On a Generalization of Tauber's Theorem

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§ 1. Tauber's classical converse of Abel's theorem [7][1]) is essentially a result concerning the Laplace transform; and it has been generalized [4, p. 9] for any transform Φ in which the kernel $\varphi(u)$ satisfies suitable conditions such as the following:

(C). For $u \geq 0$, $\varphi(u)$ is a positive, continuous and monotonic decreasing function,

$$\varphi(0) = 1 \;, \qquad \int^{\infty} \frac{\varphi(u)}{u}\, du \quad \text{is convergent;} \tag{1}$$

for $u \geq 0$, $\varphi(u)$ has a continuous derivative $-\psi(u)$ so that

$$\varphi(u) = \int_u^{\infty} \psi(x)\, dx \,.\ ^{2)} \tag{2}$$

For the Φ-transform the generalization of Tauber's theorem runs as follows.

Theorem A. *Suppose that $A(u)$ is a function of bounded variation in every finite interval of $u \geq 0$ and $A(0) = 0$* [3]). *Suppose also that*

$$\int_0^{\infty} \varphi(ut)\, d\{A(u)\} \quad \textit{is convergent for} \quad t > 0 \;,$$

$$\Phi(t) \equiv \int_0^{\infty} \varphi(ut)\, d\{A(u)\} \to A \quad \textit{as} \quad t \to +0 \,. \tag{3}$$

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[1]) Numbers within square brackets refer to the literature cited at the end.

[2]) From the monotony of φ and the convergence of the integral in (1) it follows that $\varphi(u) \to 0$ as $u \to \infty$,

$$\int_0^{\infty} \psi(u)\, du \;\; \text{exists}, \qquad \varphi(u) = \int_u^{\infty} \psi(x) dx + \text{a constant}$$

where the constant is 0 as we can see by letting $u \to \infty$.

It is usually assumed that, in addition to (1), $\int_0 \frac{1-\varphi(u)}{u}\, du$ exists. This is ensured by our conditions since

$$\lim_{u \to 0} \frac{1-\varphi(u)}{u} = \lim_{u \to 0} -\varphi'(u) \equiv \lim_{u \to 0} \psi(u) \,.$$

[3]) $A(0) = 0$ is part of the usual assumption that $A(u)$ is 'normalized'.

$$\frac{1}{u}\int_0^u x\, d\{A(x)\} = o(1) \quad as \quad u \to \infty . \tag{4}$$

Then

$$A(u) \to A \quad as \quad u \to \infty .$$

It is the main object of this note to extend Theorem A to cases where (3) is not satisfied and (4) is given with O instead of o, in much the same way as Hadwiger [2], Agnew [1] and Wintner [9] have extended Tauber's theorem.

§ 2. Tauber himself gave his theorem in two different forms of which the simpler has been extended by Hadwiger and Agnew, the other has been extended in like manner by Wintner. The former extension, in its integral form, can be further extended to any transform Φ as in

Theorem 1. *Suppose that* $s(u)$ *is integrable and*

$$S(u) = \int_0^u s(u)\, du , \tag{5}$$

$$\left.\begin{aligned} &\underline{\lim}_{u\to\infty}\ u\, s(u) = -K/p , \quad \overline{\lim}_{u\to\infty}\ u\, s(u) = K/q , \\ &K > 0 , \quad p > 0 , \quad q > 0 , \quad p^{-1} + q^{-1} = 1 \end{aligned}\right\} . \tag{6}$$

Suppose that $\varphi(u)$ *satisfies the conditions* (*C*) *of* § 1 *and*

$$F(t) = \int_0^\infty \varphi(ut)\, s(u)\, du \textit{ is convergent for } t > 0 . \tag{7}$$

Then

$$\left.\begin{aligned} &\underline{\lim}_{t\to+0}\left[S\left(\frac{\varphi^{-1}(p^{-1})}{t}\right) - F(t)\right] \geqslant -\frac{K}{p}\tau_\Phi(p) , \\ &\overline{\lim}_{t\to+0}\left[S\left(\frac{\varphi^{-1}(q^{-1})}{t}\right) - F(t)\right] \leqslant \frac{K}{q}\tau_\Phi(q) \end{aligned}\right\} . \tag{8}$$

where

$$\tau_\Phi(x) = \int_0^{\varphi^{-1}(x^{-1})} \frac{1-\varphi(u)}{u}\, du + (x-1)\int_{\varphi^{-1}(x^{-1})}^{\infty} \frac{\varphi(u)}{u}\, du . \ ^{1)} \tag{9}$$

[1]) The existence of the inverse function $\varphi^{-1}(x^{-1})$ is ensured for the values of x in question by supposing that $\varphi(x)$ is monotonic decreasing in the strict sense for at least $0 < x < 1$.

