Formal and real structural safety: influence of gross errors

Formal and Real Structural Safety. Influence of Gross Errors

Sécurité formelle et réelle des structures. Influence de fautes graves

Formale und reale Sicherheit von Tragwerken. Einfluss von groben Fehlern

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SUMMARY

Gross error occurrence cause gaps between calculated and real structural failure rates. Considerations about gross errors have, however, small influence on material consumption for given structural lay-outs. Therefore formal probabilistic reliability theory plays a meaningful role for the choice of dimensions. Concerning different lay-outs it is essential to evaluate the proneness to gross errors. Procedures for this depend on possibilities of defining some measure of proneness to failure due to gross errors. Fuzzy set theory is investigated in this view.

RÉSUMÉ

Le fait de fautes graves est la cause des différences entre la fréquence observée des défaillances de structures et la probabilité formelle de ruine. Des considérations relatives à des fautes graves n'ont cependant que peu d'importance sur la quantité de matériaux nécessaire pour une conception structurale donnée. C'est pourquoi la théorie probabiliste de la sécurité joue un rôle important dans le dimensionnement des structures. Par contre, en comparant différentes conceptions structurales, il est essentiel d'évaluer leur sensibilité aux fautes graves. Des procédures relatives à cette comparaison nécessitent la définition d'une mesure pour la sensibilité aux fautes graves. La théorie des «fuzzy sets» est étudiée dans ce sens, dans la présente contribution.

ZUSAMMENFASSUNG

Der Unterschied zwischen rechnerischer und statistisch beobachtbarer Versagenswahrscheinlichkeit von Tragwerken ist auf grobe Fehler zurückzuführen. Für ein gegebenes Tragwerk-Konzept hat jedoch die Berücksichtigung grober Fehler nur einen kleinen Einfluss auf den Materialbedarf, weshalb auch die formelle Wahrscheinlichkeitstheorie bei der Bemessung von Tragwerken durchaus ihre Berechtigung hat. Beim Vergleich verschiedener Tragwerk-Konzepte spielt jedoch die jeweilige Empfindlichkeit gegenüber groben Fehlern eine wesentliche Rolle. Voraussetzung für den Einbezug dieser Tatsache ist eine geeignete Definition eines Masses für diese Empfindlichkeit. Die „Fuzzy-Set"-Theorie wird in diesem Zusammenhang untersucht.

1. INTRODUCTION

In his contribution to the »safety concepts» session at the 1980 IABSE Congress in Vienna, [17], the author has argued that the concept of theoretical (or operational) failure probability is an indispensable tool to provide rational choices of structural dimensions. The author holds the opinion that the contributions to the observed failure rate of real structures are dominated by failures caused by gross errors that are not and should not be accounted for by the theoretical failure probability. In fact, the concluding argument of [17] supports the principle that the design value of the theoretical failure probability for a given structural lay-out should be fixed at the value which minimizes the total expected costs (where »costs» may be taken in a more general sense than just direct monetary costs), that is, the expected value of the establishing costs plus the operation and maintenance costs plus the costs of damage or failure. The question raises whether such an optimization is reasonable in consideration of the gap between real and theoretical failure rate. Fortunately the answer is confirmative in most cases. To see this let p_{th} be the theoretical probability of failure and let p_{gr} be some measure of proneness to failure due to gross errors.

The point is that for a given lay-out of the structure (including the entire plan for the building process) the proneness to failure p_{gr} is in most cases almost unaffected by variations of p_{th}, these variations only causing variation of the material consumption. Therefore, the value of p_{th} which minimizes the expected cost of the given lay-out is almost unaffected by the expected cost of failure due to gross errors (provided the costs associated with a failure are only slightly dependent on variations of the material consumption). Thus it is rational for each lay-out to choose as design value that value of p_{th} which minimizes the expected cost of the given lay-out. The validity of this argumentation seems to be the only salvation of probabilistic reliability theory from being just a plaything for university teachers. However, when the question is about choosing between different lay-outs the expected cost of failure due to gross errors must be added, that is, a cost which depends on p_{gr} must be added. While the widely accepted modern decision theory (von Neumann, Morgenstern, [13]) defines the failure cost as a function of p_{th} simply as the expected cost with respect to the given probabilistic model there is as yet no generally accepted definition of p_{gr}. Even with such a definition available it is by no means obvious how to define the expected failure cost as a function of p_{gr} except, perhaps, that it should be an increasing function of p_{gr}.

To the author's knowledge the terminology »proneness to failure» together with an attempt to define a numerical measure of it was first suggested by Pugsley, [11]. Blockley, [3], [4], has published a useful checklist for grading the quality of a project with respect to proneness to gross errors in all its stages from design to use. He applied this checklist in a grading of 23 major projects that all turned into disasters. The problem is to cook down all these gradings to a single appreciable measure of proneness to failure. He applies the fuzzy set concept which was introduced by Zadeh in 1965, [15], with the purpose of giving a precise mathematical interpretation of imprecise linguistic statements and a modelling of relations between such statements. However, in the light of the above discussion, Blockley's attempt in [4] to »fuzzify» the theoretical failure probability p_{th} seems inappropriate.

The following paragraphs of this paper analyses the fuzzy set tool in the field of interest here. What are fuzzy sets about and how should results of operations on fuzzy sets be interpreted? The possibility of an interpretation in usual probabilistic terms is discussed. This discussion reveals several open questions about basing a measure of proneness to failure due to gross errors on the established fuzzy set algebra. For the time being the author tends to give support to the view that purely pragmatic reasons justify the fuzzy set tool.

Blockley gives no justification at all. In [3] he applies one definition and in [4] another definition of fuzzy set compositions without giving any reasons for these choices. Other writers (Brown, [5], Yao, [14]) follow the line of Blockley without attempts to establish satisfying justifications. This is not surprising because a justification which satisfies an engineer of classical training may be pretty hard to give. Fact is, however, that the fuzzy set tool has been quite successful in practical engineering applications in connection with control and regulation decision procedures where the purpose is to steer some process in the right but, perhaps, imprecisely given direction, [7], [10].

The mathematical literature about fuzzy sets has grown very large since the start by Zadeh fifteen years ago. Only some very few early references will be given in the following for the purpose of documentation. Those basic concepts of fuzzy set theory (or fuzzy logics) which is needed for the discussion herein will be given from scratch in order to make the text self-contained.

