Some simple thoughts on column buckling

SOME SIMPLE THOUGHTS ON COLUMN BUCKLING

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ABSTRACT

This paper describes briefly the "physical" and "mathematical" models in physics and structural mechanics, leading to the modelling of flexural buckling of pin-ended columns; (this later aspect is presented to a certain extent as a history of ideas). Some simple thoughts on column buckling lead to a modern interpretation and generalisation of Young's formula for the "imperfect" column. A criterion for the definition of the imperfection parameter is established and it's simplest expression proposed. Various possible formulations are shown and a discussion of the "natural parameters" of the problem is followed by an example showing the potential and simplicity of the suggested approach.

The specific contents of this paper are:

This paper is dedicated to the memory
of my beloved mother,
and of my teacher and friend H. Beer,
who passed away while this paper was written.

I am thanking my wife for all her understanding, patience and help, and last but not least, for her love during these days.

I am thanking my colleagues from U.C.L. for their stimulating discussions and for their friendship. Special thanks to Professor K.O. Kemp and Dr. A.C. Walker for their helpful discussions of this paper.

1. INTRODUCTION

1.1 Modelling in Physics

The real "physical" world is extremely complex. In order to understand it, at least partially, we restrict ourselves to a particular system, within certain limits of practical interest, which we study from a specific point of view, separating - as far as possible - the major (or primary) parameters of the system from the minor (or secondary) and from the negligible ones. (Secondary parameters may be called "imperfections"). We have thus defined - somehow subjectively - a "physical model" on which we can make observations and experiments, (within the limitations of equipment and techniques). In defining the model, the question "how good is good enough?" has to be asked (and answered as far as possible) as obviously the importance and value of the expected results, have to be corelated with the features of the model and with the cost, complexity and accuracy of its investigation.

The mathematical description of the physical model (consistent with the general principles of physics) will involve necessarily idealisations and simplifications; and further simplifications will be necessary if a specific mathematical method is to be used. We have now defined - again somehow subjectively - a "mathematical model" or "theory". If we consider that the physical model is a reasonably "real" representation of the actual physical system, then the mathematical model is it's more or less "ideal" representation. Thus from the physicist's point of view the physical model is "perfect" and the mathematical model is more or less "imperfect". From the mathematicians or theoreticians point of view (assuming that the idealisations of the mathematical model are fundamental axioms) it is sometimes (wrongly) stated that he deals with a pure "perfect" model, whereas nature is "imperfect". However both points of view can be unified by defining the differences between the physical and the mathematical models as "imperfections". Obviously the question "how good is good enough?" will govern again the choice of idealisations and simplifications made in the definition of the "mathematical model".

The exaggeration of certain major parameters of the models can lead to the very useful concepts of upper and lower bounds of the problem.

A better understanding of the phenomena can be achieved through an oversimplification of the mathematical model (which sometimes can be specially constructed physically). Such models are called "Phenomenological models" or "analogues".

The three models are interdependent as they have to be checked against each other; e.g. the simple analogue may lead to the discovery of a major parameter which has been overlooked in the initial physical model, or it might explain some apparently odd behaviour of the physical model.

1.2 Modelling in Structural Mechanics

In structural mechanics the limit states of engineering structures interacting with their environment are studied from the point of view of their serviceability. This is the broad definition of the physical or "mechanical model". Various publications on measurements of loading actions and tests of structures, structural elements and structural materials cover this subject. Other publications cover the "theories" or "mathematical models" of these different topics. Phenomenological models of materials are covered in the literature on rheology etc., whereas the most extensive treatment of phenomenological models for structural components is given in a recent book by CROLL and WALKER [11] (1972). Unfortunately, there are no textbooks (as far as the author knows) covering all models in equal depth and breadth, and practically no satisfactory attempts have been made to answer the questions of "limitation of validity" and "how good is good enough".

1.3 Objective

The main objective of this introduction is to stress that it is necessary again and again to judge and assess assumptions, idealisations, simplifications etc. of the different models, their interrelationship, and to ask - and answer as far as possible - the questions about the range of validity and the question "how good is good enough". In fact every engineer acts, to a certain extent, consciously or unconsciously, in this way.

2. COLUMN MODELS

2.1 Outline

We will approach this subject (using modern terminology and notation) by following its historical development, as far as it constitutes a history of ideas (not covered in this form elsewhere), and as it will be used in this paper later on.

