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Substructures Analysis of Plate Systems

Calcul des systèmes de plaques par la méthode des substructures

Berechnung von Plattensystemen mittels Substrukturen

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Introduction

Frequently a structure to be analyzed by matrix methods possesses so many degrees of freedom that it cannot be treated within the core memory of the computer. In such cases the analyst must resort to the use of auxiliary storage facilities and accept an inevitable increase in computer time due to relatively slow access to information. For the substructures method, the structure is physically partitioned into several units, each of which can be treated in the available computer core. The books by PRZEMIENIECKI [7] and WEAVER [10] include extensive descriptions of substructures techniques. Of interest in this report is a special procedure which has proved to be very versatile when applied to plate systems with rectangular boundaries. To illustrate, an assemblage of complex finite elements subjected to bending and plane stress is used herein. This situation is typical of stiffened plate problems [6].

Substructures Analysis

For multistory framed buildings CLOUGH et al. [1, 2, 3] and WEAVER et al. [11, 12] partitioned the stiffness matrix into submatrices defining the action-displacement relations for joints at a given floor level, those at the level below, and those coupling the two levels. For planar frames all but the horizontal displacements at each level are progressively eliminated by matrix condensation as the analysis proceeds downward from the top story. The horizontal response due to static or dynamic forcing functions thus involves only one degree of freedom per floor level (assuming the floor systems perform as rigid diaphragms). All nonhorizontal displacements and member stress resultants are determined by backward substitution.

The similarity between a rectangular plane frame and a discretized rectangular plate in bending is apparent. The geometric configuration is essentially the same. Each row of finite elements in a plate is considered to be a substructure, as are the floor beams and columns of a story in a building frame. There are, however, important inherent differences. While a multistory building is restrained only at the base, the plate may be restrained arbitrarily. Furthermore, there is only one degree of freedom per story in the two-dimensional building model, whereas many lateral (normal to the plate) degrees of freedom are retained for each row of elements in the plate. Although all displacements could be eliminated for each substructure in a static analysis, a dynamic analysis is facilitated by retaining the lateral degrees of freedom.

Elimination Procedure for Rectangular Plates

Fig. 1 illustrates a substructure for a plate subjected to both bending and plane stress. Conforming bending and linear-strain membrane elements,