In the particular case $p = q = 2$ *of* (5), (8) *reduces to*

$$\overline{\lim_{t\to+0}} \left| S\left(\frac{\varphi^{-1}(2^{-1})}{t}\right) - F(t) \right| \leqslant \tau_\Phi(2) \overline{\lim_{u\to\infty}} \, | u s(u) | \, . \tag{10}$$

Proof. For any $\delta > 0$, we have the identity

$$S\left(\frac{\delta}{t}\right) - F(t) = \int_0^{\delta/t} u s(u) \frac{1-\varphi(ut)}{u} du - \int_{\delta/t}^{\infty} u s(u) \frac{\varphi(ut)}{u} du$$
$$= \int_0^{\delta} \frac{u}{t} s\left(\frac{u}{t}\right) \frac{1-\varphi(u)}{u} du - \int_{\delta}^{\infty} \frac{u}{t} s\left(\frac{u}{t}\right) \frac{\varphi(u)}{u} du$$

whence, letting $t \to +0$ and appealing to Fatou's theorem, we obtain

$$\left.\begin{aligned} -T(\delta, p) \leqslant \overline{\lim_{t\to+0}} \left[S\left(\frac{\delta}{t}\right) - F(t)\right] \leqslant T(\delta, q) \, , \\ T(\delta, x) = \frac{K}{x}\int_0^{\delta} \frac{1-\varphi(u)}{u} du + K\left(1-\frac{1}{x}\right)\int_{\delta}^{\infty} \frac{\varphi(u)}{u} du \end{aligned}\right\} . \tag{11}$$

(8) can be deduced from (11) by observing that, in the latter, it is most advantageous to have the extreme left-hand member maximum and the extreme right-hand member minimum, considering these members as functions of δ. Now it is easily verified that Min $T(\delta, q)$ corresponds to

$$\frac{K}{q}\,\frac{1-\varphi(\delta)}{\delta} - K\left(1-\frac{1}{q}\right)\frac{\varphi(\delta)}{\delta} = 0 \, , \quad \text{i.e.} \quad \delta = \varphi^{-1}(q^{-1}) \, ,$$

and similarly Max $-T(\delta, p)$ corresponds to $\delta = \varphi^{-1}(p^{-1})$. Therefore using these values of δ in (11), we get (8) as the best form of (11) and thence (10) as a particular case.

The case $\varphi(u) = e^{-u}$ of Theorem 1. In this case $\tau_\Phi(x)$ defined by (9) reduces to

$$\left.\begin{aligned} \tau_j(x) &= \int_0^{\log x} \frac{1-e^{-u}}{u} du + (x-1)\int_{\log x}^{\infty} \frac{e^{-u}}{u} du \qquad (1 < x < \infty) \\ &= \lim_{\epsilon\to 0}\left[\int_{\epsilon}^{\log x} \frac{du}{u} - \int_{\epsilon}^{\infty} \frac{e^{-u}}{u} du\right] + x\int_{\log x}^{\infty} \frac{e^{-u}}{u} du \\ &= -\lim_{\epsilon\to 0}\left(\log \epsilon + \int_{\epsilon}^{\infty} u^{-1} e^{-u} du\right) + \log\log x + x\int_{\log x}^{\infty} u^{-1} e^{-u} du \\ &= \gamma + \log\log x + x\int_{\log x}^{\infty} u^{-1} e^{-u} du \end{aligned}\right\} \tag{9'}$$

where γ is Euler's constant. Consequently, when $\varphi(u) = e^{-u}$, (10) takes the form

$$\overline{\lim_{t \to +0}} \left| S\left(\frac{\log 2}{t}\right) - F(t) \right| \leqslant \left[\gamma + \log\log 2 + 2 \int_{\log 2}^{\infty} u^{-1} e^{-u} du\right] \overline{\lim_{u \to \infty}} |u s(u)| \tag{10'}$$

which is essentially the result of Agnew [1, § 3] and Hadwiger [2] already referred to [5]).

