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Open Riemann Surfaces and Extremal Problems on Compact Subregions

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Introduction

Harmonic and analytic differentials on open Riemann surfaces have been studied by R. Nevanlinna and Myrberg [8, 9, 10, 12, 13]. The present paper will have only a slight connection with Myrberg's work which is characterized by detailed results under strong restrictions. The first part (§§ 1—3) is in fact only a new approach to Nevanlinna's theory. Whereas Nevanlinna assumes his surfaces to have a zero boundary (parabolic type) and proves that no exact harmonic differential other than zero can have finite quadratic norm, our basic assumption is the conclusion of that theorem. We cannot prove, due to the difficulty in constructing examples, that our theory is more general, but our point of view is that Nevanlinna's theorem, though intrinsically of great interest, has relatively little to do with the problems under consideration.

Our method differs from that of Nevanlinna by the use of approximating compact regions. On each compact region relevant extremal problems are solved, and the results are carried over to the open surface by a simple limiting process. In the handling of the extremal problems we are in close contact with the methods of Bergman and Schiffer, which they have developed for plane regions [3, 15]. A convenient tool for applying these methods to regions on Riemann surfaces is found in the class of Schottky differentials, and it was the recognition that Bergman's kernel-functions are in fact Schottky differentials that led us to undertake this study.

The second part of the paper (§§ 4—5) deals with an extremal problem that we have previously solved for plane regions. There are great simplifications over my original proof for which I am partly indebted to my student P. Garabedian. An interesting point is that the extremal functions are again defined by means of Schottky differentials.

It must be said that the theory of Abelian differentials on open surfaces is still very incomplete. In particular, there is no indication of how the differentials of the second and third kind should be handled in the case of infinitely many poles. If systematically developed, the ideas expounded in this paper may be expected to shed some light on this question, but the whole problem is so hazy that any precise formulation would be premature.

1. Basic notions

1.1. In this section we shall briefly recall the definition of a Riemann surface and the pertinent parts of the exterior calculus of differentials.

A Riemann surface W_0 is given by a topological Hausdorff space and a covering by open sets V together with a collection of corresponding homeomorphisms φ onto open sets $\varphi(V)$ in the complex z-plane. Whenever V_1 and V_2 overlap the function $\varphi_2 \circ \varphi_1^{-1}$, defined on $\varphi_1(V_1 \cap V_2)$, is complex analytic and hence determines a directly conformal mapping of $\varphi_1(V_1 \cap V_2)$ onto $\varphi_2(V_1 \cap V_2)$[1].

On each set V the variable z can be used as a local parameter. As long as we are careful to use only notions which are invariant under an analytic change of variable we can safely use the same letter z to denote a generic point on the surface. We shall also use the same notation W_0 for the Riemann surface and the corresponding topological space.

It is customary to consider only connected Riemann surfaces. If W_0 is compact the Riemann surface is said to be *closed*, otherwise it is *open*. It can be shown that every connected Riemann surface can be covered by a countable sequence of compact sets. Actually, the fundamental existence theorems can be proved in a manner which leads directly to this important consequence of the definition.

Riemann surfaces which can be transformed into each other by a directly conformal one to one mapping are identified.

1.2. A *first order differential* is a linear expression of the form

$$\omega = a\,dx + b\,dy \ ,$$

where a and b are complex valued functions of the parameter $z=x+iy$. There is one such expression for each local parameter, and the coefficients are supposed to depend on the choice of parameter in such a way that ω remains invariant when dx and dy are transformed in the customary

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[1] The notation is that of N. Bourbaki, Élements de Mathématique, Hermann & Cie., Paris.

manner. It will be assumed that a and b are continuous together with their first partial derivatives.

A *second order differential* is an expression

$$c\,dx\,dy$$

with a corresponding invariance property. The coefficient c is merely supposed to be continuous.

Finally, a *zero order differential* is a function f whose value at a point does not depend on the choice of parameter. When expressed in a local parameter the function shall be continuous together with its partial derivatives up to the second order.

We shall use Greek letters to denote first order differentials and Latin letters for functions. For second order differentials no particular notation will be introduced.

The sum of two differentials of the same order and the product of a function and a differential are defined in an obvious way. The product of two first order differentials $\omega_1 = a_1\,dx + b_1\,dy$ and $\omega_2 = a_2\,dx + b_2\,dy$ is, by definition,

$$\omega_1\,\omega_2 = (a_1 b_2 - a_2 b_1)\,dx\,dy\ .$$

The required invariance must of course be verified. We note that

$$\omega_2\,\omega_1 = -\,\omega_1\,\omega_2\ .$$

The *differential of a function* f is, as in classical analysis,

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy\,.$$

The differential of $\omega = a\,dx + b\,dy$ is defined as

$$d\omega = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right)dx\,dy\ .$$

Again, the invariance is easily verified. We have the important identities

$$d(df) = 0$$

and

$$d(f\,\omega) = df\cdot\omega + f\,d\omega\ .\qquad (1.1)$$

[2]

[2]) The calculus of differentials was introduced by E. Cartan and further developed by W. V. D. Hodge, G. De Rham and others. In the 2-dimensional case it is almost trivial, but the formal simplification which it affords is still of great value.

1.3. The preceding operations do not depend on the conformal structure, but are meaningful on any two-dimensional manifold of class C^2. On a Riemann surface we can define, in addition, an invariant operation which to each first order differential $\omega = a\,dx + b\,dy$ determines a *conjugate differential*

$$\omega^* = -\,b\,dx + a\,dy\ .$$

The invariance of this operation is a consequence of the Cauchy-Riemann equations.

We note the following properties:

$$\begin{aligned} \omega^{**} &= -\,\omega \\ \omega_1 \omega_2^* &= \omega_2 \omega_1^* \qquad\qquad (1.2) \\ \omega \overline{\omega}^* &= (|\,a\,|^2 + |\,b\,|^2)\,dx\,dy\ . \end{aligned}$$

In the last relation we have used the notation $\overline{\omega} = \overline{a}\,dx + \overline{b}\,dy$ for the complex conjugate of $\omega = a\,dx + b\,dy$.

1.4. In the following, when speaking of differentials we shall always mean first order differentials, unless the contrary is specifically mentioned.

We introduce the following terminology to indicate particularly important properties of differentials:

ω is *closed* if $d\omega = 0$;
ω is *exact* if $\omega = df$;
ω is *harmonic* if ω and ω^* are closed ;
ω is *pure* if $\omega^* = -\,i\,\omega\,\psi$
ω is *analytic* if it is harmonic and pure.

The corresponding classes of differentials are denoted by Γ_c, Γ_e, Γ_h, Γ_p, Γ_a respectively. All these classes are linear over the field of complex numbers. The classes of conjugate differentials are denoted by Γ_c^*, $\Gamma_e^*, \ldots$, and the complex conjugate classes by $\overline{\Gamma}_c$, $\overline{\Gamma}_e$, etc.

The following relations are trivial:

$$\begin{aligned} &\Gamma_e \subset \Gamma_c\ , \qquad \Gamma_a \subset \Gamma_h \subset \Gamma_c\ , \\ &\Gamma_h = \Gamma_c \cap \Gamma_c^*\ , \qquad \Gamma_p^* = \Gamma_p\ , \\ &\Gamma_a = \Gamma_a^* = \Gamma_c \cap \Gamma_p = \Gamma_c^* \cap \Gamma_p\ . \end{aligned}$$

A harmonic differential $\omega = a\,dx + b\,dy$ satisfies

$$\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x} ,$$
$$\frac{\partial a}{\partial x} = - \frac{\partial b}{\partial y} .$$

Hence $a - ib$ is an analytic function of the local parameter. A pure differential is of the form $a\,dz$, $dz = dx + i\,dy$. It is analytic if a is an analytic function of z.

From a harmonic differential ω we can form the analytic differential $\omega + i\,\omega^*$.

We note in this connection that the product of two pure differentials is zero. In fact, if ω_1 and ω_2 are pure we obtain by (1.2)

$$\omega_1\,\omega_2 = \overset{*}{\omega_1}\,\overset{*}{\omega_2} = - \omega_1\,\omega_2$$

and hence $\omega_1\,\omega_2 = 0$. It follows that a pure differential cannot be real unless it is identically zero. For if $\omega = \overline{\omega}$ is pure, so is $\omega^* = \overline{\omega}^*$ and $\omega\overline{\omega}^* = 0$. By the last equation (1.2) this implies $\omega = 0$.

1.5. The *norm* of ω, defined on the Riemann surface W_0, is given by

$$||\,\omega\,||^2 = \int_{W_0} \omega\overline{\omega}^*\,dx\,dy = \int_{W_0} (|a|^2 + |b|^2)\,dx\,dy .$$

In order to compute the integral, W_0 must be divided into subsets such that a single parameter can be used within each subset. Because of the invariance of the integrand the result is independent of the subdivision. The norm is $\geqq 0$, it may be positively infinite, and it is 0 only for $\omega = 0$.