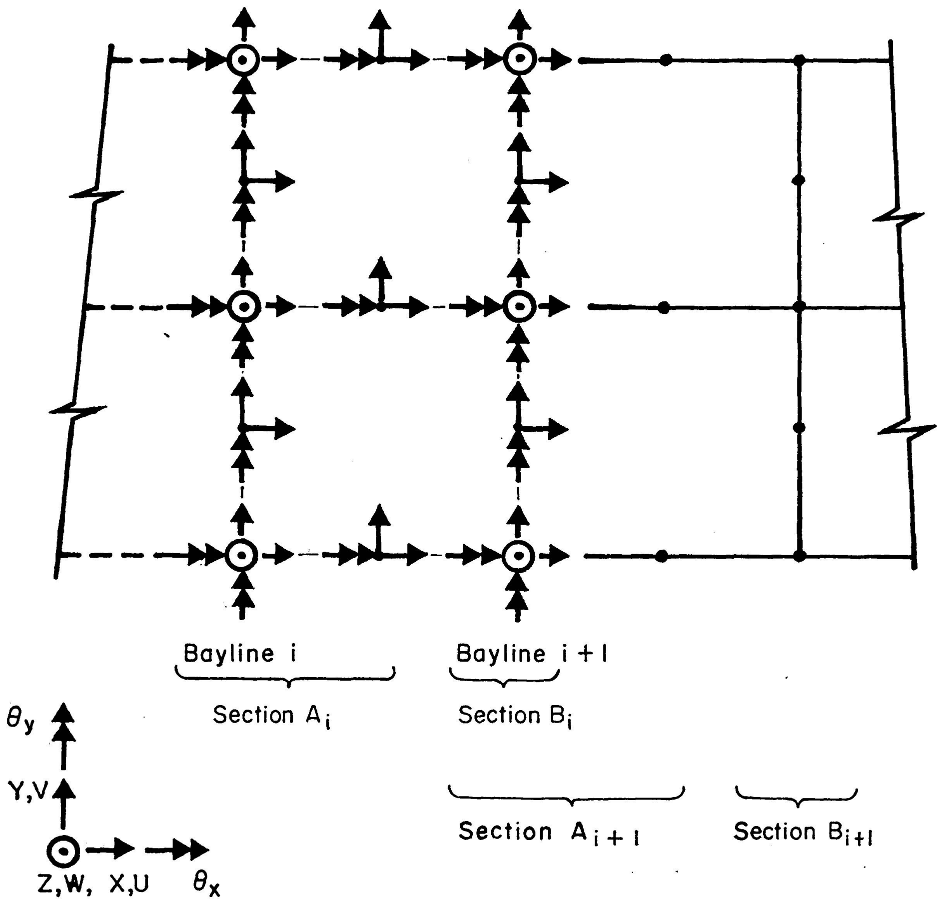

Fig. 1. A typical substructure.

suggested by FRAEIJS de VEUBEKE [4,5], have been used by the first co-author [6] for the analysis of stiffened plates. That study required a finite element with 32 nodal displacements, as indicated in Fig. 1.

The elimination proceeds in the X-direction. Each network line in the Y-direction will be called a bayline. Those nodes lying on bayline i, and those mid-edge nodes in the panel immediately following are considered to be in section A of bayline i, as indicated in Fig. 1. The nodes lying on the next bayline are considered to be in section B of bayline i. This identification is typical for any bayline. The row of plate elements delimited by baylines i and $i+1$ constitutes a substructure. Additional subscript notation is required to identify several types of displacements, as follows:

W = lateral displacement, referring to all sections in the structure;
A = nonlateral $(U, V, \theta_x, \theta_y)$ displacement associated with section A of bayline i;
B = nonlateral $(U, V, \theta_x, \theta_y)$ displacement associated with section B of bayline i.

Lower-case subscripts a and b refer to the corresponding restrained displacements associated with bayline i. The lower-case subscript w refers to the restrained lateral displacements of both sections of bayline i. It must be emphasized that upper-case W refers to all free lateral displacements in the structure, not merely to a section. All displacements of the A, B, a, b, and w types are to be eliminated in the analysis. The w-type (restrained) displacements could be included in the a and b type displacements, but they are considered separately here for convenience.

At a typical bayline in the forward-elimination process the action-displacement relation, in partitioned form, can be written as follows:

$$\begin{bmatrix} K_{AA} & K_{AB} & K_{AW} & K_{Aa} & K_{Ab} & K_{Aw} \\ & K_{BB} & K_{BW} & K_{Ba} & K_{Bb} & K_{Bw} \\ & & K_{WW} & K_{Wa} & K_{Wb} & K_{Ww} \\ & & & K_{aa} & K_{ab} & K_{aw} \\ & \text{Symmetric} & & & K_{bb} & K_{bw} \\ & & & & & K_{ww} \end{bmatrix} \begin{bmatrix} D_A \\ D_B \\ D_W \\ D_a \\ D_b \\ D_w \end{bmatrix} = \begin{bmatrix} F_A \\ F_B \\ F_W \\ F_a \\ F_b \\ F_w \end{bmatrix}. \tag{1}$$

The subscripts of the stiffness submatrices identify their association with the six displacement types. In an effort to give an uncluttered appearance, matrix symbols are not used in Eq. (1), but each item represents a submatrix. Expanding Eq. (1) produces:

$$K_{AA}\, D_A + K_{AB}\, D_B + K_{AW}\, D_W + F_{AS} = F_A, \tag{2}$$

$$K^t_{AB}\, D_A + K_{BB}\, D_B + K_{BW}\, D_W + F_{BS} = F_B, \tag{3}$$

$$K^t_{AW}\, D_A + K^t_{BW}\, D_B + K_{WW}\, D_W + F_{WS} = F_W, \tag{4}$$

$$K^t_{Aa}\, D_A + K^t_{Ba}\, D_B + K^t_{Wa}\, D_W + F_{as} = F_a, \quad (5)$$
$$K^t_{Ab}\, D_A + K^t_{Bb}\, D_B + K^t_{Wb}\, D_W + F_{bs} = F_b, \quad (6)$$
$$K^t_{Aw}\, D_A + K^t_{Bw}\, D_B + K^t_{Ww}\, D_W + F_{ws} = F_w, \quad (7)$$

in which the superscript t denotes transposition.