We can argue with (10) as Agnew has argued with (10'). Suppose that ξ' is a limit point of $S(\varphi^{-1}(2^{-1})/t)$ [or of $F(t)$] as $t \to +0$. Then there is a sequence $t_1, t_2, \ldots$ such that $t_n \to +0$ and $S(\varphi^{-1}(2^{-1})/t_n) \to \xi'$ [or $F(t_n) \to \xi'$]. From this and (10) we deduce that $F(t_n)$ [or $S(\varphi^{-1}(2^{-1})/t_n)$] is bounded and therefore there is a subsequence $T_1, T_2, \ldots$ such that $F(T_N)$ [or $S(\varphi^{-1}(2^{-1})/T_N)$] has limit ξ'' when $t \to +0$ over the subsequence, ξ'' satisfying the condition

$$|\xi' - \xi''| \leqslant \tau_\Phi(2) \overline{\lim_{u \to \infty}} |u s(u)| .$$

We thus get

Corollary 1.1. *Let $\varphi(u)$ satisfy the conditions (C) of § 1 and be strictly monotonic for at least $0 < u < 1$. Let $S(u)$ be defined as in (5) and $F(t)$ as in (7). Let L denote the set of limit points of $S(u)$ as $u \to \infty$ and L_Φ the set of limit points of $F(t)$ as $t \to +0$. Then, to each $\xi' \in L$ corresponds a $\xi'' \in L_\Phi$ and conversely such that the inequality*

$$|\xi' - \xi''| \leqslant \tau \overline{\lim_{u \to \infty}} |u s(u)|$$

holds for $\tau \geqslant \tau_\Phi(2)$, provided $0 \leqslant \overline{\lim_{u \to \infty}} |u s(u)| < \infty$.

In particular $L = L_\Phi$ when $\lim_{u \to \infty} u s(u) = 0$.

The proof of (8) establishes at the same time

Corollary 1.2. *In Theorem 1 we have, in addition to (8), the conclusion:*

$$\underline{\lim_{t \to +0}} F(t) - \frac{K}{p} \tau_\Phi(p) \leqslant \overline{\lim_{u \to \infty}} S(u) \leqslant \overline{\lim_{t \to +0}} F(t) + \frac{K}{q} \tau_\Phi(q) .$$

[5]) Hadwiger's paper is not accessible to me and all my information about it is derived from *Mathematical Reviews*.

The argument leading up to (11) shows that, in the limiting case in which $p \to \infty (q \to 1)$,

$$\varliminf_{t \to +0} F(t) - K \int_{\delta}^{\infty} \frac{\varphi(u)}{u} du \leqslant \varlimsup_{u \to \infty} S(u) \leqslant \varlimsup_{t \to +0} F(t) + K \int_{0}^{\delta} \frac{1 - \varphi(u)}{u} du$$

whence we obtain, letting $\delta \to \infty$ in the extreme left-hand member and $\delta \to 0$ in the extreme right-hand member,

$$\varliminf_{t \to +0} F(t) \leqslant \varlimsup_{u \to \infty} S(u) \leqslant \varlimsup_{t \to +0} F(t) \; .$$

This, in conjunction with the Abelian result

$$\varliminf_{u \to \infty} S(u) \leqslant \varliminf_{t \to +0} F(t) \leqslant \varlimsup_{u \to \infty} S(u) \; ^{6})$$

yields, in our limiting case,

$$\varliminf_{u \to \infty} S(u) = \varliminf_{t \to +0} F(t) \; , \quad \varlimsup_{u \to \infty} S(u) = \varlimsup_{t \to +0} F(t) \; .$$

We have therefore proved

Corollary 1.3. *If* $\varphi(u)$ *satisfies the conditions* (*C*) *of* § 1 *and if* $S(u)$, $F(t)$ *are as in* (5), (7) *respectively, then*

$$0 \leqslant \varlimsup_{u \to \infty} u\, s(u) < \infty$$

implies

$$\underset{u \to \infty}{osc}\; S(u) = \underset{t \to +0}{osc}\; F(t) \; .$$

This is a result of Minakshisundaram [5, Theorem 3].

Note. All the results proved thus far hold for the undermentioned kernels besides Laplace's kernel $\varphi(u) = e^{-u}$.