The differentials of finite norm form a Hilbert space (except for completeness) with the *inner product*

$$(\omega_1\,\omega_2) = \int_W \omega_1\,\overline{\omega}_2^* . \tag{1.3}$$

As a consequence of (1.2) the inner product satisfies

$$(\omega_2\,\omega_1) = (\overline{\omega}_1\,\overline{\omega}_2) .$$

Every open set $W \subset W_0$ is again a Riemann surface whose structure is induced by the structure of W_0, and any differential on W_0 can be considered as a differential on W. In particular, if the closure of W is compact, and if its boundary is formed by a finite number of analytic (or else sufficiently regular) curves C, we have the fundamental formula

$$\int_W d\omega = \int_C \omega \ . \tag{1.4}$$

The integral on the right has to be computed as a line integral in the parameter plane and taken in the positive sense of the boundary.

By use of (1.1) we obtain, as a particular case, the formula of integration by parts

$$\int_W f\, d\omega = \int_C f\omega - \int_W df \cdot \omega \ . \tag{1.5}$$

If W_0 is compact we can identify W with W_0 ; C vanishes, and we obtain

$$\int_{W_0} d\omega = 0 \ ,$$

$$\int_{W_0} f\, d\omega = - \int_{W_0} df \cdot \omega \ . \tag{1.6}$$

If $\omega_1 \in \Gamma_e$ and $\omega_2 \in \Gamma_c^*$ (1.3) and (1.6) imply $(\omega_1 \omega_2) = 0$, i. e. the linear subspace Γ_e is orthogonal to Γ_c^*. Conversely, if ω is orthogonal to the whole space Γ_e,

$$\int_{W_0} f\, d\omega^* = 0$$

for all f, and under our regularity assumptions this is not possible unless $d\omega^* = 0$ or $\omega \in \Gamma_c^*$. *On a compact surface Γ_c^* is the orthogonal complement of Γ_e.*

It follows that a harmonic differential can not be exact unless it is identically zero.

1.6. The topological type of a closed Riemann surface is determined by its genus p, and the 1-dimensional homology group is generated by $2p$ linearly independent cycles. For any closed differential ω and for any boundary C

$$\int_C \omega = 0$$

by (1.4). For this reason the period

$$P_C(\omega) = \int_C \omega \tag{1.7}$$

which corresponds to a cycle C depends only on the homology class of C, and there are $2p$ linearly independent periods.

By the fundamental existence theorem there are exactly $2p$ linearly independent harmonic differentials. The equation (1.7) defines a linear mapping of the space Γ_h of harmonic differentials into the $2p$-dimensional space of period vectors. The periods are all zero if and only if ω is exact. We have just seen that $\omega = 0$ is the only exact harmonic differential. Therefore the correspondence between the two spaces is one to one and there is one and only one harmonic differential with given periods.

Γ_h is spanned by the two linear subspaces Γ_a und $\overline{\Gamma_a}$. In fact, every $\omega \in \Gamma_h$ admits the representation

$$\omega = \frac{\omega + i\,\omega^*}{2} + \frac{\omega - i\,\omega^*}{2}$$

where the two terms are in Γ_a and $\overline{\Gamma_a}$ respectively. On the other hand, $\varphi_1 = \overline{\varphi}_2$, $\varphi_1, \varphi_2 \in \Gamma_a$ would imply that $\varphi_1 + \varphi_2$ and $i(\varphi_1 - \varphi_2)$ were pure and real. By the observation at the end of 1.4 we would then have $\varphi_1 = \varphi_2 = 0$. We conclude that Γ_a is p-dimensional over the field of complex numbers and $2p$-dimensional over the field of reals.

The mapping of an analytic differential onto the real parts of the periods is linear over the field of reals. This mapping is one to one, for an analytic differential with purely imaginary periods reduces to zero. There is consequently one and only one analytic differential whose periods have given real parts.

We collect the main results, which are all classical:

On a closed Riemann surface of genus p there are $2p$ linearly independent harmonic differentials and p linearly independent analytic differentials. There is one and only one harmonic differential with given periods and one and only one analytic differential whose periods have given real parts.

In this formulation linear dependence is with respect to the complex field. Our terminology differs from the customary usage by the fact that harmonic differentials are allowed to be complex. This is a superficial difference which offers some formal advantages.

1.7. The harmonic and analytic differentials which we have considered so far are all of the first kind. Differentials of the second and third kind are defined quite similarly, except for the presence of isolated singularities of a prescribed nature. In a suitable parameter the singularities of an analytic differential are given by expressions of the form

$$\left(\frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_h}{z^h}\right) dz\ .$$

Naturally, this expression varies with the parameter, but the coefficient a_1, known as the residue of the differential, is invariant, and the same is true of the order h of the pole ($a_h \neq 0$).

For a harmonic differential ω the singularities are described by the singular parts of the corresponding analytic differentials $\omega + i\omega^*$ and $\overline{\omega} + i\overline{\omega}^*$. The singularities of ω are thereby uniquely determined, for if $\omega + i\omega^*$ and $\overline{\omega} + i\overline{\omega}^*$ are regular, so is ω.

The existence theorem for differentials of the second and third kind asserts that there is an analytic differential with given singular parts, provided that the sum of the residues is zero. It is determined up to an analytic differential of the first kind. If a finite number of points and corresponding integers with a sum $m \geqq 1$ are prescribed, there are evidently $m + p - 1$ linearly independent analytic differentials whose poles are located at the given points with the additional condition that their multiplicities do not exceed the given integers. In fact, we can dispose over m coefficients a_k which have to satisfy a single linear equation.

For harmonic differentials the situation is similar. Again, the singular parts of $\omega + i\omega^*$ and $\overline{\omega} + i\overline{\omega}^*$ determine ω up to a harmonic differential of the first kind; two residue conditions have to be fulfilled. There are $2(m + p - 1)$ linearly independent harmonic differentials with poles at m given points (a pole of order h is considered as equivalent to h simple poles).

2. Schottky differentials

2.1. We consider an open region $W \subset W_0$ whose closure $\overline{W}$ is compact and whose boundary C consists of a finite number of closed analytic curves. It is assumed that C is the common boundary of W and its complement.

A differential on $\overline{W}$ is supposed to satisfy the requirements of invariance and regularity on the closed region $\overline{W}$. It is well known that $\overline{W}$ can be completed, by the process of symmetrization, to a closed Riemann surface $\hat{W}$. There is given an involutory, indirectly conformal mapping of $\hat{W}$ onto itself which leaves every point on C fixed. The image of $z \in W$ is denoted by $\tilde{z}$, the image of W by $\tilde{W}$. It is possible to choose local parameters so that, for corresponding points, $\tilde{z} = \overline{z}$.

If $\overline{W}$ has genus p and q contours, the genus of $\hat{W}$ is $P = 2p + q - 1$.

Those harmonic and analytic differentials on $\overline{W}$ which can be extended to harmonic or analytic differentials on $\hat{W}$ are called *Schottky differentials*. They constitute a very important subclass of the differentials on $\overline{W}$. It will be seen, for instance, that many extremal problems have their solution within the class of Schottky differentials.

Let ω be a differential on $\hat{W}$. We denote by $\tilde{\omega}$ the differential obtained from ω by changing the variable z to $\tilde{z}$. Explicitly, if

$$\omega = a\,dx + b\,dy$$

we set

$$\tilde{\omega} = \tilde{a}\,dx - \tilde{b}\,dy\ ,$$

where $\tilde{a}$, $\tilde{b}$ are the coefficients of ω at the point $\tilde{z}$ and in the parameter $\tilde{z}$. A similar operation is defined for functions and second order differentials.

We have performed a non-analytic change of the variable. For this reason, while multiplication and differentiation are still invariant processes, conjugation is not invariant. We find indeed

$$(\tilde{\omega})^* = \tilde{b}\,dx + \tilde{a}\,dy$$

$$(\tilde{\omega}^*) = -\,\tilde{b}\,dx - \tilde{a}\,dy$$

so that the connection is

$$(\tilde{\omega})^* = -\,(\tilde{\omega}^*)\ .$$

Writing

$$\tau_1 = \frac{\omega - \tilde{\omega}}{2}$$

$$\tau_2 = \frac{(\tilde{\omega}^*) - \omega^*}{2}$$

we have

$$\tau_2^* = \frac{\omega + \tilde{\omega}}{2}$$

and hence $\omega = \tau_1 + \tau_2^*$. Obviously, τ_1 and τ_2 vanish on or rather along C [3]), and if ω is harmonic so are τ_1 and τ_2. It follows that every harmonic Schottky differential can be represented in the form $\tau_1 + \tau_2^*$ where τ_1, τ_2 are harmonic and vanish along C. Conversely, by the Schwarz symmetry principle any such differential can be extended across C and is hence a Schottky differential.

Let us prove that the representation $\omega = \tau_1 + \tau_2^*$ is unique. If $\tau_1 + \tau_2^* = 0$ the analytic differential $\tau_1 + i\,\tau_1^* = \tau_1 + i\,\tau_2$ would vanish along C and, being of the form $a\,dz$, on C. This is clearly impossible unless $\tau_1 = \tau_2 = 0$, and the representation is consequently unique.

[3]) We say that $\omega = a\,dx + b\,dy$ vanishes *at* a point z_0 if $a(z_0) = b(z_0) = 0$ and *along* a curve $z = z(t) = x(t) + i\,y(t)$ if $a\big(z(t)\big)\,x'(t) + b\big(z(t)\big)\,y'(t) \equiv 0$.