2. GROSS ERROR EVALUATION IN TERMS OF FUZZY SETS

It is, perhaps, most illustrative to explain the fuzzy set concepts in terms of a relevant example. Blockley's grading scale and checklist of statements are well suited for our purpose, [4]. The 25 statements of the list declare ideal circumstances for occurrence of no gross errors. Some few examples from the list are:

2(b) The quantity and quality of research and development available to the designer is sufficient.

4(c) The form of structure has been well tried and tested by its use in previous structures.

6(d) The contractor is adequately experienced in this type of work.

7(a) The contractual arrangements are perfectly normal.

The grading consists of choosing a pair of characters for each statement of the list. The first member of the pair expresses the degree of confidence in the truth of the statement. The set of grades is:

1. very high confidence
2. high confidence
3. medium confidence
4. low confidence
5. very low confidence

The second member of the pair expresses the importance with respect to failure given occurrence of gross errors related to the statement. The set of grades is as above except that the word »confidence» is changed to the word »importance».

Clearly such a grading is subjective. The fact that people with insight in the subject of the statements are able to do the grading just as teachers are able to grade their pupils indicates that the grading represents some relevant information even though it is imprecise in nature. It seems to be the imprecision itself that makes the grading possible. The human brain obviously has a remarkable ability of perceiving an overall situation and to express this perception in imprecise linguistic terms. This point of view is so clearly explained by Zadeh in [16] that the author can do no better than to quote:

> »An alternative approach outlined in this paper is based on the premise that the key elements in human thinking are not numbers, but labels of fuzzy sets, that is, classes of objects in which the transition from membership to non-membership is gradual rather than abrupt. Indeed, the pervasiveness of fuzziness in human thought processes suggest that much of the logic behind human reasoning is not the traditional two-valued or even multivalued logic, but a logic with fuzzy truths, fuzzy connectives, and fuzzy rules of inference. In our view, it is this fuzzy, and as yet not well-understood, logic that plays a basic role in what may well be one of the most important facets of human thinking, namely, the ability to *summarize* information - to extract from the collections of masses of data impinging upon the human brain those and only those subcollections which are relevant to the performance of the task at hand.
>
> By its nature, a summary is an approximation to what it summarizes. For many purposes, a very approximate characterization of a collection of data is sufficient because most of the basis tasks performed by humans do not require a high degree of precision in their execution. The human brain takes advantage of this tolerance for imprecision by encoding the »task-relevant» (or »decision-relevant») information into labels of fuzzy sets which bear an approximate relation to the primary data. In this way, the stream of information reaching the brain via the visual, auditory, tactile, and other senses is eventuelly reduced to the trickle that is needed to perform a specified task with a minimal degree of precision. Thus, the ability to manipulate fuzzy sets and the consequent summarizing capability constitute one of the most important assets of the human mind as well as a fundamental characteristic that distinguishes human intelligence from the type of machine intelligence that is embodied in present-day digital computers.

In order to establish an arithmetic by which the information content of the set of pairs of grades may be summarized to give a single measure of proneness to failure due to gross errors we first introduce numerical scales of both confidence and importance. Let these scales both be the interval $\Omega = [0, 1]$. The classical way of modelling is then to define a one to one correspondence between the set of 5 grades and a subset of five numbers in Ω, e.g. the set $\{0.1, 0.3, 0.5, 0.7, 0.9\}$. The point is, however, that the linguistic gradings are imprecise in nature and even that this imprecision is an essential part of the information perceived by the human brain and expressed by it through the gradings. This essential imprecision is totally lost when applying such a one to one correspondence. An intuitively appealing alternative is to use the entire set Ω together with a so-called membership function $\mu_A : \Omega \curvearrowright [0, 1]$. The value $\mu_A(x)$ is called the degree of membership of $x \in \Omega$ in the fuzzy set A. Here A is a symbol for those imprecisely defined numerical measures that correspond to a given linguistic grading. For convenience we may identify A and the corresponding linguistic grading. As an example let A be »low confidence». We then may model the fuzzy set by a reasonable choice of a membership function μ_A that at least does not conflict our perception of the linguistic grading. Needless to say that it is hopeless to try to look for canonical principles that lead to a unique choice of such a membership function. Subjectively perceived pragmatism must be the guide for the choice.

For a moment return to the discretized scale 0.1, 0.3, 0.5, 0.7, 0.9 of degrees of confidence. These numbers may be interpreted as truth degrees for fuzzy logical statements. In classical mathematical logics a statement like 2(b) is either false or true with corresponding truth degrees 0 or 1. In human perception of imprecise statements this classical concept is obviously much too rigid. To soften the truth interpretation the idea of having truth degrees between 0 and 1 is quite natural. But while the rules of combining statements in classical logics have a canonical basis difficulties show up for the formulation of a fuzzy logic of general acceptability. We return to this discussion below. For now let us accept the idea of working with degrees of truth. As mentioned we loose the inherent imprecision of our perception, if we just assign one of the truth degrees, 0.1, 0.3, 0.5, 0.7, 0.9 to the statement 2(b), say. Assume that we judge our confidence in the truth of 2(b) to be »low». Then we might agree that the truth value is about 0.3, but not that it is precisely 0.3. So, instead we choose to represent the grade »low confidence» by a fuzzy set A defined by a membership function as, for example,

$$\mu_A(x) = \exp[-\frac{1}{2}(\frac{x-0.3}{\sigma})^2] \quad , \quad x \in [0, 1] \tag{2.1}$$

where σ is a positive constant. The larger σ is the larger is the fuzziness. Note that the extension of this function from $x \in [0, 1]$ to all real x is similar to the well-known bell-shaped normal density (Gauss curve) of mean 0.3 and standard deviation σ. Geometrically the parameter σ is the distance from the mean to the two points of inflection of the curve.

In the following let us agree to use the membership functions $\mu_{vlc}(x)$, $\mu_{lc}(x)$, $\mu_{mc}(x)$,

$\mu_{hc}(x)$, $\mu_{vhc}(x)$ corresponding to »very low»-, »low»-, »medium»-, »high»-, and »very high confidence», respectively, where these are defined as functions of the type

$$\mu(x, y; c) = \exp[-(\frac{x-y}{cy(1-y)})^2] \quad , \quad x \in [0, 1] \tag{2.2}$$

with y = 0.1, 0.3, 0.5, 0.7, 0.9, respectively. Note that the parameter σ, see (2.1), varies with y according to the formula $\sigma = cy(1-y)/\sqrt{2}$ where c is a suitable constant. Corresponding to the selected values of y we have σ/c = 0.064, 0.148, 0.177, 0.177, 0.148, 0.064. This variation reflects larger fuzziness in the medium range than at the extremes: »very low» and »very high». For c = 1 the functions are shown in figure 1.