$\varphi(u) = (1-u)^k, \; k \geqslant 1, \; 0 < u < 1; \; \varphi(u) = 0, \; u \geqslant 1$ [Riesz kernel].
$\varphi(u) = (1+u)^{-k}, \; k > 0, \; u \geqslant 0$ [Stieltjes kernel].
$\varphi(u) = u/(e^u - 1) \; , \; u > 0$ [Lambert kernel].

[6]) This can be proved on the supposition that $\varphi(u)$ is positive, differentiable with continuous derivative, monotonic decreasing in $(0, \infty)$,

$$\varphi(0) = 1 \; , \quad \varphi(\infty) = 0 \; .$$

In particular, Corollary 1.1 establishes the existence of an absolute constant $\tau_\Phi(2)$ associated with each of the Φ-transforms involving these kernels.

§ 3. The next theorem generalizes Wintner's result mentioned in the beginning of § 2, in the same way as Theorem 1 generalizes the corresponding result Hadwiger and Agnew.

Theorem 2. *Suppose that $A(u)$ is a function of bounded variation in every finite interval of $u \geqslant 0$ and $A(0) = 0$,*

$$\left.\begin{array}{l} \varliminf\limits_{u\to\infty} \dfrac{1}{u}\displaystyle\int_0^u x\,d\{A(x)\} = -\dfrac{K}{p}\,, \quad \varlimsup\limits_{u\to\infty} \dfrac{1}{u}\displaystyle\int_0^u x\,d\{A(x)\} = \dfrac{K}{q}\,, \\ K>0\,, \quad p>0\,, \quad q>0\,, \quad p^{-1}+q^{-1}=1 \end{array}\right\}. \tag{12}$$

Suppose further that $\varphi(u)$ satisfies the conditions (C) of § 1 and is such that

$$\Phi(t) \equiv \int_0^\infty \varphi(u\,t)\,d\{A(u)\} \qquad \textit{exists for} \quad t>0\,. \tag{13}$$

Then

$$\left.\begin{array}{l} \varliminf\limits_{t\to+0}\left[A\left(\dfrac{\varphi^{-1}(p^{-1})}{t}\right)-\Phi(t)\right] \geqslant -\dfrac{K}{p}\,[\tau_\Phi(p)+p]\,, \\ \varlimsup\limits_{t\to+0}\left[A\left(\dfrac{\varphi^{-1}(q^{-1})}{t}\right)-\Phi(t)\right] \leqslant \dfrac{K}{q}\,[\tau_\Phi(q)+q]\,, \end{array}\right\} \tag{14}$$

where $\tau_\Phi(x)$ is defined by (9).

In the proof of this theorem we require, in addition to Theorem 1,

Lemma 1. *If*

$$\left.\begin{array}{l} \varliminf\limits_{u\to\infty} b(u) = -\dfrac{K}{p}\,, \quad \varlimsup\limits_{u\to\infty} b(u) = \dfrac{K}{q}\,, \\ K>0\,, \quad p>0\,, \quad q>0\,, \quad p^{-1}+q^{-1}=1\,, \end{array}\right\}$$

then, for any $\delta>0$,

$$-K \leqslant \varlimsup\limits_{t\to+0}\left\{b\left(\frac{\delta}{t}\right) - t\int_0^\infty \psi(u\,t)\,b(u)\,du\right\} \leqslant K$$

where ψ is defined by (2).

The proof of this lemma is obvious and left to the reader.

Proof of Theorem 2. Comparing (12) with (6), we find that we can choose, in Theorem 1,

$$s(u) = \frac{1}{u^2} \int_0^u x \, d\{A(x)\} \tag{15}$$

$$= \frac{u A(u) - A_1(u)}{u^2}, \qquad A_1(u) \equiv \int_0^u A(x) \, dx .$$

With this choice of $s(u)$ in Theorem 1, we have

$$\left. \begin{aligned} S(u) &= \int_0^u \left\{ \frac{A(x)}{x} - \frac{A_1(x)}{x^2} \right\} dx \\ &= \left[\frac{A_1(x)}{x} \right]_0^u = \frac{A_1(u)}{u} \;{}^{7)} \end{aligned} \right\} \tag{16}$$

after an integration by parts. And

$$F(t) = \int_0^\infty \varphi(ut) \, d\left\{\frac{A_1(u)}{u}\right\} \equiv \Phi_1(t) \ , \quad \text{say} . \tag{17}$$