We have proved:

Theorem 1. *The space S_h of harmonic Schottky differentials is spanned by the subspaces S_0 and S_0^* where S_0 denotes the class of harmonic differentials which vanish along C.*

Corollary. *The space S_0 is P-dimensional over the complex field.*

For an analytic differential φ it is found that $\tau_2 = i\tau_1$ and the above representation takes the form $\varphi = \tau_1 + i\tau_1^*$. We decompose τ_1^* in its real and imaginary part: $\tau_1^* = \sigma_1 + i\sigma_2$. In this way we obtain a more convenient decomposition

$$\varphi = (-\sigma_1^* + i\sigma_1) + i(-\sigma_2^* + i\sigma_2)$$

or

$$\varphi = \theta_1 + i\theta_2 ,$$

where θ_1 and θ_2 are analytic and real along C. Alternatively, we could have written at once

$$\theta_1 = \frac{\varphi + \tilde{\bar{\varphi}}}{2} ,$$

$$\theta_2 = \frac{\varphi - \tilde{\bar{\varphi}}}{2i} .$$

The decomposition is again unique, and we have

Theorem 2. *The space S_a of analytic Schottky differentials, considered as a vector space over the real field, is spanned by S_r and iS_r, where S_r denotes the subspace of analytic Schottky differentials which are real along C.*

Corollary. *The space S_r is P-dimensional over the real field.*

The homology group of W has $P = 2p + q - 1$ linearly independent elements, but a differential in S_0 has only $2p$ periods. In fact, the periods which correspond to the contours are all zero. To compensate, there are $q - 1$ half-periods corresponding to arcs which run from one contour to another. If the periods and half-periods are all zero the extended differential is obviously exact on $\hat{W}$ and consequently identically zero. The correspondence between S_0 and the P-dimensional space of periods and half-periods is hence one to one.

The differentials in S_0^* have the full number of P periods. Again, a differential with zero periods is identically zero. Such a differential would indeed be exact, $\omega = df$, with $\omega^* = 0$ along C. By (1.3) and (1.5) we would obtain

$$\|\omega\|^2 = \int_W df\bar{\omega}^* = \int_C f\bar{\omega}^* = 0$$

and hence $\omega = 0$. We conclude:

Theorem 3. *There is one and only one differential in S_0 with given periods and half-periods, and there is one and only one differential in S_0^* with given periods.*

For analytic differentials the situation is still simpler. A differential in S_a has P periods, and if they are all zero the differential vanishes. If $\theta_1 + i\,\theta_2 = df$ we have in fact $df^* = -i\,df$ and

$$\| df \|^2 = \int_C f\,\overline{df}^* = i\int_C f\,\overline{df} = i\int_C f(\theta_1 - i\theta_2) = i\int_W df(\theta_1 - i\theta_2) = 0 ,$$

for the product of pure differentials is zero. We have proved:

Theorem 4. *There is one and only one differential in S_a with given periods.*

We note the important difference between compact bounded regions and closed surfaces. In the first case all the complex periods of an analytic differential (in the subclass S_a) can be prescribed. In the second case only the real parts of the periods can take assigned values.

2.2. We have already indicated that the class of Schottky differentials contains the solution of many extremal problems. In order to distinguish the Schottky differentials we agree to denote arbitrary differentials on $\overline{W}$ by capital letters and reserve the lower case types for Schottky differentials. Norm and inner products will be taken with respect to $\overline{W}$, and the notations Γ_c, Γ_a, Γ_e, etc., will refer to differentials on $\overline{W}$.

We begin by proving two lemmas:

Lemma 1. *The classes S_0^* and Γ_e are orthogonal complements in Γ_c.*

Suppose that $\tau \in S_0$ and $\Omega = dF$. Then

$$(\Omega, \tau^*) = -\int_W dF\bar{\tau} = -\int_C F\bar{\tau} = 0 , \qquad (2.1)$$

for $\tau = 0$ along C. Hence Γ_e and S_0^* are orthogonal. On the other hand, if Ω is closed there exists a $\tau_0^* \in S_0^*$ with the same periods as Ω. The representation $\Omega = \tau_0^* + (\Omega - \tau_0^*)$ shows that Γ_c is spanned by S_0^* and Γ_e.

Lemma 2. *The classes S_a and $\Gamma_e \cap \Gamma_a$ are orthogonal complements in Γ_a.*

Consider $\theta \in S_r$ and $\Phi = dF$, $\Phi^* = -i\Phi$. We have

$$(dF, \theta) = \int_W dF\bar{\theta}^* = \int_C F\theta^* = i\int_C F\theta = i\int_C F\theta = i\int_W dF\theta = 0 \quad (2.2)$$

for $dF\,\theta$ is identically zero. Since every $\varphi \in S_a$ can be written in the form $\theta_1 + i\,\theta_2$ with $\theta_1, \theta_2 \in S_r$ this shows that the two subspaces are orthogonal. Moreover, to every analytic Φ we can find a $\varphi_0 \in S_a$ with the same periods, and we conclude as above that the subspaces are complementary.

After these preliminaries we can solve the following problems:

Problem 1. *To determine a closed differential on $\overline{W}$ with given periods and minimum norm.*

Solution: There exists a differential $\tau^* \in S_0^*$ with the given periods. Let Ω be any closed differential with the same periods. Then $\tau^* - \Omega$ is exact, and by Lemma 1

$$(\tau^*, \tau^* - \Omega) = 0$$

or

$$\| \tau^* \|^2 = (\tau^*, \Omega) = (\Omega, \tau^*) \ .$$

Hence

$$\|\tau^* - \Omega\|^2 = \|\tau^*\|^2 - (\tau^*, \Omega) - (\Omega, \tau^*) + \|\Omega\|^2 = \|\Omega\|^2 - \|\tau^*\|^2$$

and we conclude that $\|\tau^*\|^2 < \|\Omega\|^2$ except when $\Omega = \tau^*$. Consequently, τ^* solves our problem. Observe that τ^* gives the minimum in the whole class of closed differentials and not only in the class of harmonic differentials.

Problem 2. *To determine an analytic differential on $\overline{W}$ with given periods and minimum norm.*

Solution: There exists a differential $\varphi \in S_a$ with the given periods. Let Φ be an analytic differential with the same periods. Then φ is orthogonal to $\varphi - \Phi$, and we reach the conclusion $\|\varphi\| \leqq \|\Phi\|$ exactly as above. The solution is therefore given by φ.

More general extremal problems may be solved by the same method. In particular, in either problem only part of the periods may be prescribed. In this event we denote by $S_0^{*\prime}$ the subspace of S_0^* for which the prescribed periods are zero, and by $S_0^{*\prime\prime}$ its orthogonal complement. The complement contains one and only one differential whose prescribed periods have given values. In fact, the dimension of the orthogonal complement is equal to the number of prescribed periods, and there cannot be several differentials with the same prescribed periods, for their difference would have to be orthogonal to itself.

If $\tau^{*\prime\prime} \in S_0^{*\prime\prime}$ has the prescribed periods it furnishes the solution to the generalized Problem 1. For let Ω be any closed differential with the prescribed periods and τ_0^* its projection on S_0^*. Then $\Omega - \tau_0^*$ and $\tau_0^* - \tau^{*\prime\prime}$ are both orthogonal to $\tau^{*\prime\prime}$. Hence $\Omega - \tau^{*\prime\prime}$ is also orthogonal to $\tau^{*\prime\prime}$, and the extremal property of $\tau^{*\prime\prime}$ follows as above.

The corresponding generalization of Problem 2 can be treated in the same manner. We formulate the result as:

Theorem 5. *In the class of all closed (analytic) differentials on $\overline{W}$ with prescribed values of certain periods the minimum norm is attained for the unique Schottky differential of class S_0^* (S_a) with the prescribed periods which is orthogonal to all closed (analytic) differentials whose prescribed periods are zero.*

2.3. We consider now an extended class of Schottky differentials which, being regular on $\overline{W}$, are of the second or third kind on the closed surface $\hat{W}$. An analytic differential of this kind has still a representation $\varphi = \theta_1 + i\,\theta_2$, where θ_1 and θ_2 are real along C, but θ_1 and θ_2 will possess singularities on W which cancel in the linear combination $\theta_1 + i\,\theta_2$.

A divisor is a formal product $D = a_1^{h_1} \ldots a_s^{h_s}$ of points with corresponding multiplicities. An analytic differential or function is said to be a multiple of D if the first non-vanishing coefficient of its power series development at a_i corresponds to an exponent $\geqq h_i$. We consider a divisor D on W with non-negative exponents and write $D^{-1} = a_1^{-h_1} \ldots a_s^{-h_s}$ and $\tilde{D} = \tilde{a}_1^{h_s} \ldots \tilde{a}_1^{h_s}$. By $S_a(D)$ we shall denote the class of analytic Schottky differentials whose extension to $\hat{W}$ is a multiple of $\tilde{D}^{-1}$. The following generalization of Lemma 2 can be proved:

Lemma 3. *The orthogonal complement of $S_a(D)$ in Γ_a is formed by all exact analytic differentials dF such that F is a multiple of D.*

Every $\varphi \in S_a(D)$ can be written in the form $\theta_1 + i\,\theta_2$ where θ_1 and θ_2 are multiples of D^{-1} and real on C. The inner product (θ_ν, dF) has a sense only if interpreted as a limit; we exclude the poles of θ_ν by small circles in the parametric plane and form the limit of the inner product over the remaining region for vanishing radii. It is easily seen that (2.2) remains valid provided that F satisfies the condition of the lemma. Hence $S_a(D)$ is indeed orthogonal to all such dF.