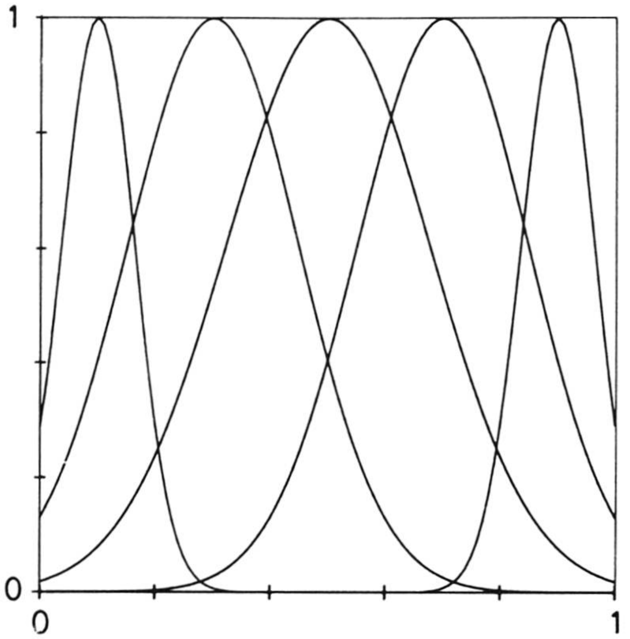

Figure 1. Suggestions of membership functions defining fuzzy sets corresponding to the five gradings of confidence in truth of a given statement.

By the way, note that for example the fuzzy set of »very low confidence» is not a fuzzy subset of the fuzzy set of »low confidence» according to the definition, [15],

$$A \subset B \Leftrightarrow \mu_A(x) \leqslant \mu_B(x) \tag{2.3}$$

This is not in conflict with intuition since the interpretation of »low confidence in truth of» is not the same as the interpretation of »low truth degrees». If A = »very low truth degrees» and B = »low truth degrees» then our perception of A and B would dictate that $A \subset B$ in the sense of (2.3). In fact, Zadeh makes automatic the operation of applying a hedge such as *very* on a primary term such as *low* simply by squaring the membership function of low to get the membership function of *very low*. That there are possibilities of *wrong* interpretation of the linguistic statements such as it is demonstrated here emphasizes the necessity of making clear the meaning of the statements before automatic operations can be executed.

For any given degree $x \in \Omega = [0, 1]$ of confidence we may next assign a fuzzy set of proneness p to failure taken as a fuzzy subset of the interval [0, 1], that is, a *conditional* fuzzy set of p given x. Corresponding to medium importance we may, for example, define the conditional membership function $\mu_{mi}(p|x)$ by

$$\mu_{mi}(p|x) = \exp[-(\frac{p-(1-x)}{kx(1-x)})^2] \quad , \quad p \in [0, 1] \tag{2.4}$$

which for x = 0 (zero confidence) and x = 1 (full confidence) degenerates to the crisp set membership functions

$$\mu_{mi}(p|0) = \begin{cases} 1 & \text{for } p = 1 \\ 0 & \text{otherwise} \end{cases} \quad , \quad \mu_{mi}(p|1) = \begin{cases} 1 & \text{for } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

Inbetween x = 0 and x = 1 the membership function is the Gauss bell corresponding to a mean of $1 - x$ and a standard deviation of $kx(1-x)/\sqrt{2}$ where k is a suitable constant. Figure 4 shows (2.4) for k = 1.

Standard fuzzy set algebra has been constructed such that if x is a member of a fuzzy set, e.g. with membership function $\mu_{lc}(x) = \mu(x, 0.3; c)$ (low confidence), then the fuzzy set of x together with the conditional fuzzy set of p given x induce a fuzzy set of p. Let S be a statement with pair of grades (»low confidence», »medium importance»). Then the membership function $\mu_S(p)$ of the corresponding fuzzy set of proneness measures becomes, [2],

$$\mu_S(p) = \max_x \min\{\mu_{lc}(x), \mu_{mi}(p|x)\} \tag{2.5}$$

We return to a discussion of the origin of the composition rules applied in (2.5).

Conditional membership functions of p given x corresponding to »very high», »high», »low», »very low» importance may be defined in terms of $\mu_{mi}(p|x)$ simply as

$$\mu_{mi}(p|1-(1-x)^q) \tag{2.6}$$

for q = 1/4, 1/2, 2, 4, respectively. These functions are all illustrated in figures 2 to 6 for k = 1. They reflect in a fuzzy way the intuitive meaning og the linguistic variables »very high», »high», »low», »very low».

For each of the statements of the checklist of n statements we may in this way via the gradings construct n fuzzy proneness sets represented by their membership functions $\mu_{S_1}(p), \mu_{S_2}(p), \ldots, \mu_{S_n}(p)$.

In this construction we have a possibility of changing the fuzziness parameters c and k from statement to statement in order to reflect our perception of doubts about how precisely we can judge the single statement.

The next question raises how to combine the n fuzzy proneness sets to a single fuzzy set expressing the total proneness to failure due to gross errors. Fuzzy set algebra suggests to join the information of the n fuzzy sets in the fuzzy *Cartesian product* with membership function

$$\mu_{S_1 \times S_2 \times \ldots \times S_n}(p_1, p_2, \ldots, p_n) = \min\{\mu_{S_1}(p_1), \mu_{S_2}(p_2), \ldots, \mu_{S_n}(p_n)\} \quad (2.7)$$

By this step the information about total proneness is still not readily interpretable. To each n-set $(p_1, p_2, \ldots, p_n)$ of proneness measures we should, in fact, assign a single total proneness p by some function f:

$$p = f(p_1, p_2, \ldots, p_n) \quad (2.8)$$

Then a fuzzy set of p is induced by this function. Its membership function is

$$\mu(p) = \max_{\bar{p} \in f^{-1}(\{p\})} \mu_{S_1 \times S_2 \times \ldots \times S_n}(p_1, p_2, \ldots, p_n) \quad (2.9)$$

where $\bar{p} = (p_1, p_2, \ldots, p_n)$ and $f^{-1}(\{p\})$ is the (crisp) set of points in $[0, 1]^n = [0, 1] \times \ldots \times [0, 1]$ which by f maps into the set $\{p\}$.