When we substitute for $S(u)$, $F(t)$ from (16), (17) in the conclusion (8) of Theorem 1, we get

$$\left. \begin{aligned} \varliminf_{t \to +0} \left[\frac{A_1(u_p)}{u_p} - \Phi_1(t) \right] &\geqslant -\frac{K}{p} \tau_\Phi(p) \ , \qquad u_p = \frac{\varphi^{-1}(p^{-1})}{t} \ , \\ \varlimsup_{t \to +0} \left[\frac{A_1(u_q)}{u_q} - \Phi_1(t) \right] &\leqslant \frac{K}{q} \tau_\Phi(q) \ , \qquad u_q = \frac{\varphi^{-1}(q^{-1})}{t} \end{aligned} \right\} . \tag{18}$$

Now $\Phi(t)$ defined by (13) and $\Phi_1(t)$ defined by (17) are connected by the relation

$$\Phi(t) - \Phi_1(t) = \int_0^\infty \varphi(ut) \, d\left\{ A(u) - \frac{A_1(u)}{u} \right\}$$

$$= \left[\varphi(ut) \left\{ A(u) - \frac{A_1(u)}{u} \right\} \right]_{u=0}^{u=\infty} + t \int_0^\infty \psi(ut) \left\{ A(u) - \frac{A_1(u)}{u} \right\} du .$$

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7) We suppose, as we may without loss of generality, that $A(x) = O(x)$ as $x \to +0$ and therefore $A_1(x) = O(x^2)$.

Since, for every $t>0$, $\varphi(u\,t) \to 0$ as $u \to \infty$ and also $A(u) - A_1(u)/u = O(1)$ as $u \to \infty$ in consequence of (12), the last step gives

$$\Phi(t) - \Phi_1(t) = t \int_0^\infty \psi(ut) \left\{ A(u) - \frac{A_1(u)}{u} \right\} du \quad . \tag{19}$$

(19) and (12) show that we can take $b(u) = A(u) - A_1(u)/u$ in Lemma 1 and reach the conclusion

$$\left.\begin{aligned} \varliminf_{t \to +0} \left\{ A(u_p) - \frac{A_1(u_p)}{u_p} - \Phi(t) + \Phi_1(t) \right\} &\geqslant -K \ , \quad u_p = \frac{\varphi^{-1}(p^{-1})}{t} \ , \\ \varlimsup_{t \to +0} \left\{ A(u_q) - \frac{A_1(u_q)}{u_q} - \Phi(t) + \Phi_1(t) \right\} &\leqslant \ \ K \ , \quad u_q = \frac{\varphi^{-1}(q^{-1})}{t} \ . \end{aligned}\right\} \tag{20}$$

Combining the first half of (20) with that of (18) and the second helf of (20) with that of (18), we establish (14) and so complete the proof.

Remark. It is obvious that Theorem 2 has a corollary analogous to Corollary 1.1, which extends Theorem A in precisely the same way as the case $\varphi(u) = e^{-u}$ of Corollary 1.1 (due to Hadwiger and Agnew) extends Tauber's simpler theorem in its integral form.

§ 4. The theorem in this section is a 'converse' theorem on the Laplace and Riesz transforms, very similar to the 'converse' theorem in § 3 on the Φ-transform. More precisely, the former theorem assumes a Tauberian condition wider than (12) and relates the behaviour of the rth Riesz transform of $A(u)$, $r \geqslant 0$, to the behaviour of the Laplace transform of $A(u)$, $A(u)$ being a function of bounded variation in every finite interval of $u \geqslant 0$. The Laplace transform of $A(u)$ is of course

$$J(t) = \int_0^\infty e^{-ut}\, d\{A(u)\} \ , \qquad t>0 \ ; \tag{21}$$

while the rth Riesz transform of $A(u)$ is defined by

$$\left.\begin{aligned} \sigma_r(u) \equiv \frac{A_r(u)}{u^r} \ , \qquad A_r(u) \equiv r \int_0^u (u-x)^{r-1} A(x)\, dx \ , \qquad r>0 \ , \\ \sigma_0(u) \equiv A_0(u) \equiv A(u) \quad . \end{aligned}\right\} \tag{22}$$

The theorem to be proved depends on Theorem 1 and Lemma 1 as well as on the two lemmas which follow.