We know already that $S_a(D)$ is $(m+P-1)$-dimensional, where $m = h_1 + \cdots + h_s$ is the degree of D. Consider the linear mapping which take every $\varphi \in S_a(D)$ into a vector whose components are the P periods of φ, the integrals

$$\int_{a_1}^{a_i} \varphi \qquad (i = 2, \ldots, m)$$

over preassigned paths, and the first $h_i - 1$ coefficients of φ at a_i. By what we have proved this correspondence must be one to one. This is sufficient to conclude that the sets in Lemma 3 are complementary, and that the class $S_a(D)$ contains the unique solution of the following problem:

Problem 3. *To find an analytic differential Φ on $\overline{W}$ of minimum norm when the periods, the integrals*

$$\int_{a_1}^{a_i} \Phi \qquad (i = 2, \ldots, m)$$

and the first $h_i - 1$ coefficients at a_i take prescribed values.

An important special case arises when the periods are zero. If all the points are simple we obtain a solution of the following interpolation problem:

Problem 3′. *To find an analytic function F with prescribed values $F(a_i)$ whose differential dF has minimum norm.*

The solution is a function f which can be continued to $\hat{W}$ with at most simple poles at the points $\tilde{a}_i$.

These problems can be generalized to the case where not all $m+P-1$ quantities are prescribed. The solution is then characterized by the fact that it is orthogonal to all differentials for which the prescribed quantities are zero.

2.4. The corresponding problem for harmonic differentials can be treated in a similar manner, but there are certain differences which command our attention.

For simplicity we shall only treat the case where $D = a_1 \ldots a_m$. We remark that (2.1) remains valid for a τ with simple poles provided that F is harmonic and vanishes at the poles. But in order to cancel the singularities we must consider differentials of the form $\tau_1 + \tau_2^*$ where τ_1 should be orthogonal to the same harmonic differentials as τ_2^*. It turns out that we must choose $\tau_1 = dg$, where $g = 0$ on C, and in this case the singular parts of τ_1 must be of the form $A\, Re \frac{dz}{z}$. In other words, g will be a linear combination of Green's functions. Differentials of this kind will be called Green's differentials.

Lemma 4. *A harmonic differential $\tau_1 + \tau_2^*$ of the extended Schottky class whose first part τ_1 is a Green's differential is orthogonal to all exact harmonic differentials dU, such that U vanishes at the poles of τ_1.*

We need only prove that $(\tau_1, \Omega^*) = 0$ for all regular closed differentials Ω. We obtain

$$(dg, \Omega^*) = -\int_W dg\overline{\Omega} = -\int_C g\overline{\Omega} = 0$$

by virtue of the fact that g is only logarithmically infinite at the poles.

There are m linearly independent dg which are multiples of D^{-1}. The residues of τ_2^* are uniquely determined by dg and have to satisfy a linear condition. Since the periods of τ_2^* are arbitrary, the number of linearly independent Schottky differentials of the type $dg + \tau_2^*$ is $m + P - 1$. It follows that the periods and the integrals

$$\int_{a_1}^{a_i} \omega$$

over preassigned paths can be prescribed.

We conclude that the class of differentials $dg + \tau_2^*$ contains the solution to

Problem 4. *To find a harmonic differential Ω of minimum norm when the periods and the values of the integrals*

$$\int_{a_1}^{a_i} \Omega \qquad (i = 1, \ldots, m)$$

over preassigned paths are given.

When all the periods are zero we obtain the solution to

Problem 4'. *To find a harmonic function U with prescribed values $U(a_i)$ whose differential dU is of minimum norm.*

We have proved that the minimum is taken for a uniquely determined harmonic function whose differential belongs to the extended Schottky class.

2.5. When there are only two points of interpolation a, b we can ask directly for the minimum of the homogeneous expression

$$\frac{\|\Omega\|^2}{\left|\int_a^b \Omega\right|^2}. \qquad (2.3)$$

It is worth while to study the explicit solutions when Ω runs through the classes Γ_h, Γ_a, $\Gamma_h \cap \Gamma_e$, $\Gamma_a \cap \Gamma_e$ respectively. In the first two cases the integration is along a preassigned path from a to b.

Let $g(z, \zeta)$ denote the Green's function with a positive logarithmic pole at ζ, and let d_z, d_ζ be symbols of differentiation with respect to z and ζ respectively. Then a minimizing differential in the class Γ_h can be given explicitly in the form

$$\omega_0(z, a, b) = d_z \int_a^b d_\zeta\, g(z, \zeta) + d_z^* \int_a^b d_\zeta^*\, g(z, \zeta)$$

or in more condensed notation

$$\omega_0 = \int_a^b (d_z d_\zeta + d_z^* d_\zeta^*)\, g(z, \zeta) \ ,$$

where the integration is with respect to the variable ζ.

It is easy to verify, first of all, that the operation $d_z d_\zeta + d_z^* d_\zeta^*$ removes the singularity $\log \frac{1}{|z-\zeta|}$ so that ω_0 is indeed regular and harmonic. Secondly, we must show that ω_0 is orthogonal to all harmonic Ω which satisfy the condition $\int_a^b \Omega = 0$. We note that the first part of ω_0 reduces to $dg(z, b) - dg(z, a)$, and we have already seen that a differential of this form is orthogonal to all harmonic Ω. The second part is temporarily denoted by σ_0^*. It is practically evident that σ_0 is exact, except for the period 2π which stems from its singularities. By simple contour integration we obtain

$$(\sigma_0^*, \Omega) = (\sigma_0, \Omega^*) = 2\pi \int_a^b \overline{\Omega}$$

and hence

$$(\omega_0, \Omega) = 2\pi \int_a^b \overline{\Omega} \ .$$

The minimum property of ω_0 is thereby established, and at the same time we obtain

$$\|\omega_0\|^2 = 2\pi \int_a^b \omega_0 \ .$$

The minimum value $d_1(a, b)$ of (2.3) can be thought of as an invariant distance between a and b. We find

$$d_1(a, b) = \frac{4\pi^2}{\|\omega_0\|^2} = \frac{2\pi}{\int_a^b \omega_0} \ .$$

We show next that $\omega_0 + i\omega_0^*$ is extremal in the class Γ_a. For $\Phi \in \Gamma_a$ we obtain indeed

$$(\omega_0 + i\omega_0^*, \Phi) = 2(\omega_0, \Phi) = 4\pi \int_a^b \overline{\Phi}$$

and consequently $\omega_0 + i\omega_0^*$ is orthogonal to all analytic Φ with $\int_a^b \Phi = 0$. By the same formula we obtain

$$\| \omega_0 + i\omega_0^* \|^2 = 4\pi \int_a^b \omega_0 - i\omega_0^*$$

which shows that $\int_a^b \omega_0^* = 0$. The minimum of (2.3) in the class Γ_a is

$$d_2(a, b) = 2d_1(a, b) .$$

In order to find the minimum in $\Gamma_h \cap \Gamma_e$ we need only subtract from ω_0 a $\tau_0^* \in S_0^*$ which makes $\omega_0 - \tau_0^*$ exact. Since $(\tau_0^*, dU) = 0$ for all exact dU we find

$$(\omega_0 - \tau_0^*, dU) = 2\pi(\overline{U(b)} - \overline{U(a)}) ,$$

and the extremal character of $\omega_0 - \tau_0^*$ is evident. We have moreover

$$\| \omega_0 - \tau_0^* \|^2 = \| \omega_0 \|^2 - \| \tau_0 \|^2 = 2\pi \int_a^b \omega_0 - \tau_0^*$$

and the minimum itself is

$$d_3(a, b) = \frac{4\pi^2}{\| \omega_0 \|^2 - \| \tau_0 \|^2} .$$

We note that $\sigma_0^* - \tau_0^*$ is exact and write $\sigma_0^* - \tau_0^* = dG(z, a, b)$.

The function $G(z, a, b)$ has logarithmic poles of different sign at a and b, and its normal derivative vanishes on C. It is known as the Neumann function with these singularities. We have shown that the harmonic function

$$U(z, a, b) = g(z, b) - g(z, a) + G(z, a, b)$$

makes the quantity

$$\frac{\| dU \|^2}{| U(a) - U(b) |^2}$$

a minimum.

In $\Gamma_a \cap \Gamma_e$ the minimum is attained for an exact differential of the form $\omega_0 + i\omega_0^* - \varphi_0$ where $\varphi_0 \in S_a$. The value of the minimum is

$$\frac{16\pi^2}{2\| \omega_0 \|^2 - \| \varphi_0 \|^2} .$$

2.6. Returning to Problem 1 we remark that the Schottky differential τ^* with the same periods as Ω is also the best approximation of Ω in the class S_0^*, in the sense that it makes $||\Omega - \tau^*||$ a minimum. In fact, for any $\tau_1^* \epsilon S_0^*$, $\tau_1^* - \tau^*$ is orthogonal to $\tau^* - \Omega$, and we obtain

$$|| \Omega - \tau_1^* ||^2 = || \Omega - \tau^* ||^2 + || \tau^* - \tau_1^* ||^2 .$$

Hence Problem 1 has the same solution as

Problem 5. *To find the best approximation of a closed differential Ω in the class S_0^*.*

If Ω is harmonic the best approximation in S_0 is found by determining the best approximation of Ω^* in S_0^*.