2.2 The Mechanical Model

A straight, axially loaded, slender rod is called a column (strut, or stanchion) and its primary behavioural feature (in the context of this paper) is its flexural deformation. This loose description of flexural buckling goes back to HERON of Alexandria [19] (~A.D. 75) and to LEONARDO da VINCI [21] (1452-1519). The limit state of the column can be defined either through failure MUSSCHENBROEK [24] (1729) or the onset of large deformations EULER [15] (1744) . MUSSCHENBROEK [23] (1726) was the first to define material properties as: "hard, perfectly hard, soft, perfectly soft, flexible, elastic and perfectly elastic", and to design testing machines permitting systematic variations of parameters for the testing of material properties and structural components (1729-op.cit) EULER (1744) (op. cit.) considers elastic deformations and later (1757) [16] inelastic bending "... because it occurs in all bodies that resist flexure, whether they are elastic or not". He suggests to determine the flexural stiffness through bending tests under similar boundary conditions (as for the column). This concept was rediscovered 132 years later by CONSIDÈRE [9] (1889) and ENGESSER [14] (1889) and marked the beginning of modern research into inelastic buckling. THOMAS YOUNG [30] (1807) had an even clearer understanding of inelastic deformations, stating: "... a permanent alteration of form ... limits the strength of materials with regard to practical purposes, almost as much as fracture, since in general the force which is capable of producing this effect, is sufficient, with a small addition, to increase it till fracture takes place". He notices the different behaviour of stocky and slender columns and gives limits for various materials for the two types of behaviour. His understanding of what we call today inhomogeneity of material properties and imperfections is amazing, and has its origin in analyzing experimental results; "... considerable irregularities may be observed in all the experiments ... and there is no doubt but some of them were occasioned by the difficulty of applying the force precisely at the extremities of the axis, and others by the accidental inequalities of the substances, of which the fibres must often have been in such directions as to constitute originally rather bent than straight columns". This concept was rediscovered by several authors, but is usually attributed to AYRTON and PERRY [1] (1886), 79 years later. The importance of "past history of the material" has been demonstrated by B. BAKER [3] (1888) and only since WILSON and BROWN [29] (1935) showed the importance of residual stresses (47 years later), began to be a subject for modern research. R.H. SMITH [27] (1878) (who rediscovered Young's concept of imperfections) recognized that "... the whole question of the strength of struts is one of probability", a concept which gained acceptance only after its rediscovery 72 years later by DUTHEIL [12] (1950).

It can be seen that the physical model of flexural buckling was reasonably well established almost 100 years ago (or even longer) but unfortunately not well known or understood. Rayleigh's remarks about Young as quoted by TIMOSHENKO [28] (1953) that he "... did not succeed in gaining due attention from his contemporaries. Positions which he had already occupied were in more than one instance reconquered by his successes at great expense of intellectual energy", apply equally well to the 18th and 19th century scientists mentioned above.

2.3 The Mathematical Model (theory)

We restrict ourselves to the elementary case of the pin-ended straight column with constant cross-section. In his second memoir EULER (1757) (op.cit), gives his general formula for the "ideal inelastic column", of length ℓ :

$$N_B = \frac{\pi^2 B}{\ell^2} \qquad (1)$$

where B is termed "stiffness moment", or in todays terminology "flexural stiffness" and includes such more recent concepts as tangent modulus, or deteriorated stiffness etc. In his first memoir (1744) (op.cit) he calls this term the "elastic moment", and in his third memoir (1778)[17] gives it's more precise version, (for the classical "elastic Euler-load")

$$N_E = \frac{\pi^2 EI}{\ell^2} \qquad (2)$$

Euler's definition of the elastic modulus E (usually attributed to Young) and of the second moment of area I,are correct, but he ignores JAKOB BERNOULLI's (1695)[7] correct definition of the position of the neutral axis. YOUNG (1807) (op.cit) gives the correct value for I (with the correct position of the neutral axis) and gives also a, clumsy but correct, derivation for the mechanical model of his physical model (see chapter 22). He considers pin-ended elastic columns with an initial sinusoidal curvature of amplitude e , and a straight column with a load N applied with an eccentricity e_o. We will transcribe his results for these "imperfect columns" into a more modern form so as to express the second-order moment M^{II} by multiplying the first order moment

$$M^{I} = Ne_o \qquad (3)$$

with an amplification factor α , so that:

$$M^{II} = M^{I}\alpha \qquad (4)$$

For the initially curved column:

$$\alpha = \frac{1}{1-\overline{N}_E} \qquad (5)$$

and for the eccentrically loaded column:

$$\alpha = \sec \frac{\pi}{2}\sqrt{\overline{N}_E} \qquad (6)$$

with the non-dimensional parameter:

$$\overline{N}_E \equiv N/N_E \qquad (7)$$

where N_E is the elastic Euler load, equ.(2). A numerical comparison of (5) and (6), by AYRTON and PERRY (1886) (op.cit) shows that for practical values of $\bar{N}_E$ the algebric expression (5) is a good approximation of (6); so that the initial curvature can be considered as a "generalised imperfection". NAVIER (1826-1833) [25] writes the first modern textbook on engineering mechanics, and gives an elegant mathematical derivation for the eccentrically loaded column, (without indicating the concept of the "imperfect" column). Comparing theoretical results with tests he concludes that the elastic Euler-load N_E and a suitable failure load N_0 for the stocky column are upper bounds for the experimental results. Relatively late, MERCHANT (1954) [22] suggested empirically (in the more general context of frame-buckling) that the special form of the well-known Rankine-formula:

$$\bar{N}_0 + \bar{N}_E = 1 \qquad (8)$$

with the non-dimensional parameter

$$\bar{N}_0 \equiv N/N_0 \qquad (9)$$

is a lower bound. HORNE (1963) [20] has proved theoretically that this is correct under certain conditions (which are satisfied for the pin-ended column). Young did not apply his findings to bridge the gap between the experimental results and his theory. The first best-known attempts in this direction are due to AYRTON and PERRY (1886) (op.cit) who admitted as limit state yield in the extreme fibre; and suggested various expressions for imperfections. This approach was adopted in various codes of practice mainly due to work by ROBERTSON (1925) [26] and DUTHEIL (1950) (op.cit). The development of theories for inelastic buckling including the effect of imperfections, and the extensive study of residual stresses, are well known and will not be discussed here.

2.4 The Phenomenological Model (analogue)

JAKOB BERNOULLI, or G. CRAMER (the editor and commentator of his works) (1744) [10] have imagined the two-spring model for the bending of a cross-section, which may be considered as the predecessor of the well-known Shanley-model. EULER (1778) [18] when faced with the problem of self-weight buckling of a column, considers two rigid links connected by a torsional spring. For the modern treatment of such analogues see the book by CROLL and WALKER (1972) (op.cit).

3. SOME SIMPLE THOUGHTS ON COLUMN BUCKLING

3.1 The Interaction Diagram for Column Behaviour

The mechanical model of the slender column has been defined in (2.2). Obviously a very stocky column (under identical loading and support conditions) has to be represented by a different mechanical model. For a solid cross-section the column degenerates into a block and for a built-up cross-section into a plate-assemblage. For a ductile material like steel, to which we restrict ourselves in this paper, barreling and plate-deformation will be the respective primary behavioural features. For a certain region of slendernesses the behaviour of the slender and very stocky columns may interact, and for a certain value of slenderness (treating the problem as it has been done tacitly till here, as a deterministic-one) buckling will be predominant. We could thus identify by experiment the practical limit of column behaviour, and will call this limit the "stocky column" with a buckling load N_E^o for which $N \equiv N_o$. (The very stocky columns fall outside our present object of study, and as their behaviour depends on different parameters, cannot be included in the same diagrams or tables as the slender columns). When the length of the column tends to zero, the column degenerates into a "sheet", and ends up as a "mathematical fiction" which might be used eventually as a "conventional" value; but could be dangereously misleading in understanding column behaviour. As forces are readily measureable in experiments, we shall call N_o , N_E and N_E^o the "primary natural column parameters" which can be studied experimentally and evaluated from a probabilistic point of view. With the non-dimensional parameters $0 \leq \overline{N}_o \leq 1$ as defined in (9) and $\overline{N}_E^o \leq \overline{N}_E \leq 1$ as defined in (7), with

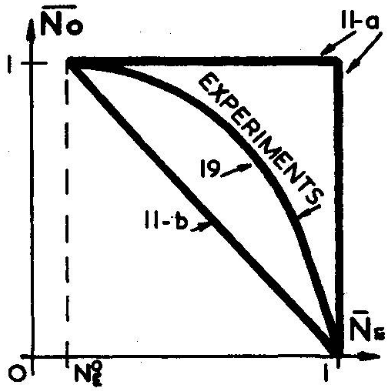

FIG.1

$$\overline{N}_E^o \equiv N_E^o/N_E \qquad (10)$$

the equations for the bounds can now readily be written:

$$\overline{N}_o + \overline{N}_E - \overline{N}_o\overline{N}_E = 1 \text{ (upper bound)} \qquad (11\text{-a})$$

$$\overline{N}_o + \overline{N}_E - \overline{N}_o\overline{N}_E^o = 1 \text{ (lower bound)} \qquad (11\text{-b})$$

The equations (11) define a triangle which will contain all experimental results. We will assume that the experimental results can be represented by a curve. (See the interaction diagram in figure 1).