Lemma 2. *$A_r(u)$ defined by (22) is integrable for $r > 0$ and*

$$A_{r+1}(u) = \frac{1}{r+1}\int_0^u A_r(u)\,du \quad .$$

This result is well-known [3].

Lemma 3. *Let $\sigma_k(u)$, $k \geqslant 1$, be defined by (22) and $J(t)$ by (21). Then*

$$J_k(t) = t\int_0^\infty e^{-ut}\sigma_k(u)du \quad \textit{is convergent (absolutely) for } t > 0 \;, \tag{23}$$

$$\varliminf_{t\to+0} J(t) \leqslant \varlimsup_{t\to+0} J_k(t) \leqslant \varlimsup_{t\to+0} J(t) \;. \tag{24}$$

Proof. Since we may suppose that $A(0) = 0$ [8]), we can write $J(t)$ in the form

$$J(t) = t\int_0^\infty e^{-ut}A(u)\,du = \frac{t^{k+1}}{\Gamma(k+1)}\int_0^\infty e^{-ut}A_k(u)\,du \quad , \quad k \geqslant 1 \;,$$

either integral converging (absolutely) for $t > 0$. From this (23) follows at once.

To prove (24) we note that, when $t > 0$,

$$\left.\begin{aligned} \Gamma(k+1)\int_t^\infty \frac{J(x)}{x^{k+1}}(x-t)^{k-1}\,dx &= \int_t^\infty (x-t)^{k-1}\,dx\int_0^\infty e^{-ux}A_k(u)\,du \\ &= \int_0^\infty A_k(u)\,du\int_t^\infty e^{-ux}(x-t)^{k-1}\,dx \\ &= \Gamma(k)\int_0^\infty e^{-ut}\frac{A_k(u)}{u^k}\,du \end{aligned}\right\} \tag{25}$$

the interchange of the order of integration in (25) being justified by Fubini's theorem. From (25)

[8]) The supposition just means, in case $A(0) \neq 0$, we consider $A(u) - A(0)$ instead of $A(u)$ and consequently $A_k(u) - A(0)\,u^k$, $\sigma_k(u) - A(0)$ instead of $A_k(u)$, $\sigma_k(u)$ respectively. It is, however, usual to assume that $A(u)$ is normalized and so $A(0) = 0$.

$$
\left.\begin{aligned}
J_k(t) &= kt\int_t^{t^{1/2}} \frac{J(x)}{x^{k+1}}(x-t)^{k-1}\,dx + kt\int_{t^{1/2}}^{\infty} \qquad (t<1)\\
&= I_1 + I_2\ ,
\end{aligned}\right\} \tag{26}
$$

where

$$
I_1 \leqslant \overline{\text{bound}}_{t\leqslant y\leqslant t^{1/2}}\ J(x)\,(1-t^{1/2})^k\ , \qquad I_2 \leqslant \overline{\text{bound}}_{t^{1/2}\leqslant x<\infty}\ J(x)\left[1-(1-t^{1/2})^k\right]. \tag{27}
$$

We may first suppose that $J(t)$ is bounded as $t \to +0$ [9]; and use (27) in (26) letting $t \to +0$. We then obtain

$$
\overline{\lim_{t\to+0}}\ J_k(t) \leqslant \overline{\lim_{t\to+0}}\ J(t)\ .
$$

and complete the proof of (24) in the case $J(t) = O(1)$, $t \to +0$, by a repetition of the above argument involving $\underline{\text{bound}}\ J(x)$.

The case $\underline{\lim}\, J(t) = -\infty$, $\overline{\lim}\, J(t) = \infty$, is trivial. The case $\underline{\lim}\, J(t) = -\infty$, $\overline{\lim}\, J(t) =$ finite, requires merely the proof of $\overline{\lim}\, J_k(t) \leq \overline{\lim}\, J(t)$ which is supplied above; and the case $\underline{\lim}\, J(t) =$ finite, $\overline{\lim}\, J(t) = \infty$, is dealt with similarly. In the case $\underline{\lim}\, J(t) = \overline{\lim}\, J(t) = -\infty$, our argument shows that $\overline{\lim}\, J_k(t) = -\infty$, i. e. $\lim J_k(t) = -\infty$ which is the conclusion sought. The conclusion in the remaining case $\overline{\lim}\, J(t)$ $\underline{\lim}\, J(t) = \infty$ follows from a similar argument.