Quite similarly, Problem 2 has the same solution as

Problem 6. *To find the best approximation of an analytic differential Φ by an analytic Schottky differential.*

For a harmonic Ω Problem 5 is perhaps not the most natural. It may indeed be more important to determine the best approximation in the class of all harmonic Schottky differential:

Problem 7. *To find the best approximation of a harmonic differential Ω in the class S_h.*

Let φ_1 and φ_2 be the analytic Schottky differentials with the same periods as $\Omega + i\Omega^*$ and $\overline{\Omega} + i\overline{\Omega}^*$ respectively. Then $\frac{1}{2}(\varphi_1 + \overline{\varphi}_2)$ is the solution to our problem. We need only show that $\frac{1}{2}(\varphi_1 + \overline{\varphi}_2) - \Omega$ is orthogonal to all τ and τ^*. Computing the periods we see at once that this differential is exact: hence it is orthogonal to all τ^*. Its conjugate differential $-\frac{i}{2}(\varphi_1 - \overline{\varphi}_2) - \Omega^*$ is likewise exact, and the given differential is also orthogonal to all τ.

The problem of closest approximation to an analytic differential Φ has a meaning also when Φ has singularities. The approximating Schottky differential φ must have the same singularities as Φ. In general we cannot require that φ be regular on $\widetilde{W}$, for such a differential would exist only if the sum of the residues is zero. But there is clearly a unique Schottky differential φ with the same singularities and periods as Φ which has at most a simple pole at a preassigned point $\bar{z}_0$. In the class of Schottky differentials with this behavior on $\overline{W}$ the differential φ gives the closest approximation of Φ. If φ_1 is another differential of the same class we conclude by Lemma 3 with $D = z_0$ that $\varphi_1 - \varphi$ is orthogonal to $\varphi - \Phi$ and the minimum property of φ follows immediately.

3. Open surfaces

3.1. On an open surface W_0 we must distinguish between two kinds of homology. A finite cycle can be the boundary of a compact chain, or it can be the boundary of a non-compact chain. Accordingly, we make a difference between homology in the compact sense and homology in the non-compact sense.

If a cycle γ which is homologous to zero in the non-compact sense is contained in a compact subregion $\overline{W}$, then γ is homologous on $\overline{W}$ to a cycle on the boundary C of $\overline{W}$. In other words, it is homologous to a linear combination of the contours which make up C.

We consider on W_0 a harmonic differential Ω_0 of finite norm. For each compact $\overline{W}$ we can determine the Schottky differential τ^* which has the same periods as Ω_0^* on $\overline{W}$. Since τ^* is the solution of Problem 1 we have

$$\| \tau \|_W^2 \leqq \| \Omega_0 \|_W^2 , \tag{3.1}$$

where the norms are taken with respect to W.

Suppose that $W \subset W'$ and let τ, τ' be the differentials which correspond to these regions. By the minimum property of τ^*

$$\| \tau \|_W^2 \leqq \| \tau' \|_W^2$$

and *a fortiori*

$$\| \tau \|_W^2 \leqq \| \tau' \|_{W'}^2 .$$

In other words, $\| \tau \|_W^2$ is an increasing function of W and the limit

$$d = \lim_{W \to W_0} \| \tau \|_W^2$$

must exist. More precisely, to every $\varepsilon > 0$ there exists a compact set $A \subset W_0$ such that

$$\| \tau \|_W^2 > d - \varepsilon$$

for all $W \supset A$.

From (3.1) we conclude that $d \leqq \| \Omega_0 \|^2$. We consider the following two possibilities:

1. $d = \| \Omega_0 \|^2$. Since

$$\| \Omega_0 - \tau \|_W^2 = \| \Omega_0 \|_W^2 - \| \tau \|_W^2$$

we must have

$$\lim_{W \to W_0} \| \Omega_0 - \tau \|_W^2 = 0$$

and particularly

$$\lim_{W \to W_0} \| \Omega_0 - \tau \|_A^2 = 0$$

for any fixed compact region A. This implies, by a well-known reasoning which makes use of the harmonicity of τ, the pointwise convergence

$$\Omega_0 = \lim_{W \to W_0} \tau ,$$

and this convergence is uniform on every compact set.

We can conclude even more. Let γ be a finite cycle which is homologous to zero in the non-compact sense. Choose W so large that $\gamma \subset W$. We know that γ is homologous to a cycle on the boundary C. But $\tau = 0$ along C. Consequently,

$$\int_\gamma \tau = 0$$

and as W tends to W_0 we obtain by uniform convergence

$$\int_\gamma \Omega_0 = 0 .$$

Hence the period of Ω_0 over any cycle which is homologous to zero in the non-compact sense must vanish.

2. $d < \| \Omega_0 \|^2$. The limit of τ will still exist. Suppose that W' and W'' both contain W. If the corresponding Schottky differentials are denoted by τ, τ', τ'' we have

$$\| \tau' - \tau \|_W^2 = \| \tau' \|_W^2 - \| \tau \|_W^2 \leqq d - \| \tau \|_W^2$$

$$\| \tau'' - \tau \|_W^2 = \| \tau'' \|_W^2 - \| \tau \|_W^2 \leqq d - \| \tau \|_W^2$$

and consequently

$$\| \tau'' - \tau' \|_W^2 \leqq 4(d - \| \tau \|_W^2) .$$

It follows immediately that

$$\tau_0 = \lim_{W \to W_0} \tau$$

exists and is harmonic.

Moreover, $\Omega_0^* - \tau_0^*$ is exact and its norm $\| \Omega_0 \|^2 - \| \tau_0 \|^2 = \| \Omega_0 \|^2 - d$ is > 0. The following theorem has been proved:

Theorem 7. *If* 0 *is the only exact harmonic differential of finite norm on* W_0, *then every harmonic differential* Ω_0 *of finite norm is the limit of Schottky differentials* τ, *and the periods of* Ω_0 *over* 0*-homologous cycles in the non-compact sense are all zero.*

The class of open Riemann surfaces which satisfy the hypothesis of this theorem will be denoted by N_1 (Nevanlinna). A necessary and sufficient condition for a surface to be of class N_1 can easily be given in a rather implicit form.

Let a, b be two points on W_0. For each compact region $\overline{W}$ which contains a and b we determine $d_3(a, b)$ as defined in 2.5, i. e. the minimum of the quantity

$$\frac{\| dU \|^2}{| U(a) - U(b) |^2}$$

for all harmonic functions in $\overline{W}$.

Theorem 8. *W_0 is of class N_1 if and only if*

$$\lim_{W \to W_0} d_3(a, b) = \infty \tag{3.2}$$

for all a, b.

It is quite obvious that the invariant distance $d_3(a, b)$ increases with W. Hence $\lim d_3(a, b)$ will always exist. Suppose first that there exists a non-constant harmonic function U_0 on W_0 with $\| dU_0 \| < \infty$, and chose a, b so that $U_0(a) \neq U_0(b)$. Then we shall evidently have

$$d_3(a, b) \leqq \frac{\| dU_0 \|^2}{| U_0(a) - U_0(b) |^2}$$

for all W, and the limit (3.2) will be finite.

Suppose conversely that the limit is finite. Let $W \subset W'$ and let U, U' denote corresponding extremal functions, normalized by the condition $U(a) - U(b) = U'(a) - U'(b) = 1$. Then dU is orthogonal to $dU - dU'$ over W, and we obtain

$$\| dU' - dU \|_W^2 = \| dU' \|_W^2 - \| dU \|_W^2 \leqq \| dU' \|_{W'}^2 - \| dU \|_W^2 .$$

But $\| dU \|_W^2 = d_3(a, b)$ tends to a finite limit, and the existence of $dU_0 = \lim_{W \to W_0} dU$ follows by familiar reasoning. dU_0 is naturally exact and the square of its norm is equal to the limit of $d_3(a, b)$.

We note that the sufficiency of the condition (3.2) has been proved only when it is known to hold for all a, b. If it holds for a single pair a, b our proof shows that every single valued harmonic function U with $\| dU \| < \infty$ must satisfy the condition $U(a) = U(b)$.

R. Nevanlinna [12] has proved that a surface which has no Green's function is necessarily of class N_1. For a plane region this condition is also necessary. We represent W_0 as the complement of a compact set E in the plane. The Green's function exists if and only if E is of positive capacity. If the capacity is positive a closed subset $E_1 \subset E$ can be found whose capacity is likewise positive and smaller than that of E. The difference $g - g_1$ of the Green's functions which belong to the complements of E and E_1 respectively is a non-constant harmonic function on W_0, and it is easy to verify that $\| dg - dg_1 \|$ is finite. Therefore, if E is of positive capacity, its complement is not of class N_1. It is an open question whether in the general case the class N_1 is effectively larger than the class of surfaces without a Green's function.

3.3. Consider now the problem of determining a harmonic differential with given periods. There is given a finite or infinite set of cycles γ_n, which are linearly independent in the sense of compact homology, and corresponding periods P_n. Does there exist a harmonic differential of finite norm with these periods?