Even the function f may be selected on basis of rules from standard fuzzy set (or logic) algebra. If we take $1 - p$ as the truth value of the statement »the structure is safe with respect to failure caused by gross errors of any source» and $1 - p_i$ as the truth value of the statement »the structure is safe with respect to failure caused by gross errors of the source corresponding to statement i» then, obviously, the conjunction of the statement corresponding to $i = 1, \ldots, n$ is the total safety statement of truth value $1 - p$. Standard fuzzy logic then defines $1 - p = \min\{1-p_1, 1-p_2, \ldots, 1-p_n\}$, or

$$p = \max\{p_1, p_2, \ldots, p_n\} \quad (2.10)$$

which defines the function f in (2.8).

The last step in the proneness to failure analysis is to translate the fuzzy set defined by (2.9) into a linguistic statement taken from a finite set of standardized statements. A natural procedure is to determine the area and the centre of the area below the curve defined by the membership function (2.9). The translation of the fuzzy set into a linguistic statement as for example: »the structural lay-out has above medium proneness to failure due to gross errors with low degree of fuzziness», may then be done by use of the lists next to figure 9.

Blockley's gradings of the 23 fatal structural lay-outs have in common that there is at least one statement for each structural lay-out that has the worst possible combination of gradings, i.e. very low confidence together with very high importance. This pair of gradings will always dominate all other pairs and place the structural lay-out

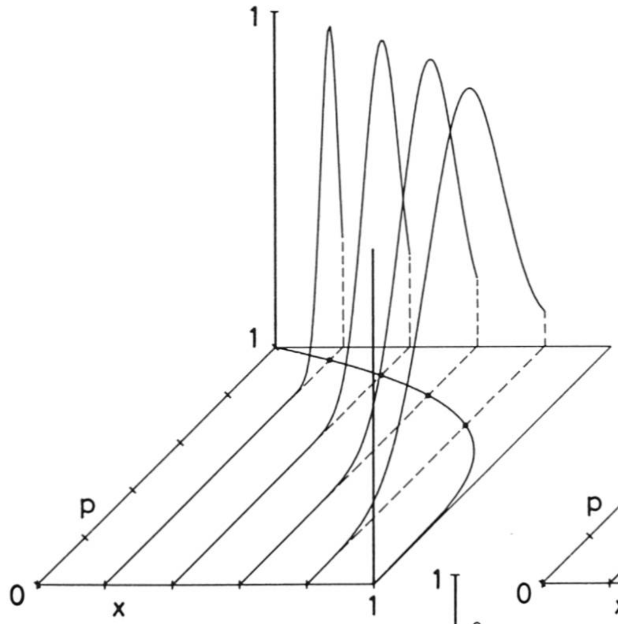

Figure 2. $\mu_{vhi}(p|x)$
(very high importance)

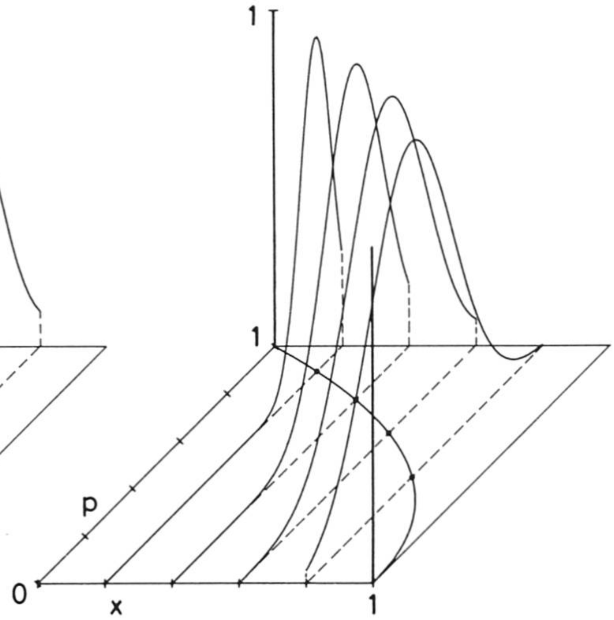

Figure 3. $\mu_{hi}(p|x)$
(high importance)

Figures 2 to 6 show suggestions of conditional membership functions of the proneness p to failure due to gross errors when given the confidence x in the truth of a given statement that declares ideal circumstances.

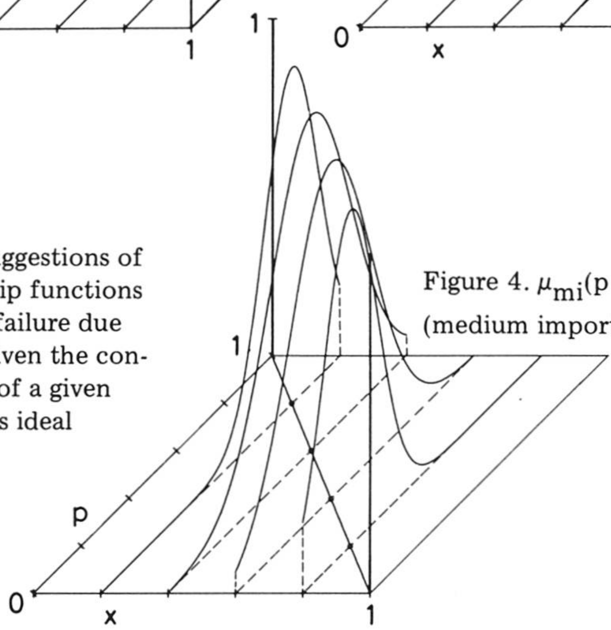

Figure 4. $\mu_{mi}(p|x)$
(medium importance)

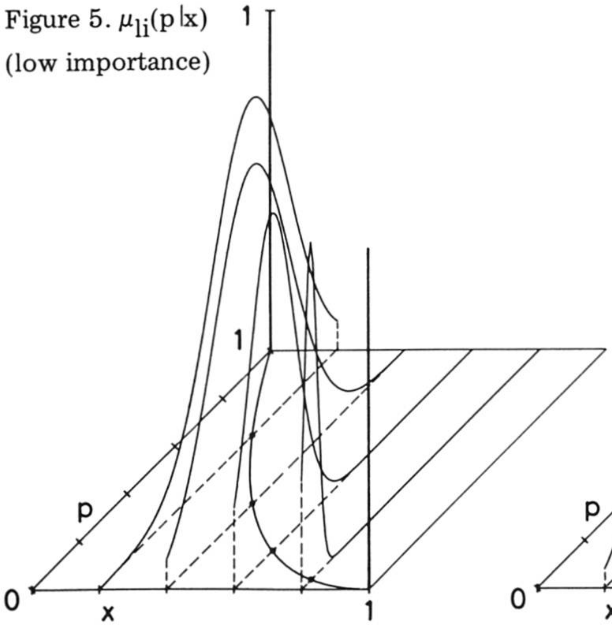

Figure 5. $\mu_{li}(p|x)$
(low importance)