We can now prove

Theorem 3. *Let $A(u)$ be a function of bounded variation in every finite interval of $u \geqslant 0$ and let $A_r(u)$ defined by* (22) *satisfy*

$$
\left.\begin{aligned}
&\underline{\lim_{u\to\infty}}\ u^{-k-1}B_k(u) = -K/p\ , \quad \overline{\lim_{u\to\infty}}\ u^{-k-1}B_k(u) = K/q\ .\\
&B_k(u) \equiv (k+1)\{uA_k(u) - A_{k+1}(u)\}\ . \qquad k \geqslant 0\ .\\
&K>0\ ,\quad p>0\ ,\quad q>0\ ,\quad p^{-1}+q^{-1}=1\ .
\end{aligned}\right\} \tag{28}
$$

Then $J_k(t)$ defined by (23) *is convergent for $t>0$ even if $0<k<1$ and related as below to $\sigma_k(u)$ defined by* (22):

$$
\left.\begin{aligned}
\underline{\lim_{t\to+0}}\left\{\sigma_k\!\left(\frac{\log p}{t}\right) - J_k(t)\right\} &\geqslant -\frac{K}{p}\left\{\tau_j(p) + \frac{p}{k+1}\right\},\\
\overline{\lim_{t\to+0}}\left\{\sigma_k\!\left(\frac{\log q}{t}\right) - J_k(t)\right\} &\leqslant \frac{K}{q}\left\{\tau_j(q) + \frac{q}{k+1}\right\},
\end{aligned}\right\} \tag{29}
$$

[9]) This means, by virtue of the known fact $J(\infty) = A(+0)$, $|J(x)| < K$ for $0 < x < \infty$.

where $\tau_j(x)$ *is defined by* (9'). *Further*

$$\varliminf_{t\to+0} J(t) - \frac{K}{p}\left\{\tau_j(p) + \frac{1}{k+1}\right\} \leqslant \varlimsup_{u\to\infty} \sigma_k(u) \leqslant \varlimsup_{t\to+0} J(t) + \frac{K}{q}\left\{\tau_j(q) + \frac{1}{k+1}\right\} . \tag{30}$$

Proof. From Lemma 3 we know that when either $k = 0$ or $k \geqslant 1$, $J_k(t)$ defined by (23) is convergent for $t > 0$. When $0 < k < 1$, the same conclusion follows from the identity

$$J_k(t) = J_{k+1}(t) + \frac{t}{k+1}\int_0^\infty e^{-ut} u^{-k-1} B_k(u)\, du$$

in which the two integrals on the right are (absolutely) convergent by virtue of (23) and (28).

To prove (29) we note that, on account of (28), (6) is satisfied with

$$s(u) = u^{-k-2} B_k(u) .$$

This choice of $s(u)$, along with $\varphi(u) = e^{-u}$, defines the $S(u)$ and $F(t)$ of Theorem 1 thus.

$$\begin{aligned} S(u) &= \int_0^u \{(k+1)A_k(x)x^{-k-1} - (k+1)A_{k+1}(x)x^{-k-2}\}\, dx \\ &= [A_{k+1}(x)x^{-k-1}]_0^u = \sigma_{k+1}(u) \text{ [10]} \end{aligned}$$

after an integration by parts in which we use Lemma 2;

$$F(t) = \int_0^\infty e^{-ut} u^{-k-2} B_k(u)\, du = t\int_0^\infty e^{-ut}\sigma_{k+1}(u)\,du = J_{k+1}(t) .$$

After this, we can complete the proof of (29), arguing exactly as in the proof of Theorem 2, but with $\sigma_{k+1}(u)$, $\sigma_k(u)$, $J_{k+1}(t)$, $J_k(t)$, $\tau_j(x)$ instead of $A_1(u)/u$, $A(u)$, $\Phi_1(t)$, $\Phi(t)$, $\tau_\Phi(x)$ respectively.