For any compact $\overline{W}$ we can find the Schottky differential τ^* of minimum norm which has the given periods on all γ_n contained in W. It is immediately clear that $\| \tau \|$ increases with W. If our problem has a solution Ω_0, then $\| \tau \| \leqq \| \Omega_0 \|$ for all W, and $\lim\limits_{W \to W_0} \| \tau \| < \infty$. Conversely, if $\lim\limits_{W \to W_0} \| \tau \| < \infty$ we can again prove that $\tau_0^* = \lim\limits_{W \to W_0} \tau^*$ exists and has the prescribed periods. Hence, at least theoretically, the question of existence can always be decided.

If W_0 is of class N_1, and if there exists a linear combination of the γ_n which is homologous to zero in the non-compact sense, while the corresponding combination of the periods is different from zero, then we know that the problem has no solution and we conclude that $\lim \| \tau \| = \infty$.

With regard to uniqueness we can assert that the solution is unique if W_0 is of class N_1 and if the γ_n form a homology basis in the non-compact sense.

In order to conclude that $\lim \| \tau \| < \infty$ it is sufficient to know that there exists a *closed* differential Ω_0 of finite norm with the given periods. We assume that the γ_n are linearly independent in the non-compact sense. Then, if only a finite number of the prescribed periods are different from zero, it is possible to find, by elementary construction, a closed differential Ω_0 with the given periods which vanishes identically outside of a compact set. In this case our problem has therefore always a solution. If W_0 is not equivalent with a plane region it contains at least one cycle which is not homologous to zero in the non-compact sense, and we conclude that the class of harmonic differentials of finite norm does not reduce to the zero differential.

If W_0 is of class N_1 there exists a unique harmonic differential π_n of finite norm with the period 1 over γ_n and all other periods equal to zero. Whenever the series $\sum_1^\infty P_n \pi_n$ converges in norm, it will obviously represent a differential with the periods P_n, but the converse is probably not true. For this reason the differentials π_n are not of very great use in the study of differentials with infinitely many periods.

3.2. The corresponding problems for analytic differentials can be treated in a similar manner. We consider an analytic differential Φ_0 of finite norm and determine on each compact $\overline{W}$ the Schottky differential φ with the same periods as Φ_0. We conclude as before that $\|\varphi\|_W$ increases to a limit $d \leqq \|\Phi_0\|$. If $d = \|\Phi_0\|$, Φ_0 is the limit of φ as W tends to W_0. If $d < \|\Phi_0\|$, $\varphi_0 = \lim \varphi$ will still exist and $\varphi_0 - \Phi_0$ is an exact analytic differential of finite positive norm. We obtain the following counterpart of Theorem 7:

Theorem 8. *If 0 is the only exact analytic differential of finite norm on W_0, then every analytic differential Φ_0 of finite norm is the limit of Schottky differentials φ.*

The class of Riemann surfaces with this property will be denoted by N_2. It is obvious that a surface of class N_1 is also of class N_2, but the class N_2 is actually larger. A sufficient and necessary condition is given by

Theorem 9. *W_0 is of class N_2 if and only if*

$$\lim_{W \to W_0} d_4(a, b) = \infty$$

for all a, b.

The proof parallels that of Theorem 8 and need not be given.

4. Bounded analytic functions

4.1. The class of bounded analytic functions $F(z)$ on a Riemann surface neither contains nor is contained in the class with $\|dF\| < \infty$. This class must hence be made the object of a separate treatment.

In the case of functions on a compact subregion $\overline{W}$ we are led to consider an extremal problem which is closely related to the classical lemma of Schwarz. For plane regions I have already considered this problem in an earlier publication [1][4]). In the present treatment considerable simplifications will be introduced which have their root in an important remark of my student P. Garabedian [4].

We state our problem in the following form:

Problem 8. *Let a and b be two interior points on a compact region* $\overline{W}$. *Among all analytic functions* $F(z)$ *on* $\overline{W}$ *which satisfy the condition* $|F(z)| \leqq 1$ *and vanish at a it is required to find the one which makes* $|F(b)|$ *a maximum.*

It was noted by Garabedian that problems of this type are closely connected with certain associated extremal problems in which an analytic differential Φ with corresponding properties is to be determined. In the present case the associated problem is the following:

Problem 9. *Among all analytic differentials* Φ *on* $\overline{W}$ *with simple poles at a and b which satisfy the condition*

$$\int_C |\Phi| \leqq 2\pi$$

it is required to find the one whose residue at b is of maximum absolute value.

The connection between these problems is brought to light by the following reasoning. Suppose that it is possible to find two analytic differentials φ, θ and an analytic function f on $\overline{W}$, connected by the relation

$$\theta = f\varphi \,, \tag{4.1}$$

with the following properties:

1. θ is real and $\geqq 0$ along C, and its only singularity is a simple pole at b;
2. f is regular with $f(a) = 0$ and $|f| = 1$ on C;
3. φ satisfies
$$\int_C |\varphi| = 2\pi$$

and is regular except for simple poles at a and b.

From these conditions it follows that f and φ are solutions of our problems. In fact, let F and Φ satisfy the conditions of Problems 8 and 9 respectively. Evaluating the residue of $F\Phi$ at b we obtain

[4]) In this paper the proof in section 2.2 is incomplete. The gap can be filled in an obvious, if somewhat inelegant way. The proof given in the present paper is very direct, but not the shortest possible.

$$|F(b)| \cdot |\mathrm{Res}_b \Phi| = \frac{1}{2\pi}\left|\int_C F\Phi\right| \leqq \frac{1}{2\pi}\int_C |\Phi| \leqq 1 ,$$

while on the other hand

$$|f(b)| \cdot |\mathrm{Res}_b \varphi| = \frac{1}{2\pi}\int_C \theta = \frac{1}{2\pi}\int_C |\varphi| = 1 .$$

These relations obviously imply, in a symmetric way,

$$|F(b)| \leqq |f(b)| \tag{4.2}$$

and

$$|\mathrm{Res}_b \Phi| \leqq |\mathrm{Res}_b \varphi| . \tag{4.3}$$

Moreover, in (4.2) equality holds only if $F\varphi$ has a constant argument on C and, simultaneously, $|F| = 1$ on C. It follows that F/f must reduce to a constant of absolute value 1, and the solution to Problem 8 is unique, except for multiplication with such a constant. For Problem 9 the uniqueness of the solution is not asserted.

4.2. We proceed to show that f, φ and θ can be found in accordance with the conditions 1 to 3 of the preceding section. Our method will be to solve Problem 8 within a compact subclass of functions with $|F| = 1$ on C and take f equal to the extremal function in that subclass. It is then shown, by a direct variational argument, that corresponding φ and θ must exist, and hence that f is extremal in the wider class of all bounded functions $(|F| \leqq 1)$.

It must first be proved that the class of functions with $F(a) = 0$ and $|F| = 1$ on C is not empty. In other words, we must show that $\overline{W}$ can be mapped onto a full covering surface of the unit circle. The method by which this is proved will be simpler if we replace the unit circle by the upper halfplane.

We suppose again that $\overline{W}$ is of genus p with $q \geqq 1$ contours and write $P = 2p + q - 1$. Choose $P + 1$ distinct points z_j on C and corresponding real numbers A_j. It is possible to construct an analytic differential θ_0 which is real on C and whose only singularities are double poles at the points z_j with the singular parts

$$A_j \frac{dz}{(z - z_j)^2} ;$$

the local variable z at z_j is chosen so as to map C onto the real axis and points in W into points of the upper half-plane. θ_0 is determined up to a regular differential $\theta \in S_r$. By a proper choice of the latter we can make the periods and half-periods of the imaginary part of θ_0 vanish (cf. the remark which follows immediately after the Corollary to Theorem 2). With this determination of θ_0 we can write $\theta_0 - \bar{\theta}_0 = i\,dG$, where G vanishes on C except at the points z_j.

The inner product (θ_0, θ), where θ is a regular Schottky differential, has a sense if interpreted as a limit over a region obtained from W by leaving out small semi-circles about the z_j. We find easily

$$(\theta_0, \theta) = (\theta_0 - \bar{\theta}_0, \theta) = -\int_C G\bar{\theta} \tag{4.4}$$

where the contour integral is also interpreted as a limit. If we write $\theta = \alpha\, dz$ in a neighborhood of z_j and observe that

$$G \sim i A_j \left(\frac{1}{z - z_j} - \frac{1}{\bar{z} - z_j} \right)$$

it is seen that the singularity at z_j contributes the amount $-\pi A_j \alpha(z_j)$ to the right hand member of (4.4). We obtain

$$(\theta_0, \theta) = -\pi \sum_1^{P+1} A_j \alpha(z_j) \;. \tag{4.5}$$

The necessary and sufficient condition that θ_0 be exact is that $(\theta_0, \theta) = 0$ for all $\theta \in S_r$. Suppose first that $\theta_0 = dF$. Then the same computation as above leads to

$$(\theta_0, \theta) = (\theta_0, \theta + \bar{\theta}) = i \int_C F(\bar{\theta} - \theta) = \pi \sum_1^{P+1} A_j \alpha(z_j)$$

and comparison with (4.5) shows that $(\theta_0, \theta) = 0$. Suppose on the other hand that $(\theta_0, \theta) = 0$ for all $\theta \in S_r$, and let φ be the analytic Schottky differential which makes $\theta_0 - \varphi$ exact. Repeating the above argument, which does not make use of the fact that θ_0 is real along C, we obtain $(\theta_0 - \varphi, \theta) = 0$ and consequently $(\varphi, \theta) = 0$ for all $\theta \in S_r$. This implies $\varphi = 0$ and we conclude that θ_0 is exact.