3.2. Young's Formula, Modern Version

Young's approach (see 2.2 and 2.3) will be used to find an expression for the experimental curve in fig. 1. Considering the initial curvature e_o as the "generalised imperfection", the second order moment can be written, using equs. (3 to 5):

$$M^{II} = \frac{Ne_o}{1-\overline{N}_E} \qquad (12)$$

The interaction diagram in figure 2 shows (as an example for the idealised I-section) the well-known (conventional) elastic and rigid-plastic failure-conditions (limit-states). It can be seen that any failure-condition for any cross-section can be approximated by the linear interaction-formula:

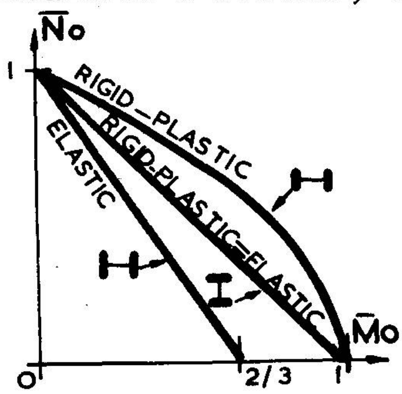

FIG.2

$$\overline{N}_o + c_o\,\overline{M}_o = 1 \qquad (13)$$

where,

$$\overline{M} \equiv M/M_o \qquad (14)$$

is the ratio of the actual moment M to the (rigid-plastic) ultimate moment M_o and c_o is a suitable approximation const-

On substituting (12) into (13) we obtain with (14), after some algebric manipulations:

$$(1 - \overline{N}_o)(1 - \overline{N}_E) - \eta \overline{N}_o = 0 \qquad (15)$$

with the "non-dimensional (generalised) imperfection"

$$\eta = \frac{c\, e_o}{M_o/N_o} \qquad (16)$$

Equation (15) with (16) represents the modern version of Young's formula. (In view of the historical account in (2) we consider this name more appropriate than the current name of Perry's formula).

3.3 A Criterion for the Definition of the Non-dimensional Imperfection

Young's epigones found it difficult to define a suitable expression for η. The interaction-diagram in figure 1 is now very useful in providing a necessary criterion for the definition of η. This criterion is: "Upon specialisation, the expression (16) for η and equ. (15) should yield the bounds (11), and so automatically contain the corner points of the interaction diagram (fig. 1)". (There is no proof that this condition is also sufficient).

3.4 The Simplest Expression for the Non-dimensional Imperfection

It is obviously possible to define many expressions for η which satisfy the criterion given above. The simplest expression will be linear in the three primary natural parameters of the problem (as defined in 3.1):

$$\eta = c\,(\overline{N}_E - \overline{N}_E^o) \qquad (17)$$

with the "imperfection parameter" c, where: $0 \leq c \leq 1$. The upper bound is obtained for $c \equiv 0$, and the lower bound for $c \equiv 1$. We consider c to be the fourth natural parameter of the problem.

3.5 Discussion of Other Expressions for the Non-dimensional Imperfection

The most popular expressions for η are due to ROBERTSON (1925) (op.cit.) (although his expression is implied already in the paper by AYRTON and PERRY) and DUTHEIL (1950) (op. cit.) both expressions will be generalised so that they contain the corner point ($\overline{N}_o = 1$, $\overline{N}_E = \overline{N}_E^o$). The ROBERTSON parameter can be written in non-dimensional form (this is a further generalisation of his concept) in our variables

$$\eta_{R1} = c_{R1}\sqrt{\frac{\overline{N}_E - \overline{N}_E^o}{\overline{N}_o}} \qquad (18\text{-a})$$

or

$$\eta_{R2} = c_{R2}\left(\sqrt{\overline{N}_E/\overline{N}_o} - \sqrt{\overline{N}_E^o/\overline{N}_o}\right) \qquad (18\text{-b})^*$$

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* An expression of the type (18-b) has been suggested by DWIGHT in his contribution to this colloquium.

Both expressions pass through both corner points; and with $c \equiv 0$ yield the upper bound, but do not yield upon specialisation the lower bound. DUTHEIL was the first to use non-dimensional parameters, and his generalised parameter is:

$$\eta_D = \varsigma_D \frac{\overline{N}_E - \overline{N}_E^{\circ}}{\overline{N}_o} \qquad (18\text{-c})$$

Equation (15) with (18-c) will not pass through the lower corner-point and will not yield upon specialisation the lower bound. A combination of the satisfactory parameter (17) with the Robertson and (or) Dutheil parameters (18) will obviously not satisfy the criterion defined in 3.3.