Equivalent generalized forces, associated with the specified displacements, appear in Eqs. (2), (3), and (4). They carry the following definitions:

$$F_{AS} = K_{Aa}\, D_a + K_{Ab}\, D_b + K_{Aw}\, D_w, \quad (8)$$
$$F_{BS} = K_{Ba}\, D_a + K_{Bb}\, D_b + K_{Bw}\, D_w, \quad (9)$$
$$F_{WS} = K_{Wa}\, D_a + K_{Wb}\, D_b + K_{Ww}\, D_w. \quad (10)$$

F_{as}, F_{bs}, and F_{ws} in Eqs. (5), (6), and (7) are similarly defined. It is convenient to abbreviate the subsequent development by the substitutions:

$$\bar{F}_A = F_A - F_{AS}, \quad (11)$$
$$\bar{F}_B = F_B - F_{BS}, \quad (12)$$
$$\bar{F}_W = F_W - F_{WS}. \quad (13)$$

The six vectors to be determined in Eqs. (2) through (7) are the unrestrained displacements D_A, D_B, and D_W, and the generalized nodal restraint forces F_a, F_b, and F_w.

Solve for D_A in Eq. (2):

$$D_A = K^{-1}_{AA}\,(\bar{F}_A - K_{AB}\, D_B - K_{AW}\, D_W). \quad (14)$$

By substitution of Eq. (14) into Eqs. (3) and (4), D_A is eliminated. The resulting equations are:

$$K^*_{BB}\, D_B + K^*_{BW}\, D_W = \bar{F}^*_B, \quad (15)$$
$$K^{*t}_{BW}\, D_B + K^*_{WW}\, D_W = \bar{F}^*_W, \quad (16)$$

in which,

$$K^*_{BB} = K_{BB} - K^t_{AB}\, K^{-1}_{AA}\, K_{AB}, \quad (17)$$
$$K^*_{BW} = K_{BW} - K^t_{AB}\, K^{-1}_{AA}\, K_{AW}, \quad (18)$$
$$K^*_{WW} = K_{WW} - K^t_{AW}\, K^{-1}_{AA}\, K_{AW}, \quad (19)$$
$$\bar{F}^*_B = \bar{F}_B - K^t_{AB}\, K^{-1}_{AA}\, \bar{F}_A, \quad (20)$$
$$\bar{F}^*_W = \bar{F}_W - K^t_{AW}\, K^{-1}_{AA}\, \bar{F}_A \quad (21)$$

are the condensed stiffness and load matrices.

The displacements of type B for the current bayline become displacements of type A for the following bayline. In the process, matrix K^*_{BB} assumes the role of initializing matrix K_{AA} for the following bayline, where it is augmented by the contributions from that bayline. The initialization is accomplished simply by shifting the contents of K^*_{BB} into matrix K_{AA} at the end of an elimination step. A similar treatment applies to the load vectors.

The elimination (or condensation) procedure is performed for every bayline in the plate. At the last bayline section B is undefined, and section A contains only those nodes along this last bayline. Therefore, the vector of lateral displacements can be directly determined from

$$K^*_{WW} D_W = \bar{F}^*_W. \tag{22}$$

The nonlateral displacements can be calculated by backward substitution, beginning at the last bayline and working toward the first. Displacements D_A for the last bayline are found from Eq. (14) with $D_B = 0$. These displacements D_A become the D_B for the next bayline. This simple procedure is repeated for each bayline until all displacements have been determined.

During the backward-substitution phase, the generalized restraint forces F_a, F_b, and F_w are found directly from Eqs. (5), (6), and (7) as the displacement vectors D_A and D_B become available. If any loads are directly applied to the restraints, then their contribution must be included in the final reactions. Such direct contributions occur when the consistent load vector for a distributed or concentrated load on a finite element is calculated and assigned to the restrained nodes. The generalized forces so calculated are useful for checking overall equilibrium, although they are not of interest themselves because they represent fictitious forces. If desired, the stress resultants can be described in terms of these nodal forces.

The stresses, or stress resultants, are also calculated during the back-substitution phase. A detailed computer algorithm is available from the first co-author on request.

Vibrational Analysis

When a lumped-mass approach is considered adequate for a free-vibration study, d'Alembert's principle gives

$$M_W \ddot{D}_W + K^*_{WW} D_W = 0, \tag{23}$$

where M_W is a diagonal mass matrix. For harmonic motion with circular frequency p, Eq. (23) becomes the eigenvalue problem:

$$K^*_{WW} D_{W_n} = p^2 M_W D_{W_n}, \tag{24}$$

where D_{W_n} denotes the vector of peak amplitudes (mode shape).