We show now that it is possible to choose points z_j and non-negative coefficients A_j so that

$$\sum_1^{P+1} A_j \alpha(z_j) = 0 \tag{4.6}$$

for all $\theta \in S_r$. Let θ_i run through a basis of S_r and let the corresponding $\alpha(z_j)$ be denoted by $\alpha_i(z_j)$. Clearly, (4.6) can be satisfied with non-negative coefficients if and only if the simplex with the vertices

$$\left(\alpha_1(z_j), \ldots, \alpha_P(z_j)\right), \qquad j = 1, \ldots, P+1$$

contains the origin. If this were not true for any choice of the z_j, the convex hull of the set of points $\left(\alpha_1(t), \ldots, \alpha_P(t)\right)$ for $t \in C$ would fail to contain the origin. The whole set would then lie in a half-space, and there would exist constants $a_1, \ldots, a_P$, not all equal to zero, such that

$$a_1 \alpha_1(t) + \cdots + a_P \alpha_P(t) \geqq 0$$

for all $t \in C$. In other words, $\theta = a_1 \theta_1 + \cdots + a_P \theta_P$ would be $\geqq 0$ along C in contradiction with the fact that

$$\int_C \theta = 0 \ .$$

We conclude that there must exist a choice of z_j and $A_j \geqq 0$ which satisfies (4.6).

The corresponding θ_0 is exact: $\theta_0 = dF$. F can be chosen real on the boundary, except at the points z_j where the imaginary part becomes positively infinite. It follows from the maximum principle that Im $F > 0$ on the whole of W. The function

$$\frac{F(z) - F(a)}{F(z) - \overline{F(a)}}$$

maps $\overline{W}$ onto a covering surface of the unit circle, and a standard argument shows that every point is covered exactly $P + 1$ times.

We have proved that the class of functions on $\overline{W}$ which are of absolute value 1 on C, vanish at a and take every value at most $P+1$ times is not empty. This class is certainly compact. For reasons which will become apparent we shall let $f(z)$ denote the function which maximizes $|f(b)|$ in the larger class of functions with the same properties which take every value at most $2P + 2$ times. It will be shown that f has in fact at most $P + 1$ zeros and solves the extremal problem in the class of all bounded functions.

4.3. Consider an analytic function $h(z)$ on $\overline{W}$ with the zeros $a_1, \ldots, a_m$. It is clear at once that

$$\log |h(z)| = -\sum_1^m g_j(z) \qquad (4.7)$$

where $g_j(z) = g(z, a_j)$ is the ordinary Green's function with a positive pole at a_j. Conversely, if the a_j are given (4.7) defines a single-valued function $h(z)$ if and only if the periods of

$$\sum_1^m dg_j^* \tag{4.8}$$

are multiples of 2π.

This condition can be expressed in a more convenient form. Let τ be a harmonic Schottky differential of class S_0 ($\tau = 0$ along C). We have already noted that $(\tau, dg_j) = 0$ (cf. the proof of Lemma 4). Consider first the case where τ is exact with a single half-period $\neq 0$. We can write $\tau = du$, where $u = 0$ on all contours except one, C_k, on which the value is a constant u_k. By contour integration we obtain

$$(du, dg_j) = u_k \int_{C_k} dg_j^* - 2\pi u(a_j) = 0$$

and hence

$$2\pi \sum_1^m u(a_j) = u_k \sum_{j=1}^m \int_{C_k} dg_j^* .$$

Consequently, the period of (4.8) over C_k is a multiple of 2π if and only if $\sum_1^m u(a_j)$ is a multiple of u_k.

A similar reasoning applies when τ has one non-vanishing period while the remaining periods and half-periods are zero. In this case u is no longer single-valued, but there is a familiar method of cutting the surface and applying contour integration which yields the following general criterion:

Let the points a_j be joined to C by arcs γ_j. The points a_j are the zeros of a single-valued function $h(z)$ with $|h| = 1$ on C if and only if for every $\tau \in S_0$ the expression

$$\sum_1^m \int_{\gamma_j} \tau$$

is a linear combination with integral coefficients of the periods and half-periods of τ.

It is clear that this condition depends only on the location of the points a_j and not on the choice of the curves γ_j.

4.4. We return to the function $f(z)$ which we assumed to be extremal for Problem 8 within the class of functions with $|F| = 1$ on C and at most $2P+2$ zeros. Let the zeros of $f(z)$ be a and $a_1,\dots,a_m$. It follows from the preceding result that $a, a'_1,\dots,a'_m$ will be the zeros of a competing function if

$$\sum_1^m \int_{a_j}^{a'_j} \tau = 0 \tag{4.9}$$

for all $\tau \in S_0$. By (4.7) and the extremal property of $f(z)$ (4.9) must imply

$$\sum_1^m g(b, a'_j) \geqq \sum_1^m g(b, a_j) \ . \tag{4.10}$$

In other words, the function $\sum_1^m g(b, a'_j)$ has a conditional minimum for $a'_j = a_j$. We observe that this is true regardless of the choice of the paths of integration in (4.9). If a'_j lies in a fixed neighborhood of a_j we can choose the path of integration within that neighborhood, and if we set $\tau = du$ in the respective neighborhoods (4.9) takes the form

$$\sum_1^m u(a'_j) = \sum_1^m u(a_j) \ . \tag{4.11}$$

Let τ run through a set of linearly independent differentials τ_i, $i = 1,\dots,P$, and let the corresponding integrals be denoted by u_i. Using the method of Lagrangean multipliers we conclude that there exists a linear combination

$$u = \lambda_0 g(z, b) + \lambda_1 u_1 + \cdots + \lambda_P u_P$$

whose partial derivatives with respect to the real and imaginary part of each a'_j vanish for $a'_j = a_j$. This means that there exists a differential

$$\lambda_0 \, dg(z, b) + \tau \tag{4.12}$$

with zeros at $a_1,\dots,a_m$. It is understood that the λ_i are not all zero, and we shall show later that $\lambda_0 \neq 0$.

If $f(z)$ has multiple zeros we wish of course to prove that the zeros of (4.12) have at least the same multiplicity[5]). A simple device is needed to take care of this detail. Suppose, for instance, that a_1 is a zero of order r. We set $a_1 = \cdots = a_r$ and apply the preceding argument with $a'_1,\dots,a'_r$ in a neighborhood of a_1, and for the sake of simplicity we choose the

[5]) This minor point was overlooked in [1].

parametric representation so that $a_1 = 0$. Let us introduce the new variables

$$\left.\begin{aligned} s_1 &= a_1' + a_2' + \cdots + a_r' \\ s_2 &= a_1'^2 + a_2'^2 + \cdots + a_r'^2 \\ &\cdots \\ s_r &= a_1'^r + a_2'^r + \cdots + a_r'^r \end{aligned}\right\} .$$

Since the left hand members of (4.10) and (4.11) are symmetric functions of $a_1', \ldots, a_r'$ they can be expressed in terms of $s_1, \ldots, s_r$. It is therefore possible to write down the conditions of Lagrange with $s_1, \ldots, s_r$ as independent variables. Setting

$$u(a_j') - u(a_j) = \mathrm{Re}(\alpha_1 a_j' + \cdots + \alpha_r a_j'^r + \cdots)$$

we obtain

$$\sum_1^m \left(u(a_j') - u(a_j)\right) = \mathrm{Re}\,(\alpha_1 s_1 + \cdots + \alpha_r s_r + \cdots) ,$$

where only the linear terms have been displayed. The fact that the partial derivatives vanish implies $\alpha_1 = \cdots = \alpha_r = 0$, and it follows that (4.12) vanishes at least to the order r. We have proved:

There exists a differential of the form (4.10) *which vanishes at all the zeros of the function $f(z)$, and at least with the same multiplicity.*

The total number of zeros of the differential (4.10) is well determined. In fact, the Schottky differential

$$\theta = \lambda_0(-\,dg^* + i\,dg) + (-\,\tau^* + i\,\tau) \qquad (4.13)$$

has twice as many zeros and poles on $\hat{W}$. On the other hand each differential divisor is known to be of degree $2P - 2$. We conclude that (4.10) has exactly P zeros if $\lambda_0 \neq 0$ and $P - 1$ zeros if $\lambda_0 = 0$. It follows that $m \leqq P$, and $f(z)$ has at most $P + 1$ zeros.

4.5. We shall finally show that there exists a differential of the form (4.13) with $\lambda_0 > 0$ which vanishes at the zeros of f and, in addition, satisfies the condition $\theta \geqq 0$ on C. The proof will be based on a less familiar form of Lagrange's method, which we formulate as

Lemma 6. *Let the functions $h_i(x_1, \ldots, x_n)$ with $h_i(0, \ldots, 0) = 0$, $i = 1, \ldots, m$ be of class C^1 in a neighborhood of the origin, and assume that* *the functional matrix* $\left\| \frac{\partial h_i}{\partial x_k} \right\|$ *is of rank m at the origin. For sufficiently small* $\varrho = (\xi_1^2 + \cdots + \xi_n^2)^{1/2}$ *the system of equations*

$$h_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} \xi_j \frac{\partial h_i(0, \ldots, 0)}{\partial x_j}$$

has then a solution

$$x_j = \xi_j + \varepsilon_j(\varrho) \ ,$$

where $\varepsilon_j(\varrho)/\varrho \to 0$ *for* $\varrho \to 0$. *In particular, if* $\xi_j \neq 0$ *and* ϱ *is sufficiently small* x_j *will have the same sign as* ξ_j.