3.6 The Buckling Curve and an Alternative Interpretation

Substitution of (17) into (15) yields the hyperbola:

$$1-\overline{N}_E-(1-c\overline{N}_E^{\circ})\overline{N}_o+(1-c)\overline{N}_E\overline{N}_o = 0 \qquad (19\text{-a})$$

or

$$\overline{N}_o = \frac{1-\overline{N}_E}{(1-c\overline{N}_E^{\circ})-(1-c)\overline{N}_E} \quad (19\text{-b}) \qquad \overline{N}_E = \frac{1-(1-c\overline{N}_E^{\circ})\overline{N}_o}{1-(1-c)\overline{N}_o} \quad (19\text{-c})$$

An alternative interpretation of $\overline{N}_E$ can be obtained if we consider N to be identical with N_B as defined by equ. (1) Dividing equ. (1) and (2) yields then

$$\overline{N}_E \equiv B/EI \qquad (20)$$

i.e. $\overline{N}_E$ can be used to define the ratio of actual and elastic stiffness (or in a more specialised interpretation, the ratio of tangent and elastic moduli).

3.7 The Introduction of Alternative Variables

In our previous formulation both non-dimensional variables $\overline{N}_o$ and $\overline{N}_E$ were load-dependent. By taking the ratio

$$\Psi \equiv \overline{N}_E/\overline{N}_o \qquad (21\text{-a})$$

or it's equivalent:

$$\Psi \equiv N_o/N_E \qquad (21\text{-b})$$

we obtain a load-independent, non-dimensional variable which we may call the "strength-stiffness-ratio". Accordingly we will have

$$\Psi^{\circ} \equiv \overline{N}_E^{\circ} \qquad (22)$$

so that $\Psi^{\circ} \leq \Psi \leq \infty$. With (21) and (22) the equations (19) becomes the cubic:

$$1-(1-c\Psi^{\circ})\overline{N}_o-\Psi\overline{N}_o+(1-c)\Psi\overline{N}_o^2 = 0 \qquad (23\text{-a})$$

or, explicitly:

$$\Psi = \frac{1-(1-c\Psi^{\circ})\overline{N}_o}{\overline{N}_o\left[1-(1-c)\overline{N}_o\right]} \quad (23\text{-b}) \qquad \overline{N}_o = \frac{[(1-c\Psi^{\circ})+\Psi]-\sqrt{[(1-c\Psi^{\circ})+\Psi]^2-4(1-c)\Psi}}{2(1-c)} \quad (23\text{-c})$$

An alternative form of (23-c), more suitable for numerical evaluation is:

$$\overline{N}_o = \frac{2}{[(1-c\Psi^o)+\Psi]+\sqrt{[(1-c\Psi^o)+\Psi]^2-4(1-c)\Psi}} \qquad (23\text{-d})$$

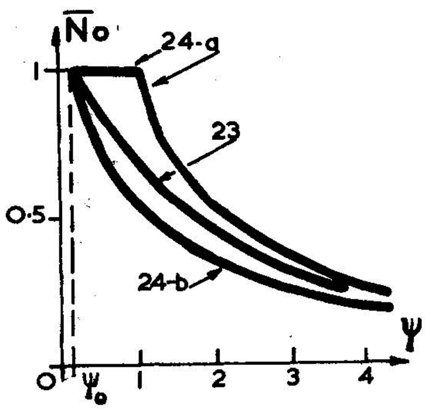

FIG.3

A representation of equ. (23) together with the corresponding upper bound:

$$(1-\overline{N}_o)(1-\Psi N\ ) = 0 \qquad (24\text{-a})$$

and lower bound:

$$1-(1-\Psi^o)\overline{N}_o - \Psi\overline{N}_o = 0 \qquad (24\text{-b})$$

is given in figure 3.
Usually the slenderness-ratio $\lambda = \ell/i$ (i = radius of gyration of the cross-section) is used as parameter for buckling problems. It can be easily shown that the "non-dimensional slenderness-ratio is:

$$\overline{\lambda} \equiv \sqrt{\Psi} \qquad (25\text{-a})$$

or

$$\overline{\lambda} \equiv N_o \lambda^2/\pi^2 EA \qquad (25\text{-b})$$

and it's corresponding limiting value:

$$\overline{\lambda}^o \equiv \sqrt{\Psi^o} \qquad (26)$$

Substituting (25) and (26) in (23) we obtain the quartic:

$$1-(1-c\,\overline{\lambda}^{o2})\overline{N}_o - \overline{\lambda}^2\overline{N}_o + (1-c)\overline{\lambda}^2\overline{N}_o^2 = 0 \qquad (27\text{-a})$$

or explicitly:

$$\overline{\lambda} = \sqrt{\frac{1-(1-c\,\overline{\lambda}^{o2})\overline{N}_o}{\overline{N}_o[1-(1-c)\overline{N}_o]}} \qquad (27\text{-b})$$

