1]) N. I. MUSKEHELISHVILI: Some Basic Problems of the Mathematical Theory of Elasticity, translated from the Russian by J. R. M. Radok, P. Noordhoff, Ltd., Groningen 1963.

$$\Phi(z) = -\frac{X+iY}{2\pi}\frac{1}{z}, \qquad \Psi(z) = \frac{X-iY}{2\pi}\frac{1}{z}. \tag{7}$$

Similarly the known solution of the infinite plate under a concentrated load gives

$$\Phi(z) = -\frac{X+iY}{2\pi(1+\chi)}\frac{1}{z}, \qquad \Psi(z) = \frac{\chi(X-iY)}{2\pi(1+\chi)}\frac{1}{z}, \tag{8}$$

where $\chi = 3-\nu$ for plane strain conditions

and $\chi = \frac{3-\nu}{1-\nu}$ for generalized plane stress conditions,

ν being the Poisson's ratio.

The remaining unknown constants A, B, C, D, can be determined from the relations (5), (6), (7), (8) as follows:

$$A = \frac{X}{2\pi(1+\chi)}, \quad B = \frac{Y}{2\pi(1+\chi)}, \quad C = \frac{\chi X}{2\pi(1+\chi)}, \quad D = \frac{\chi Y}{2\pi(1+\chi)}. \tag{9}$$

Finally, by substituting the expressions for the coefficients (9) in the relations (3), the functions $\tau(z)$ and $\psi(z)$ are found as follows.

$$\begin{aligned} \tau(z) &= \frac{1}{2\pi(1+\chi)}\left[-(X+iY)(\log z+\chi\log(z-2i\rho)+2i\rho\frac{X-iY}{z-2i\rho}\right], \\ \psi(z) &= \frac{X+iY}{2\pi(1+\chi)}\left[\chi\log z+\log(z-2i\rho)+\frac{2i\rho}{z-2i\rho}\right]. \end{aligned} \tag{10}$$

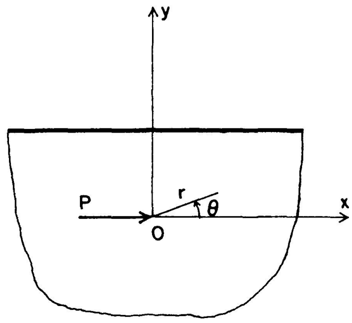

Fig. 2.

Example 1. Let the force P acting at the point 0 be horizontal, Fig. 2. Muskhelishvili gives the following expressions for the stresses in polar coordinates[2]).

$$\sigma_\rho+\sigma_\theta = 2[\phi(z)+\overline{\phi(z)}], \qquad \sigma_\theta-\sigma_\rho+2i\tau_{\rho\theta} = 2[\bar{z}\phi'(z)+\Psi(z)]e^{2i\theta}. \tag{11}$$

Taking the derivatives of the functions $\tau(z)$ and $\psi(z)$ in (10) and substituting them in (11), the stress components are obtained in polar coordinates as follows:

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[2]) Please refer to footnote 1.

$$\sigma_\theta = \frac{P}{2\pi(1+\chi)}\left[\frac{(\chi-1)\cos\theta}{r} + \frac{(1-2\chi)\,r\cos\theta - 2\rho\sin 2\theta}{r^2 - 4\rho r\sin\theta + 4\rho^2}\right.$$
$$+ \frac{[\chi r^2 + 4(6-\chi)\rho^2]\,r\cos\theta - 4(r^2+2\rho^2)\rho\sin 2\theta}{(r^2 - 4\rho r\sin\theta + 4\rho^2)^2}$$
$$\left. + 4\rho r\frac{r^3\sin 2\theta - 6\rho r^2\cos\theta + 8\rho^3\cos\theta}{(r^2 - 4\rho r\sin\theta + 4\rho^2)^3}\right],$$