The condensed stiffness matrix K^*_{WW} can be obtained by the forward-elimination process described above for static analysis. This array, together with the diagonal mass matrix, may then be used to determine the natural frequencies and the associated mode shapes for the plate.

Numerical Example

A computer program based on the substructures approach has been applied to the analysis of both stiffened and unstiffened plates. For purposes of illustration, the static and free-vibration analysis of a square, simply-supported plate subjected only to bending is presented here (see Fig. 2).

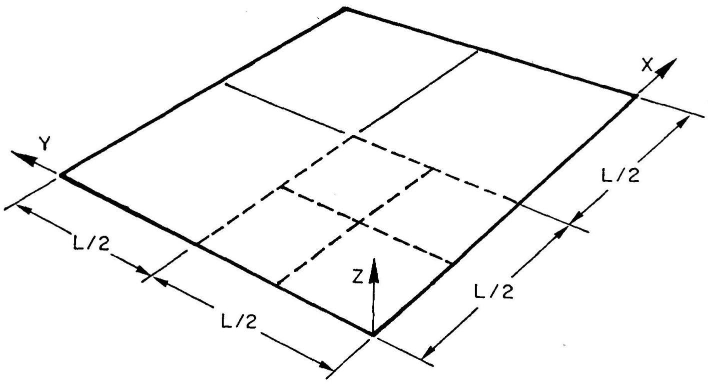

Fig. 2. Square, simply-supported plate ($N = 2$).

The conforming quadrilateral element with mid-edge nodes [5] is used to idealize the plate. Associated with each corner node are a translation and two rotations; associated with each mid-edge node is a normal slope. The letter N denotes the number of square elements along half the side of the plate. For example, $N = 2$ means that four elements are used to represent one-quarter of the plate, as shown in Fig. 2.

For a uniform load q, the maximum deflection is given in Table 1, in terms of a coefficient α, where

$$W_{max} = \frac{\alpha q L^4}{D}. \qquad (25)$$

The symbol D in Eq. (25) represents the flexural rigidity, and L is the side length of the plate.

Table 1. Maximum Deflection of a Square, Simply-Supported Plate under Uniform Load

N	Maximum Deflection Coefficient α	
	Consistent Load	Lumped Load
1	0.0040824	0.0027189
2	0.0040624	0.0036870
4	0.0040617	0.0039663
6	0.0040615	0.0040189
Exact [8]	0.004062	0.004062

Usually the bending moments are of greater interest to the structural engineer than are the deflections. The convergence characteristics for the bending moment M_{xx} along the centreline are given in Table 2, in terms of a coefficient β, where:

$$M_{xx} = \beta q L^2. \tag{26}$$

Table 2. Moment M_{xx} on Centreline of a Square, Simply-Supported Plate under Uniform Load

x/L	Moment Coefficient β			
	$N=1$	$N=2$	$N=4$	Exact [8]
0.0	0.01036	0.00400	0.00117	0.0
0.125			0.02591 0.02560	0.02488
0.250		0.04201 0.04019	0.03968 0.03940	0.03891
0.375			0.04637 0.04622	0.04582
0.500	0.05616	0.04933	0.04830	0.04789

Two values of β are given for the intermediate values of x/L to show the moment discontinuity to be expected at the node. The moment variation is linear between the nodes, and follows the exact curve [8] very closely.

From the static load analysis the condensed stiffness matrix K^*_{WW} is available. For $N=1$ and $N=2$ respectively:

$$K^*_{WW} = [2.1049], \tag{27}$$

$$M_W = [0.0625] \tag{28}$$

and

$$K^*_{WW} = \begin{bmatrix} 52.1821 & & & \text{Symmetric} \\ -19.1298 & 26.0910 & & \\ -19.1298 & 2.0763 & 26.0910 & \\ 2.0763 & -9.5649 & -9.5649 & 13.0455 \end{bmatrix}, \tag{29}$$

$$M_W = \begin{bmatrix} 0.0625 & 0 & 0 & 0 \\ 0 & 0.03125 & 0 & 0 \\ 0 & 0 & 0.03125 & 0 \\ 0 & 0 & 0 & 0.015625 \end{bmatrix}. \tag{30}$$

Assumed for this example are a plate length of 1.0, a flexural rigidity D of 0.091575, and a mass per unit area of 1.0. With the lumped-mass approach the lowest natural frequency is found to be 0.9236 cps for $N=1$, and 0.9501 cps for $N=2$. The exact solution, given by TIMOSHENKO and YOUNG [9] is 0.9506 cps.