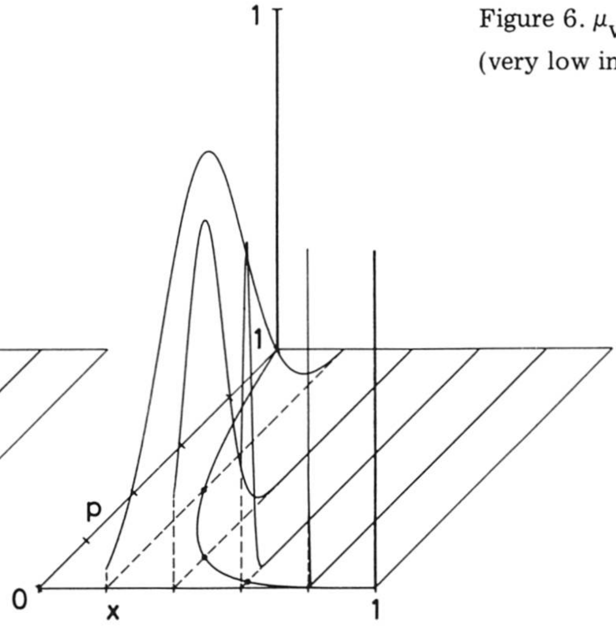

Figure 6. $\mu_{vli}(p|x)$
(very low importance)

in the class with extremely high proneness to failure. This is quite reasonable when noting the fact that all structures failed and that the gradings were all worked out post failure. A closer look reveals that the fuzzy set model given above generally has a tendency to pick out one or some few pairs of gradings to dominate all other of the given pairs of gradings.

The practical calculations naturally become overwhelming if (2.9) is used directly for a large n. However, the calculation can be done recursively by first taking S_1 and S_2 together and calculating $\mu(p)$ from (2.9) for n = 2 with $p = \max\{p_1, p_2\}$. Using the calculated $\mu(p)$ as proneness to failure membership function for the statement combined of S_1 and S_2 this may next be taken together with S_3 and a new $\mu(p)$ may be calculated for the statement combined of S_1, S_2, and S_3. In this way we may proceed recursively until all n statements are included. The proof that this recursive algorithm leads to the result defined by (2.9) follows from the complete distributive law of the lattice of real numbers of the interval [0, 1], see [8, p. 151]:

$$\min\{a, \max_{i \in I} b_i\} = \max_{i \in I} (\min\{a, b_i\}) \qquad (2.13)$$

Here I is any index set having such properties that $\max_{i \in I} b_i$ exists.

It follows that if $\mu_{S_1}(p) = \mu_{S_2}(p)$ for all p then $\mu(p)$ for the statement combined of S_1 and S_2 equals this common membership function. Thus all statements with identical pairs of gradings have no more effect on the proneness to failure than just one of the statements provided the fuzziness parameters c and k are the same from pair to pair. In other words, only the set of all different pairs of gradings need be taken into account. Among these the algorithm picks out one or some few to be determining for the membership function of the total proneness to failure.

Figure 7 shows $\mu(p)$ for Blockley's gradings of the Tay bridge. It is totally dominated by the worst possible pair of gradings. Figure 8 shows $\mu(p)$ for Blockley's single example of a structural lay-out which has not failed. It turns out to be dominated by the grading pair (medium confidence, high importance). Figure 9 shows a constructed example where a single pair of gradings does not dominate.

The fuzzy set analysis illustrated herein is entirely different from that used by Blockley in [4]. This is in itself not a criticism of Blockley's analysis but it emphasizes that the fuzzy set tool is a tool for modelling subjective perception of imprecise information. Blockley expresses simply a different perception of his gradings than the author's. The fuzzy set concept with its compositional rules as used herein is no more than a language of subjectivistic reasoning helping the reasoning person to keep due consistency in his process of condensation of different perceptions that he believes contain some relevant information about the problem at hand. The scientific minded person may very well claim that the whole procedure is so swayed by arbitrary definitions and choices that the result of it must be quite arbitrary. In traditional ob-

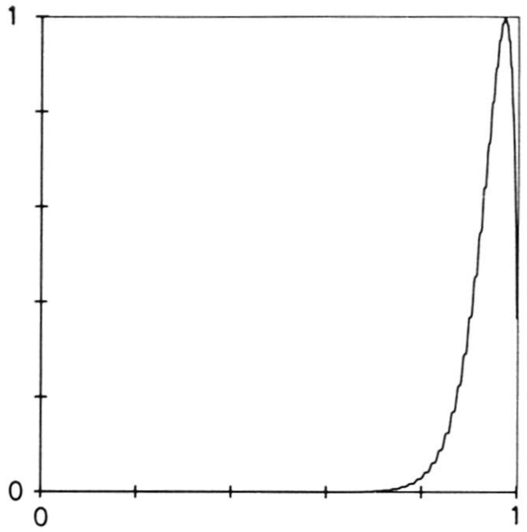

Figure 7. Blockley's grading of Tay Bridge, [4]. The structural lay-out has extremely high proneness to failure with very low degree of fuzziness. Dominating grading is (very low confidence, very high confidence).

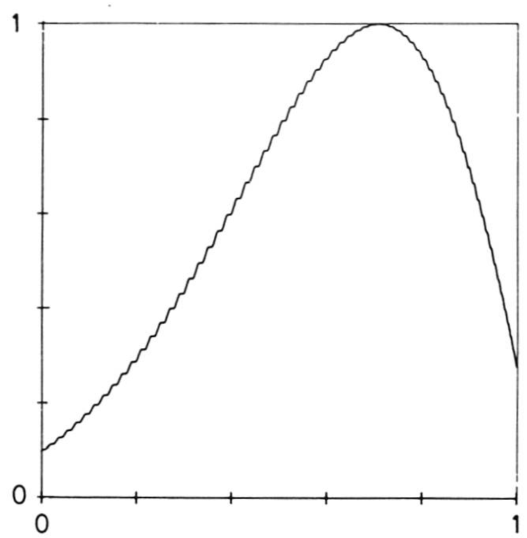

Figure 8. Blockley's non-failure example, [4]. The structural lay-out has above medium proneness to failure with high degree of fuzziness. Dominating grading is (medium confidence, high importance).