$$\sigma_\rho = \frac{P}{2\pi(1+\chi)}\left[(3+\chi)\frac{\cos\theta}{r} + \frac{(1+2\chi)\,r\cos\theta - 2\rho\sin 2\theta}{r^2 - 4\rho r\sin\theta + 4\rho^2}\right.$$
$$+ \frac{[\chi r^2 - 4(\chi+2)\rho^2]\,r\cos\theta + 4(r^2-2\rho^2)\rho\sin 2\theta}{(r^2 - 4\rho r\sin\theta + 4\rho^2)^2}$$
$$\left. + 4\rho r\frac{r^3\sin 2\theta - 6\rho r^2\cos\theta + 8\rho^3\cos\theta}{(r^2 - 4\rho r\sin\theta + 4\rho^2)^3}\right],$$

$$\tau_{\rho\theta} = \frac{P}{2\pi(1+\chi)}\left[(\chi-1)\frac{\sin\theta}{r} + \frac{r\sin\theta + 2\rho\cos 2\theta}{r^2 - 4\rho r\sin\theta + 4\rho^2}\right.$$
$$+ \frac{[4\rho^2 r(2-\chi) - \chi r^3]\sin\theta + 8\rho^3\cos 2\theta - 2\rho r^2(1-2\chi)}{(r^2 - 4\rho r\sin\theta + 4\rho^2)^2}$$
$$\left. + 4\rho r\frac{r^3\cos 2\theta + 6\rho r^2\sin\theta + 8\rho^3\sin\theta - 12\rho^2 r}{(r^2 - 4\rho r\sin\theta + 4\rho^2)^3}\right].$$

Example 2. The concentrated load P and the Poisson ration ν are taken

$$P = 100\,\pi\, e^{\frac{i\pi}{6}}, \qquad \nu = 0.3.$$

For a possible comparison with the results of photo-elastic experiments the calculation is made of isoclinic and isochromatic lines. The former (Fig. 3) keep constant angle with the directions of principal stresses at all points, and the latter (Fig. 4) correspond to lines along which the difference of principal stresses is constant.

The calculations have been made with the IBM 7040 computer.

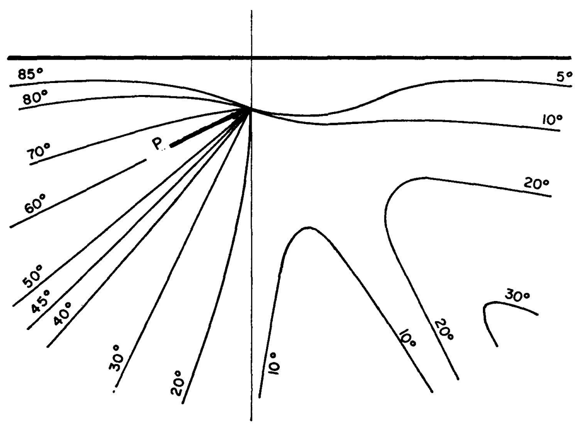

Fig. 3. Isoclinics.

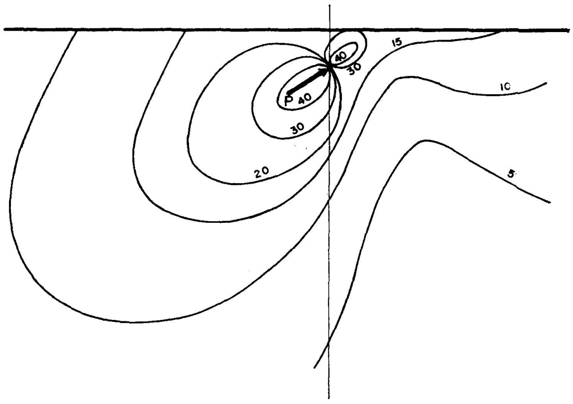

Fig. 4. Isochromatics.

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Summary

An attempt has been made to obtain the solution of the presented problem by considering the solution of the semi-infinite plate loaded at the boundary and the solution of infinite plate loaded at the internal point are its limit cases.

Résumé

On a essayé d'obtenir la solution du problème présenté en examinant la solution d'une plaque semi-infinie chargée sur son bord, et la solution d'une plaque infinie chargée sur un point intérieur qui sont ses cas limites.

Zusammenfassung

Für das vorgelegte Problem ward der Versuch unternommen, eine Lösung zu finden, in Anbetracht dessen, daß die Lösungen der Halbebene für Lasten einmal am Rand und im Innern die Grenzfälle darstellen.