If the complete stiffness matrix were used to find the natural frequencies, then the order of the eigenvalue problem to be solved would be 5 by 5, 20 by 20, and 80 by 80 for $N=1$, 2, and 4 respectively instead of the reduced orders of 1 by 1, 4 by 4, and 16 by 16. The reduction of order is even more dramatic for the case of eccentrically-stiffened plates where in-plane displacements in the x and y directions must be considered at every node. The original orders of the eigenvalue problem would then be 5 by 5, 52 by 52, and 200 by 200 for $N = 1$, 2, and 4 respectively. Thus, the eigenvalue problem to be solved is of much lower order when all but the lateral degrees of freedom have been eliminated. Because the elimination is done in a row-by-row manner, large numbers of nodal displacements do not create a great problem.

It should be noted that only the symmetric modes of vibration can be obtained when one-quarter of the plate is used, and a number of elements sufficient to represent all desired mode shapes must be specified.

Conclusions

A substructures approach to the static and dynamic analysis of plate systems has been described. The order of the set of equations to be solved at any stage is relatively low; so a large core capacity in a digital computer is not required. Specified support displacements are easily considered in the general treatment presented. A great advantage of this substructures technique is that a lumped-mass, free-vibration analysis, involving only the lateral degrees of freedom in the plate, may be performed with little extra effort once a static analysis has generated the reduced stiffness matrix. This advantage would be sacrificed if all of the degrees of freedom in each substructure were eliminated. It should be noted that no information need be transferred to auxiliary storage if the analyst is concerned only with natural frequencies and mode shapes. The appropriate segments of the computer program can simply be by-passed. Results for an example of a simply-supported square plate compare closely with known exact solutions [8].

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Summary

A substructures approach to the static and dynamic matrix analysis of rectangular plate systems is described. For each substructure the nodal displacement vector is partitioned to consider both unconstrained and constrained displacements in the plane of the plate and perpendicular to it. The order of the set of simultaneous equations to be solved for any substructure is relatively low; so a large capacity computer core is not required. Furthermore, by retention of only the lateral degrees of freedom, a lumped-mass, free-vibration analysis may be performed with little effort once a static analysis has generated the reduced stiffness matrix. The static and free-vibration behaviour of a square, simply-supported plate is presented to illustrate the technique.

Résumé

Les auteurs décrivent une méthode des substructures pour le calcul matriciel, statique et dynamique, des systèmes de plaques rectangulaires. Pour chaque substructure, on sépare le vecteur des déplacements en déplacements forcés et déplacements libres, dans le plan de la plaque et dans la direction perpendiculaire. Le nombre des équations simultanées à résoudre pour chaque substructure étant relativement limité, on peut donc renoncer à l'usage d'un ordinateur de grande capacité. De plus, en ne retenant que les déplacements perpendiculaires à la plaque, on peut effectuer aisément une analyse des vibrations libres, une fois que l'on dispose de la matrice de rigidité réduite. Pour illustrer cette technique, on décrit le comportement statique et les vibrations libres d'une plaque carrée simplement supportée.

Zusammenfassung

Die Berechnung der statischen und dynamischen Matrizen für rechteckige Plattensysteme mittels Substrukturen wird beschrieben. In jeder Substruktur ist der Vektor, der die Knotenverschiebungen darstellt, in zwei Komponenten unterteilt, die sowohl die erzwungenen als auch die freien Verschiebungen in der Plattenebene sowie die senkrecht dazu darstellen. Die Anzahl der für jede Substruktur zu lösenden simultanen Gleichungen ist verhältnismäßig gering, so daß ein relativ kleiner Digitalrechner benutzt werden kann. Da nur die Freiheitsgrade normal zur Plattenebene verwendet werden, ist eine einfache Analyse der freien Schwingungen mittels konzentrierter Punktmassen möglich, die eine durch statische Analyse berechnete reduzierte Steifheitsmatrix benützt. Als Beispiel sind das statische Verhalten und die freien Schwingungen einer quadratischen, frei unterstützten Platte dargestellt.