$$\overline{N}_o = \frac{[(1-c\,\overline{\lambda}^{o2})+\overline{\lambda}^2]-\sqrt{[(1-c\,\overline{\lambda}^{o2})+\overline{\lambda}^2]\ -4(1-c)\overline{\lambda}^2}}{2(1-c)\overline{\lambda}^2} \qquad (27\text{-c})$$

and the more suitable form:

$$\overline{N}_o = \frac{2}{[(1-c\,\overline{\lambda}^{o2})+\overline{\lambda}^2]+\sqrt{[(1-c\,\overline{\lambda}^{o2})+\overline{\lambda}^2]\ -4(1-c)\overline{\lambda}^2}} \qquad (27.\text{d})$$

Similarly, substituting (25) and (26) in (24) we have the upper bound:

$$(1-\overline{N}_o)(1-\overline{\lambda}^2\overline{N}_o) = 0 \qquad (28\text{-a})$$

and the lower bound:

$$1-(1-\overline{\lambda}^{o2})\overline{N}_o - \overline{\lambda}^2\overline{N}_o = 0 \qquad (28\text{-b})$$

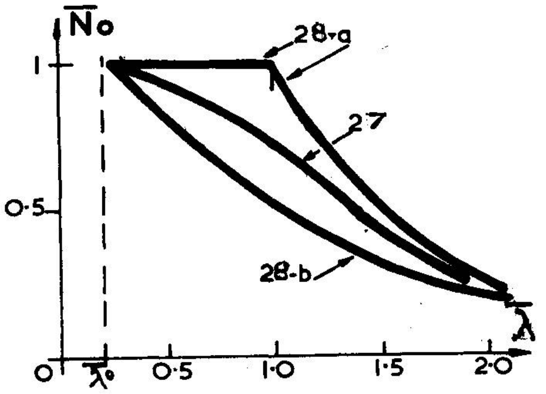

FIG. 4

The equations (27) and (28) are shown in figure 4; this is the well known presentation of the buckling curve. It should be noted that the order of the equivalent equations (19)(23) and (27) is increasing by one, and that the corresponding curves in figures (1)(3) and (4) get more complicated.

3.8 On Some Simplified Formulae

Considering only relatively small values of $\bar{\lambda}$, (27-d) may be expanded into power series, and retaining only the major terms, we obtain

$$N = 1-c(\bar{\lambda}^2 - \bar{\lambda}^{o2}) \qquad (29\text{-a})$$

or

$$N = \frac{1}{1+c(\bar{\lambda}^2 - \bar{\lambda}^{o2})} \qquad (29\text{-b})$$

Equ. (29-a) is a generalised form of the JOHNSON-parabola (as used in the USA with $\bar{\lambda}^o \equiv 0$) and equ. (29-b) is the generalised form of the well known RANKINE-formula.

3.9 The "Natural" Column Parameters

The natural column parameters, defined earlier will be discussed again. The "elastic" Euler load N_E (equ. 2) is well understood and reasonably well known experimentally. For the coupled parameters N_o and N_E^o of the "stocky column" (chapter 3.1) there is little experimental evidence and few theoretical studies available, and values are adopted at present through some kind of intuitive extrapolation. The imperfection parameter c should be studied in the region of highest "imperfection sensitivity". The scatter of experimental results, and column behaviour in this region has been explained, on an analogue, by CHILVER and BRITVEC (1963)[8], and there is sufficient experimental evidence available. It seems that a single imperfection parameter c is good enough for the description of column behaviour. Obviously a probabilistic study of all these parameters is desirable for design purposes. As the number of parameters is small, such an approach is feasible.

3.10 The European Column Curves and an Example for the Use of the Proposed Formulae

The late Professor H. BEER chaired and inspired Commission 8 (Buckling) of the European Convention for Structural Steelwork. The main results on column buckling obtained by this commission are reported in several papers in the September 1970 issue of "Construction Métallique". The theoretical foundations for the European Column Curves are given in a paper by BEER and SCHULZ (1970)[5]. At a meeting of Commission 8 in London (April 1971) J.B. DWIGHT and B.W. YOUNG (1971)[13] summarized their work, on similar lines but adopting the concept of $N_E^\circ \neq 0$*). At the discussions at this meeting BARTA (1971)[4] proposed the use of Young's formulae (equs. 15 and 16) suggesting as generalised imperfection the sum of the Dutheil term (equ. 18-c) and of his term (17), both with the assumption $N_E^\circ \equiv 0$. Barta's final formula is practically identical with an algebric approximation formula reported by BEER and SCHULZ (1971)[6] at the same meeting and due to BAAR (1970)[2], and unknown to the author at that time. (Baar investigated four algebric approximation formulae, without any attempt of a theoretical justification). In the search for a generally accepted approach, the concept $N_E^\circ \neq 0$ has been adopted, but (to the authors knowledge) the final curves are still subject to discussions. In order to show the potential of the proposed simple approach, we reproduce in figure (5) the European curves ($N_E^\circ \equiv 0$) from a paper by BEER and SCHULZ (1971)[6] to which we have ($N_E^\circ \equiv 0$) added the Merchant lower-bound curve. For the present purpose it is "good enough" to use equ. (27-d) with $\bar{\lambda}^\circ \equiv 0$, and to determine the imperfection parameter c for the most imperfection-sensitive value of $\bar{\lambda}$; (i.e. $\bar{\lambda} = 1$); This results in the following values for the three curves