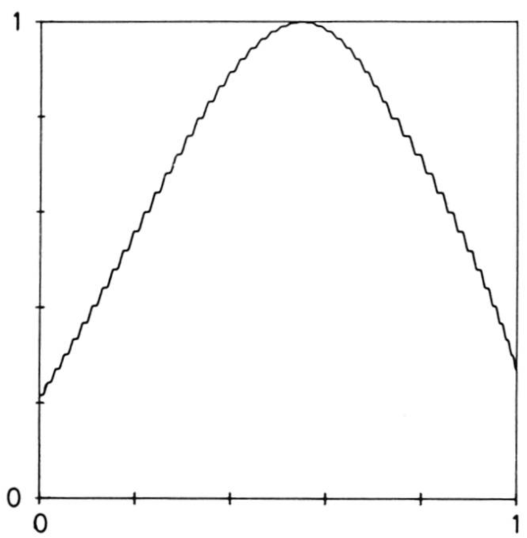

Figure 9. Constructed example. It has medium proneness to failure with high degree of fuzziness. The gradings are (see page 4): (2, 2), (3, 3), (4, 4). The dominating gradings are underlined.

Centre of area is closest to	*Proneness to failure due to gross errors is*
0.05	extremely low
0.1	very low
0.2	rather low
0.3	low
0.4	below medium
0.5	medium
0.6	above medium
0.7	high
0.8	rather high
0.9	very high
0.95	extremely high

Area is closest to	*Degree of fuzziness is*
0.1	very low
0.3	low
0.5	medium
0.7	high
0.9	very high

Figures 7 to 9 show membership functions of proneness to failure due to gross errors. The proneness measures are fuzzy subsets of the interval [0, 1].

jectivistic sense this is true, no doubt. The point is, however, that the sources of fuzzy information are non-objectivistic and non-reproducible in their very nature like the process of perception in the human brain. The result of such a perception process in the human brain is objectivistically arbitrary but it is not arbitrary to the percepting person. It is the brain's condensation of all the fuzzy information available to the brain and it is the basis on which the person acts.

Undoubtedly, it is so that the composition rules applied by the brain are not consistently the same at different times and for different types of information. In certain matters of small degree of fuzziness this causes evident confusion. To help the human brain being more rational the human brain itself has invented mathematical modelling with impressive success. Thinking of this it seems quite natural to take the step also to set up a model to secure rationality in fuzzy reasoning in order to reduce confusion and thus improve decision (even though confusion due to bad perception and condensation of fuzzy information from an intellectual point of view is much easier to excuse than in the case of precise information). However, a difficulty is that the rules of fuzzy logic cannot be established on the same kind of clear evidence as characterizes classical mathematical logic. We look into this in the next two sections.

3. THE FUZZY SET OPERATIONS

We will first look at the basis for the rule leading to (2.7). In fact, it is based on Zadeh's definition, [15],

$$\mu_{A \cap B}(x) = \min \{\mu_A(x), \mu_B(x)\} \tag{3.1}$$

for the membership function of the fuzzy intersection $A \cap B$ of two fuzzy sets A and B with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively. By comparison with the definition of the fuzzy containment relation $\subset$, see (2.3), it is easily proven that $A \cap B$ as defined by (3.1) is the *largest* fuzzy set which is contained in both A and B. In this respect the definition (3.1) is unique. In the same spirit the fuzzy union $A \cup B$ is defined as the *smallest* fuzzy set that contains both A and B. Uniquely the membership function becomes

$$\mu_{A \cup B}(x) = \max \{\mu_A(x), \mu_B(x)\} \tag{3.2}$$

It is interesting to note that there is a set of »canonical» conditions formulated by Bellman and Giertz, [1], which also uniquely lead to the definitions (3.1) and (3.2). The most fundamental condition of these is that

1. The membership values $\mu_{A \cap B}(x)$ and $\mu_{A \cup B}(x)$ depend solely on the membership values $\mu_A(x)$ and $\mu_B(x)$.

Once this condition is accepted (for doubts, see next section), it seems difficult not to accept the following conditions about the functions f and g which are postulated by condition 1 to give

$$\mu_{A\cap B}(x) = f(\mu_A(x), \mu_B(x)) \quad , \quad \mu_{A\cup B}(s) = g(\mu_A(x), \mu_B(x)) \tag{3.3}$$

These further conditions are

2. f and g are non-decreasing and continuous in both variables,
3. f and g are symmetric,
4. f(t, t) and g(t, t) are strictly increasing in t,
5. $f(s, t) \leqslant \min\{s, t\}$ and $g(s, t) \geqslant \max\{s, t\}$,
6. f(1, 1) = 1, g(0, 0) = 0,
7. equivalent set formulations such as $A \cup (B\cap C)$ and $(A\cup B) \cap (A\cup C)$ have equal membership values (3 is a special case of this).

Condition 5 may need a comment: Accepting x as a member of both A and B requires more than accepting x as a member of A alone. Thus $\mu_{A\cap B}(x) \leqslant \mu_A(x)$. By symmetry it follows that

$$\mu_{A\cap B}(x) \leqslant \min\{\mu_A(x), \mu_B(x)\} \tag{3.4}$$

Analogously, accepting x as a member of A and/or B requires less than accepting x as a member of A alone or B alone. Hence

$$\mu_{A\cup B}(x) \geqslant \max\{\mu_A(x), \mu_B(x)\} \tag{3.5}$$

While these conditions as proved by Bellman and Giertz, [1], uniquely give the definitions (3.1) and (3.2), it seems difficult to impose an analogously simple set of canonical conditions for unique determination of Zadeh's definition of the fuzzy complement

$$\mu_{\complement A}(x) = 1 - \mu_A(x) \tag{3.6}$$

except that it seems to be a very natural definition. It was used in the derivation of (2.10) (in its fuzzy logic version).

That (2.7) is a consequence of (3.1) follows by interpreting $S_1, \ldots, S_n$ as cylinder sets in $[0, 1]^n$ in which case the Cartesian product $S_1 \times S_2 \times \ldots \times S_n$ is the same as the intersection of all cylinder sets.