The proof of the Lemma follows classical lines and will be omitted.

We choose points a_j' near to the zeros $a_1, \ldots, a_m$ of $f(z)$ just as before. In addition we consider arbitrary points $c_1, \ldots, c_n$ on C and let c_k' denote an interior point near c_k. A sufficient condition under which the points a, a_j' and c_k' are the zeros of a function $h(z)$ with $|h| = 1$ on C can be expressed in the form

$$\sum_j \left(u_i(a_j') - u_i(a_j)\right) + \sum_k \left(u_i(c_k') - u_i(c_k)\right) = 0 \ , \qquad (4.14)$$

where u_i is a local integral of τ_i. By the extremal property of f these conditions must imply

$$\sum_j \left(g(a_j', b) - g(a_j, b)\right) + \sum_k \left(g(c_k', b) - g(c_k, b)\right) \geqq 0 \qquad (4.15)$$

provided that $m + n \leqq 2P + 1$, and therefore certainly if $n \leqq P + 1$.

The local variable at a boundary point will be denoted by $s + it$; we assume that the boundary corresponds to the real axis and that interior points correspond to $t > 0$. If c_k corresponds to s_k we choose $c_k' = s_k + i t_k$ so that the location of c_k' depends on only one real variable t_k. The local variables of a_j' will be denoted by (x_j, y_j) and it is no loss of generality to assume that $a_j' = a_j$ corresponds to the combination (0, 0).

We apply Lemma 6 with the left hand members of (4.14) and (4.15) as functions and x_j, y_j, t_k as independent variables. We know that the functions cannot take the value combination $(0, \ldots, 0, -\varrho)$ for positive values of the t_k. We conclude that there *cannot* exist numbers ξ_j, η_j and μ_k with $\mu_k > 0$ (corresponding to the ξ_j of the Lemma) such that

$$0 = \sum_j \left(\xi_j \frac{\partial u_i}{\partial x_j} + \eta_j \frac{\partial u_i}{\partial y_j}\right) + \sum_k \mu_k \frac{\partial u_i}{\partial t_k}$$

$$-\varrho = \sum_j \left(\xi_j \frac{\partial g}{\partial x_j} + \eta_j \frac{\partial g}{\partial y_j}\right) + \sum_k \mu_k \frac{\partial g}{\partial t_k}$$

unless the vectors

$$V_i = \left(\frac{\partial u_i}{\partial x_j}, \frac{\partial u_i}{\partial y_j}, \frac{\partial u_i}{\partial t_k}\right)$$

$$W = \left(\frac{\partial g}{\partial x_j}, \frac{\partial g}{\partial y_j}, \frac{\partial g}{\partial t_k}\right)$$

are linearly dependent.

This condition expresses that the convex cone spanned by the vectors $\pm\left(\frac{\partial u_i}{\partial x_j}, \frac{\partial g}{\partial x_j}\right)$, $\pm\left(\frac{\partial u_i}{\partial y_j}, \frac{\partial g}{\partial y_j}\right)$ and $\left(\frac{\partial u_i}{\partial t_k}, \frac{\partial g}{\partial t_k}\right)$ does not contain the direction $(0, \ldots, 0, -1)$ in its interior. This is true for an arbitrary choice of the points c_k provided that their number does not exceed $P+1$. The vectors lie in a $(P+1)$-dimensional space, and the convex cone spanned by an arbitrary collection of such vectors is the union of convex cones spanned by $P+1$ vectors each. We can therefore conclude that the convex cone spanned by $\pm\left(\frac{\partial u_i}{\partial x_j}, \frac{\partial g}{\partial x_j}\right)$, $\pm\left(\frac{\partial u_i}{\partial y_i}, \frac{\partial g}{\partial y_j}\right)$ and *all* the vectors $\left(\frac{\partial u_i}{\partial t}, \frac{\partial g}{\partial t}\right)$ fails to contain the vector $(0, \ldots, 0, -1)$ in its interior. It follows that this cone is contained in a closed half-space which does not contain the direction $(0, \ldots, 0, -1)$. If the inner normal of this half-space is denoted by $(\lambda_1, \lambda_2, \ldots, \lambda_P, \lambda_0)$, the function

$$U = \lambda_1 u_1 + \cdots + \lambda_P u_P + \lambda_0 g$$

must satisfy the conditions

$$\frac{\partial U}{\partial x_j} = \frac{\partial U}{\partial y_j} = 0$$

at the points $a_1, \ldots, a_m$ and

$$\frac{\partial U}{\partial t} \geqq 0$$

at all boundary points. On the other hand, the fact that the half-space does not contain the direction $(0, \ldots, 0, -1)$ implies $\lambda_0 \geqq 0$.

The corresponding analytic differential

$$\theta = \lambda_0(-\,dg^* + i\,dg) + \sum_1^P \lambda_j(-\,\tau_j^* + i\,\tau_j)$$

vanishes at $a_1, \ldots, a_m$, and on the boundary $\theta = \frac{\partial U}{\partial t}\,ds \geqq 0$. Since θ cannot be identically zero we obtain

$$\lambda_0 = \frac{1}{2\pi}\int_C \theta > 0\ ,$$

and by way of normalization we can take $\lambda_0 = 1$. We have thus found a differential

$$\theta = i(dg + i\,dg^*) + i(\tau + i\,\tau^*)$$

which together with f and $\varphi = \theta/f$ satisfies the conditions 1 to 3 set forth in 4.1. Consequently, f and φ are solutions of Problems 8 and 9.

4.6. We have shown that the extremal function $f(z)$ has at most $P+1$ zeros. On the other hand, $w = f(z)$ must map each contour onto the full circle $|\,w\,| = 1$. It follows that the number of sheets is at least q, and we have the double inequality

$$q \leqq m + 1 \leqq 2p + q\ .$$

In the case of a plane region $p = 0$ and we find $m + 1 = q$.

If $m + 1$ attains its maximal value $2p + q$ the extremal differential φ is uniquely determined. In fact,

$$|\,f(b)\,| \cdot |\,\mathrm{Res}_b\,\Phi\,| = \frac{1}{2\pi}\left|\int_C f\Phi\right|$$

equals its maximum 1 only if the argument of $f\Phi$ is constant on C. If $m = P$ all the zeros of θ would be zeros of f and

$$\frac{\Phi}{\varphi} = \frac{f\Phi}{\theta}$$

would be a regular function whose argument were constant on C. It follows that Φ/φ must reduce to a constant.

These results are summed up as follows:

Theorem 10. *Problem* 8 *has a unique solution* $f(z)$ *which maps* $\overline{W}$ *onto a full covering surface of the unit circle with at least* q *and at most* $2p + q$ *sheets. The zeros of* f, *other than* a, *are zeros of a differential* θ *which is real* *and non-negative along C and has a simple pole with the residue* 1 *at* b. *The differential* $\varphi = \theta / f$ *is a solution of Problem* 9, *and this solution is unique whenever* f *has the maximum number of zeros.*

It has also been proved that the two maxima satisfy the relation

$$\max |F(b)| \cdot \max |\operatorname{Res}_b \Phi| = 1 . \tag{4.16}$$

If W is replaced by a larger region, the first maximum will obviously decrease. By (4.16) the maximum of $|\operatorname{Res}_b \Phi|$ will therefore increase.

4.7. We shall now make use of Theorem 10 for the study of open surfaces. An open surface W_0 will be said to be of class P if every single-valued and bounded analytic function on W_0 reduces to a constant.

For fixed a, b and compact $\overline{W}$ containing a, b as interior points we write

$$M(W) = \operatorname{Max} |F(b)|$$

where F runs through all analytic functions on $\overline{W}$ satisfying $|F(z)| \leqq 1$ and $F(a) = 0$. Because of the monotonic property

$$M = \lim_{W \to W_0} M(W)$$

exists, and it is clear that W_0 belongs to P if and only if $M = 0$ for all a, b.

It follows from (4.16) that on the other hand

$$M(W) = \operatorname{Min} \frac{1}{2\pi} \int_C |\Phi| ,$$

where Φ ranges over all analytic differentials on $\overline{W}$ with simple poles at a, b and the residue 1 at b. If $M > 0$, we conclude that there does not exist any differential with two simple poles with the property that

$$\lim_{W \to W_0} \int_C |\Phi| = 0 .$$

If $M = 0$, i. e. if $W_0 \in P$, we would expect that such a differential exists. Unfortunately, there is no criterion of compactness which would allow us to assert that the extremal differentials φ converge, and even if they converge we cannot show that the limiting differential has the required property. This means that we must be content with the following formulation :

Theorem 11. *An open surface W_0 is of class P if and only if the following holds: Given $\varepsilon > 0$, points a, b and a compact set E, there exists a differential Φ with simple poles at a, b and the residue* 1 *at b, which is defined in a compact region $\overline{W}$ containing a, b and the set E and which satisfies the condition*

$$\int_C |\Phi| < \varepsilon ,$$

where C denotes the boundary of $\overline{W}$.

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