$$c_a = 0.232 \; ; \qquad c_b = 0.444 \; ; \qquad c_c = 0.743$$

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*) Dwight's and Young's contribution to this colloquim represent a more detailed version of this report.

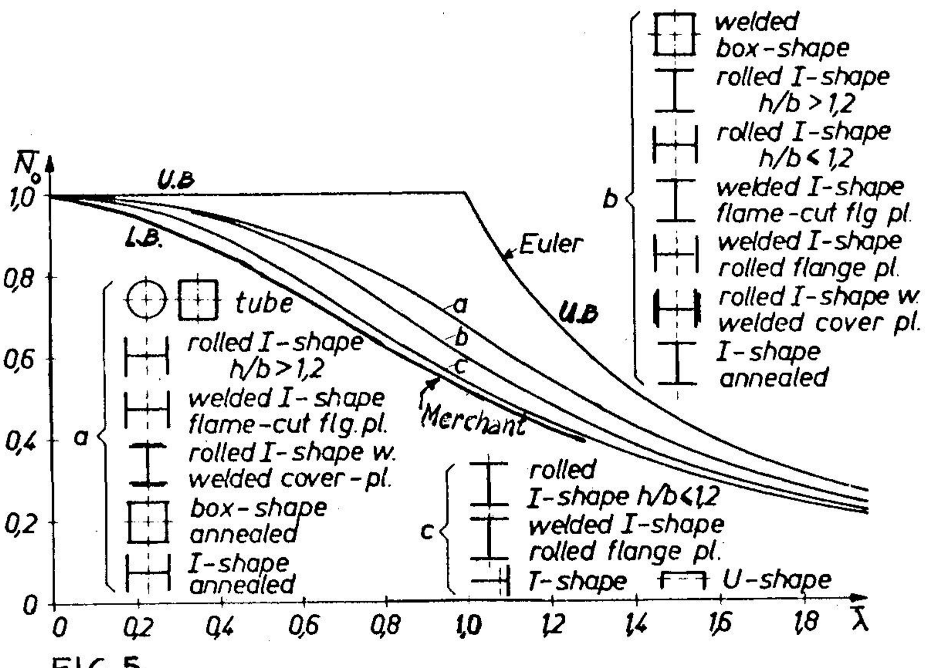

FIG.5

The results differ so little from those of Beer and Schulz that the curves could not be clearly traced together. We give therefore a numerical table of comparison:

Values of $\overline{N}_o$

curve \ $\overline{\lambda}$		0	0.25	0.50	0.75	1.00	1.50	2.00
U.B.		1	1	1	1	1	0.444	0.250
a	B-S	1	0.990	0.923	0.821	0.675	0.381	0.228
	B	1	0.985	0.934	0.831	0.675	0.388	0.234
b	B-S	1	0.983	0.885	0.757	0.600	0.343	0.207
	B	1	0.972	0.887	0.754	0.600	0.357	0.222
c	B-S	1	0.975	0.884	0.687	0.537	0.323	0.202
	B	1	0.955	0.836	0.683	0.537	0.327	0.209
L.B.		1	0.941	0.800	0.640	0.500	0.308	0.200

U.B. = upper bound; L.B. = lower bound
B-S = results by Beer and Schulz*)
B = present paper

3.11 Concluding Remarks

The approach shown in this paper will be extended in further publications to other materials, boundary conditions and loading cases. It's extension to post-buckling of structures will be given in joint paper with Dr. A.C. Walker.

*) The numerical values have been obtained in a private communication from Professor H. Beer.

References

1. AYRTON, W.E. and PERRY, J.
   On Struts
   The Engineer, 1886, p. 464-465; and 573-515.

2. BAAR, S.
   Courbes de flambement de la CECM. - Recherche d'une équation simple représentant au mieux les points donnés. Rapport to Committee 8 of CECM - SERCOM No. SB/1W st 26, 1970.