To prove (30) we substitute, in the conclusion of Corollary 1.2, the expressions for $S(u)$, $F(t)$ in the preceding paragraph. The result is

$$\varliminf_{t\to+0} J_{k+1}(t) - \frac{K}{p}\tau_j(p) \leqslant \varlimsup_{u\to\infty} \sigma_{k+1}(u) \leqslant \varlimsup_{t\to+0} J_{k+1}(t) + \frac{K}{q}\tau_j(q) .$$

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[10]) As in footnote [7]) we may suppose that $A(x) = O(x)$ as $x \to +0$ and thus $A_r(x) = O(x^{r+1})$.

This, in conjunction with (28) in the form

$$\underline{\lim}_{u\to\infty} \{\sigma_k(u) - \sigma_{k+1}(u)\} = -\frac{K}{p(k+1)}, \quad \overline{\lim}_{u\to\infty} \{\sigma_k(u) - \sigma_{k+1}(u)\} = \frac{K}{q(k+1)}$$

and (24) in the form

$$\underline{\lim}_{t\to+0} J(t) \leqslant \underline{\lim}_{t\to+0} J_{k+1}(t) \leqslant \overline{\lim}_{t\to+0} J(t) , \qquad k \geqslant 0 ,$$

leads us to (30).

The case $K = 0$ of Theorem 3 deserves separate mention as

Corollary 3.1. *If, in Theorem* 3, (28) *takes the particular form*

$$\lim_{u\to\infty} u^{-k-1} B_k(u) = 0 , \qquad k \geqslant 0 ,$$

the conclusion (29) *becomes the assertion that the limit points of* $\sigma_k(u)$ *as* $u \to\infty$ *are identical with those of* $J_k(t)$ *as* $t\to +0$ *and* (30) *reduces to*

$$\operatorname*{osc}_{u\to\infty} \sigma_k(u) = \operatorname*{osc}_{t\to+0} J(t) .$$

Another noteworthy case of Theorem 3 is

Corollary 3.2. *In Theorem* 3 *we can write the hypothesis* (28) *briefly in the form*

$$\overline{\lim}_{u\to\infty} | u^{-k-1} B_k(u) | < \infty , \qquad k \geqslant 0 ,$$

and, in addition thereto, supposing that

$$\lim_{t\to+0} J(t) = \pm \infty ,$$

deduce from (30) *the conclusion*

$$\lim_{u\to\infty} \sigma_k(u) = \pm \infty .$$

The above corollaries are in the same class as a theorem I have proved elsewhere [6] to the effect that *if* (*in the notation of Theorem* 3) $\underline{\lim}\, u^{-k-1} B_k(u) > -\infty$, *then* $J(t) \to A$ *as* $t \to +0$ *involves* $\sigma_{k+1}(u) \to A$ *as* $u \to \infty$.

Theorem 2 shows that, in the case $k = 0$ of Corollary 3.2, $J(t)$ defined as in (21) can be replaced by $\Phi(t)$ defined as in (13). Results of this kind for the Φ-transform can be reformulated expressly for series. For instance, the particular result just referred to can be reformulated as under.

Let

$$\Phi(t) = \sum_{n=1}^{\infty} a_n \varphi(\lambda_n t) , \quad t>0 , \quad 0<\lambda_1<\lambda_2<\cdots , \quad \lambda_n \to\infty ,$$

where $\varphi(u)$ *is the kernel of either Laplace or Stieltjes or Lambert.*

Let $A(u)$ *be the* λ*-step function:*

$$A(u)=a_1+a_2+\cdots+a_n \text{ for } \lambda_n\leq u<\lambda_{n+1} , \quad A(u)=0 \text{ for } 0\leq u<\lambda .$$

Then the hypothesis

$$\text{either} \quad \overline{\lim_{n\to\infty}} \, | a_1 \lambda_1 + a_2 \lambda_2 + \cdots + a_n \lambda_n |/\lambda_n<\infty ,$$

$$\text{or (more particularly)} \quad \overline{\lim_{n\to\infty}} \, | a_n \lambda_n |/(\lambda_n - \lambda_{n-1})<\infty ,$$

$$\text{and} \quad \lim_{t\to+0} \Phi(t) = \pm\infty ,$$

together lead to the conclusion

$$\lim_{u\to\infty} A(u) = \pm\infty .$$

The last result reduces to a theorem of Vijayaraghavan [8, Theorem 1] when $\varphi(u) = e^{-u}$, $\lambda_n = n$.

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