We are now ready to discuss the basis for the rule

$$\mu_B(y) = \max_x \min\{\mu_A(x), \mu_B(y|x)\} \tag{3.7}$$

which was, in fact, used in (2.5). Here $\mu_B(y|x)$ is a given membership function of a fuzzy subset B(x) of y space depending on x. We may for a fixed $x = x_0$ interpret this fuzzy set as a cylinder fuzzy set in the (x, y) space with membership function

$$\mu_{B(x_0)}(x, y) = \mu_B(y|x_0) \tag{3.8}$$

The point is now that x is not given but rather has a membership value $\mu_A(x)$ of a fuzzy set A. Let A_{x_0} be the fuzzy subset of A defined by that membership function which is zero everywhere except at $x = x_0$ where it takes the value $\mu_A(x_0)$. We may interpret A_{x_0} as a cylinder fuzzy set in the (x, y) space with membership function

$$\mu_{A_{x_0}}(x, y) = \mu_A(x)\, 1_{x=x_0}(x, y) \tag{3.9}$$

where $1_{x=x_0}$ is the characteristic function of the crisp set $\{(x, y)\,|\,x=x_0\}$.

The intersection of the fuzzy cylinder sets A_{x_0} and $B(x_0)$ gets the membership function

$$\mu_{A_{x_0}\cap B(x_0)}(x, y) = \min\{\mu_A(x)\, 1_{x=x_0}(x, y), \mu_B(y|x_0)\} \tag{3.10}$$

according to (3.1). By a direct generalization of (3.2) the union of all these fuzzy sets $A_{x_0}\cap B(x_0)$ over all x_0 has the membership function

$$\max_{x_0} \{\mu_{A_{x_0}\cap B(x_0)}(x, y)\} = \max_{x} \min\{\mu_A(x), \mu_B(y|x)\} \tag{3.11}$$

where we recognize the right hand side as the same as the right hand side of the rule (3.7). Since these membership values are independent of x the union is a cylinder set in (x, y) space. As such it only carries information about y. Thus it is quite natural to define the *marginal* fuzzy set B in terms of the given conditional fuzzy sets B(x) and the given marginal fuzzy set A by

$$B = \bigcup_x (A_x \cap B(x)) \tag{3.12}$$

which, as it is proven, has the membership function (3.7).

In order to explain the rule used in (2.9) let X be a space in which a fuzzy subset A is defined by the membership function $\mu_A(x)$ (x may be a vector, e.g.). Further, let $f : X \curvearrowright Y$ be a mapping of X onto Y where Y is some space, and let A_x signify the same type of fuzzy subset of A as above. By f the crisp set $\{\xi\} \subset X$ is mapped onto the crisp set $\{\eta\} \subset Y$, where $\eta = f(\xi)$. If we, by generalization, accept that the fuzzy set A_ξ by f is mapped onto the fuzzy set $f(A_\xi)$ defined by the membership function

$$\mu_{f(A_\xi)}(y) = \mu_A(\xi)\, 1_{y=\eta}(y) \tag{3.13}$$

the rule of *mapping induced fuzzy sets* as used in (2.9) follows from the generalization of (3.2) and the generalization of the elementary mapping rule for crisp sets »map of union» = »union of maps» to fuzzy sets. Since $A = \bigcup_x A_x$ we simply get

$$f(A) = f(\bigcup_{\eta} \bigcup_{x \in f^{-1}(\{\eta\})} A_x) = \bigcup_{\eta} \bigcup_{x \in f^{-1}(\{\eta\})} f(A_x) \quad (3.14)$$

which according to (3.13) and (3.2) has the membership function

$$\mu_{f(A)}(y) = \max_{\eta} \{ \max_{x \in f^{-1}(\{\eta\})} \{\mu_A(x) \, 1_{y=\eta}(y)\}\} = \max_{x \in f^{-1}(\{y\})} \mu_A(x) \quad (3.15)$$

This concludes the explanation of those fuzzy set rules which are used in the last section. Several more concepts and rules are in the collection of tools of fuzzy set theory (e.g. fuzzy relations and their compositions). In fact, by the acceptance of the composition rules (3.1) and (3.2) the algebra of fuzzy sets fits to a thoroughly developed branch of mathematical algebra called *lattice theory*, [8], which includes Boolean algebra. The existence of this theory has in itself made it attractive to choose the definitions (3.1) and (3.2) for fuzzy set intersection and union, respectively.

Before concluding this section it should be mentioned that omission of condition 7 leads to several other solutions of f and g. One of these is $f(x, y) = xy$ and $g(x, y) = x + y - xy$. In fact, Zadeh, [15], defines a composition of A and B called *algebraic product* (written as AB) by the membership function

$$\mu_{AB}(x) = \mu_A(x) \, \mu_B(x) \quad (3.16)$$

and a corresponding dual composition $A \oplus B = \complement(\complement A \complement B)$ given by the membership function

$$\mu_{A \oplus B}(x) = 1 - (1 - \mu_A(x))(1 - \mu_B(x)) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x) \quad (3.17)$$

Except that $AB \subset A \cap B$ and $A \oplus B \supset A \cup B$ there is no obvious interpretation of AB and $A \oplus B$ as extensions of usual set operations. The square A^2, for example, has no logical equivalence to A as required by condition 7.

Next section will throw some light on the composition rules (3.1), (3.2) and (3.16), (3.17) from a probabilistic modelling point of view.

4. PROBABILISTIC INTERPRETATION OF FUZZY SET CONCEPTS

In this section the goal is to discuss the fuzzy set composition rules in the light of a probabilistic interpretation of the membership function concept.

Consider the following example. Several competent engineers are asked to judge independently of each other the length of a crack in a given structure with respect to whether it reduces the safety of the structure or not. They are asked to answer solely yes or no to the question. The fraction of »yes» of the total number of answers may, perhaps, be considered as a guidance to what membership value to choose of the considered crack in the fuzzy set of dangerous cracks. On the other hand, the usual modelling of such a fraction is to consider it as a probability. A probabilistic model suited to deal with this kind of subjectivistic classification in two classes, »yes» or »no», is constructed as follows. Imagine that any one of the engineers who are confronted with the structural situation decides to hold the opinion that all cracks of length belonging to a certain set A of lengths (presumably A is an interval) are dangerous while the cracks corresponding to the complement of this set are harmless. The judging engineer, and thus his or her set, is imagined to be drawn from some population, that is, from a sample space of a probability space. The expected fraction of »yes» is the probability that the random set A covers the length value of the considered crack. In the following, let us interpret membership functions of fuzzy sets in this probabilistic way and let us use the notation $\mu_A(x) = P(x \in A)$.