3. BAKER, B.
   Discussion on Open-hearth Steel for Boilers.
   Proc. Inst. Civ. Eng. London, Vol. XCII p. 44 (1888).

4. BARTA, T.A.
   A simple equation for the buckling curve.
   U.C.L. Communication (unpublished) to Committee 8 CECM - London Meeting 1971.

5. BEER, H. and SCHULZ, G.
   Bases théoriques des courbes européennes de flambement. Construction Métallique 7, (1970) p. 37-57.

6. BEER, H. and SCHULZ, G.
   Die neuen Knickspannungslinien der Europäischen Konvention der Stahlbauverbände.
   Wiss. Zeitschr. T.U. Dresden, Vol. 20 (1971) p. 817-821.

7. BERNOULLI, J.
Expliciationes, annotationes et additiones ad ea, quae in actis sup. anni de curva elastica ...
Acta Erud. Dec. 1695 p. 537-553.

8. CHILVER, A.H. and BRITVEC, S.J.
The plastic buckling of aluminium columns.
Proc. Symposium on Aluminium in Structural Engineering, Published by the Aluminium Federation London, 1963. .

9. CONSIDÈRE, A.
Résistance des pièces comprimées.
Congrès international de procedés de construction, 1889.

10. CRAMER, G.
Jakob Bernoulli: Opera
Geneva 1744.

11. CROLL, J.A. and WALKER, A.C.
Elements of structural stability
Macmillan, London, 1972.

12. DUTHEIL, J.
Le flambement des éléments comprimès dans la construction métallique; le flambement et le deversement.
Bull. Soc. Roy. Belge des Ingénieurs et des Industriels No. 3, 1950.

13. DWIGHT, J.B. and YOUNG, B.W.
Comparison of European and British Column Curves.
Report (unpublished) at meeting of Committee 8 of the European Convention for Structural Steelwork, London 1971.

14. ENGESSER, F.
Über die Knickfestigkeit gerader Stäbe
Zeitschr. des Arch. u. Ing. Vereins, Hannover, 1889, p. 455.

15. EULER, L.
Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes. Additamentum I: De Curvis Elasticis.
M.M. Bousquet, Lausanne & Geneve 1744.

16. EULER, L.
Sur la force des colonnes,
Mem. Acad. Berlin. Vol. 13 1759 p. 252-282 (presented in 1757).

17. EULER, L.
Determinatio onerum, quae columnae gestare valent.
Acta Acad. Sci. Petrop. 2 (1778) p.121-145 (1780).

18. EULER, L.
De altitudine columnarum sub proprio pondere curruentium.
Acta Acad. Sci. Petrop. 2 (1778) p. 163-193 (1780).

19. HERON of ALEXANDRIA (A.D. 75?)
Mechanics II (34-h) in Arabic, ed. with german translation by L. Nix and W. Schmidt Leipzig 1900.

20. HORNE, M.R.
Elastic-plastic Failure Loads of Plane Frames.
Proc. Roy. Soc. A 274, (1963) p. 343.

21. LEONARDO DA VINCI (1452-1519).
Inst. France Ms. A. f 45v.

22. MERCHANT, W.
The failure load of rigidly jointed frameworks as influenced by stability.
Structural Engineer, Vol. 32, 1954, p. 185.

23. MUSSCHENBROEK, P.
Epitome elementorum physico-mathematicoıum conscripta in usus academicos, S. Lugtmans, Lugdumi Batavorum, 1726 (p. 380-395).

24. MUSSCHENBROEK, P.
Phisicae Experimentales et Geometricae Dissertationes.
Introductio ad coherentiam corporum firmorum. p.466-662.
S. Luchtmans, Lugduni Batavorum (Leyden) 1729.

25. NAVIER, L.
Résumé des Leçons données a l'École des Ponts et Chaussées sur l'Application de la Mecanique. Part I, second, corrected and augmented edition. Carillan-Goeury - Paris 1833 (first ed. 1826).

26. ROBERTSON, A.
The strength of struts
Selected Engineering Paper No. 28 Inst. of Civ. Eng. London, 1925.

27. SMITH, R.H.
The Strength of Struts.
Proceedings Edinburgh and Leith Engineering Society, Vol. 4, 1878, p. 42-67.

28. TIMOSHENKO, S.P.
History of strength of materials.
McGraw-Hill Publishing Co. Ltd., New York 1953.

29. WILSON, W.M. and BROWN, R.L.
The effect of residual longitudinal stresses upon the load-carrying capacity of steel columns.
Univ. of Illinois Bull. No. 280 1935.

30. YOUNG, Th.
A course of lectures on natural philosophy and the mechanical arts. (Vol. I and II).
J. Johnson, London 1807.