Assume next that the engineers are asked also to judge by yes or no whether it pays to repair the crack (the alternatives being either to do nothing or to replace the part of the structure which contains the crack). To this question corresponds in a similar way as above a random set B of crack lengths and we write $\mu_B(x) = P(x \in B)$. The intersection membership function then becomes the probability

$$\mu_{A \cap B}(x) = P(x \in A \wedge x \in B) \leqslant \min\{P(x \in A), P(x \in B)\} = \min\{\mu_A(x), \mu_B(x)\} \quad (4.1)$$

Thus the definition (3.1) is simply an upper bound on the probability that both A and B cover x. Correspondingly we have

$$\mu_{A \cup B}(x) = P(x \in A \vee x \in B) \geqslant \max\{P(x \in A), P(x \in B)\} = \max\{\mu_A(x), \mu_B(x)\} \quad (4.2)$$

It is not possible, however, to construct a probability function P which makes strict equality valid for any pair of random sets. For example, we have trivially that $P(x \in A \wedge x \in \complement A) = 0$ while $P(x \in A) + P(x \in \complement A) = 1$. This example also shows that it is not possible to construct a nontrivial probability function P that obeys condition 1 of the seven conditions in the last section.

On the other hand, if for some particular random sets A and B the equation

$$P(x \in A \cap B) = \min\{\mu_A(x), \mu_B(x)\} \quad (4.3)$$

is valid then also the equation

$$P(x \in A \cup B) = \max\{\mu_A(x), \mu_B(x)\} \quad (4.4)$$

is valid and vice versa, as it is easily proven. Trivially, (4.4) is true for A = B. While random events $x \in A$ and $x \in B$ that satisfy (4.4) may be characterized as being *strongly dependent* the composition rules (3.17) and (3.18) correspond to *independence* between the events $x \in A$ and $x \in B$:

$$P(x \in A \cap B) = P(x \in A)\, P(x \in B) = \mu_{AB}(x) \qquad (4.5)$$

$$P(x \in A \cup B) = P(x \in A) + P(x \in B) - P(x \in A \cap B) = \mu_{A \oplus B}(x) \qquad (4.6)$$

Composition rules of fuzzy sets which have been defined probabilistically as in this section are, naturally, given completely from the underlying probability space. Except for trivial cases, however, it seems extremely difficult to choose such a probability space that, first, fits subjective judgements and, second, admits practicable compositions. At least one can claim that such a probabilistic fuzzy set model will be swayed with arbitrariness, perhaps more than that of Zadeh, even though it admits a relative frequency interpretation.

The set of the seven canonical conditions of last section is a shortcut through all these problems. The resulting composition rules (3.1) and (3.2) make up a much simpler algebra than that of probability theory. Since the rules are inconsistent with a probability space formulation we must in general abandon probabilistic interpretations of membership functions resulting from use of these rules. Their usefulness can solely be judged by experiencing in practice that we are doing better with them than without. Next section touches the problem of associating real world behaviour with fuzzy set membership functions.

5. PRONENESS TO FAILURE AND REAL WORLD BEHAVIOUR. THE DECISION PROBLEM

A touchstone for the practical value of the fuzzy set grading of a structural lay-out with respect to failure due to gross errors as suggested in section 2 would be to make a comparison with observed failure rates in the real world. This may, at least in principle, be done in the following way. The lay-outs for a large number of realized structures are graded by a panel of competent engineers. For each of the proneness classes the failure frequency is calculated among all the structural lay-outs put into the class by the panel. In a given class let n be the total number of structural lay-outs and let r be the number of these that have failed due to gross errors. The corresponding gross error failure probability p may be estimated in terms of the posterior distribution of p given that the prior distribution is uniform on the interval [0, 1], say. The expected value of p then becomes $(r + 1)/(n + 2)$. If such an investigation shows that these expected failure probabilities increase with increasing proneness to failure the fuzzy set model obviously behaves appropriately.

Assuming that this conclusion results from the investigation we may define a mapping from the set of proneness classes to the set of expected failure probabilities or, more generally, to the set of posterior distributions. From here on usual decision theory applies to decide between alternative structural lay-outs.

Obviously the investigation program outlined above is up against overwhelming practical difficulties because of the very small failure rates that fortunately are observed in practice. The author has doubts that it will be possible to get a real competent panel of engineers to do the job. To this add the difficulties of getting sufficient documentation about the single structural lay-outs. However, there might exist a reachable possibility of trying these ideas with respect to classification about proneness to fire. Possibilities may also exist with respect to investigations concerning proneness to serviceability damage since the rate of such damage is not very small in practice.

Anyhow, sequential proneness to failure judgements carried out systematically to guide sequential decisions during the whole process of design, bidding, construction, and use may turn out to improve the quality of professional civil engineering behaviour. A study of fuzzy system analysis, [2], within the integrated field of civil engineering management may turn out to be fertile.

6. CONCLUSIONS

It is a fact that there is a considerable gap between theoretical probability of failure and real failure rate for several important classes of realized structures. Nevertheless, it is claimed in the paper that formal probabilistic reliability theory is perfectly meaningful as a decision tool for choosing structural dimensions whatever be its ability to predict real failure rates. The essential explanation of the gap is that most experienced failures are caused by gross errors. That probabilistic reliability theory is more than just desk entertainment follows from an argument that says that the existence of circumstances having a potential of producing gross errors has only secondary importance for the choice of dimensions. A measure of proneness to failure due to gross errors is, however, of great importance for the decision problem of choosing between several possible different lay-outs of the structure.

It is investigated whether it is possible to define a measure of proneness to failure due to gross errors by use of Zadeh's fuzzy set algebra. This is done together with a detailed discussion of the fuzzy set concept and its standard composition rules in the light of the applications of interest here. Attempts to make probabilistic interpretations of fuzzy sets illustrate clearly the differences between probabilistic modelling and fuzzy set modelling. The discussion shows that the two views cannot be consistently united. However, the last is by far the most simple to work with in this imprecise field of human perception.

It can be concluded that a subjective measure of proneness to failure due to gross errors can be formulated in the language of fuzzy sets and fuzzy logic. The proneness measure suggested herein is based on a grading procedure reported by Blockley but the fuzzy set algebra is applied quite differently here than done by Blockley and his followers, Brown and Yao.

In spite of its triviality it is pointed out how, in principle, to establish a connection between the measure of proneness to failure due to gross errors and the expected cost of failures of this type. Unfortunately this important link is doomed to be very weak for long time to come. This is due to the foretellable big practical problems of getting sufficient information in order that a competent panel of engineers can make the necessary grading of a large number of structural lay-outs from practice.

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