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# A Theorem on Orientable Surfaces in Four-Dimensional Space

By SHIING-SHEN CHERN and E. SPANIER, Chicago

1. *Introduction.* Let $M$ be a closed oriented surface differentiably imbedded in a Euclidean space $E$ of four dimensions. Let $G$ denote the Grassmann manifold of oriented planes through a fixed point $O$ of $E$. It is well known that $G$ is homeomorphic to the topological product $S_1 \times S_2$ of two 2-spheres. By mapping each point $P$ of $M$ into the oriented plane through $O$ parallel to the oriented tangent plane to $M$ at $P$, we define a mapping $t$: $M \to G$. If $M$, $S_1$, $S_2$ denote also the fundamental cycles of the respective manifolds and $t_*$ denotes the homomorphism induced by $t$, we have

$$t_*(M) \sim u_1 S_1 + u_2 S_2 \ .$$

In a recent paper[1]) Blaschke studied the situation described above by methods of differential geometry and proved that the sum $u_1 + u_2$ equals the Euler characteristic of $M$. He also asserted that $u_1 = u_2$. The object of this note is to give a proof of this assertion, as well as a new proof of the theorem on $u_1 + u_2$.

2. *Review of some known results on sphere bundles.* Let $B$ be an oriented sphere bundle of $d$-spheres over a base space $X$ with projection $f$. The relation between the homology properties of $B$ and $X$ are summarized in the following exact sequence[2]);

$$\cdots \to H^p(X) \overset{f^*}{\to} H^p(B) \overset{\psi}{\to} H^{p-d}(X) \overset{\cup\,\Omega}{\to} H^{p+1}(X) \to \cdots$$

where each $H$ denotes a cohomology group relative to a coefficient group which is the same for all the terms of the sequence. The homomorphisms that occur in the sequence can be described briefly as follows:

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1) *Blaschke, W.*, Ann. Mat. Pura Appl. (4) 28, 205—209 (1949).

2) *Gysin, W.*, Comm. Math. Helv. 14, 61—122 (1942). — *Chern, S. S. and Spanier, E. H.*, Proc. Nat. Acad. Sci., U. S. A. 36, 248—255 (1950).

$f^*$ is the dual homomorphism induced by the projection $f$; $\psi$ is a mapping which amounts to „integrating over the fiber"; the third homomorphism is the cup product with the characteristic class $\Omega$ (with integer coefficients) of the bundle. From this sequence we see that if, for every coefficient system, the fiber $S^d \nsim 0$ in $B$ then the unit element 1 of the integral cohomology ring of $X$ is in the image of $\psi$ and $\Omega = 0$.

Let $E$ be oriented. Over the oriented surface $M \subset E$ there are two vector bundles, the *tangent bundle* of tangent vectors and the *normal bundle* of normal vectors. By taking unit vectors we get two bundles of circles over $M$. According to a theorem of Seifert and Whitney[3]) the characteristic class of the normal bundle is zero. Since this theorem holds in a more general situation and can be proved in a simple way, we state and prove the theorem for the general case[4]).

**Theorem.** Let $M$ be an orientable manifold imbedded in a Riemann manifold $M'$. If $M \sim O$ in $M'$, then the characteristic class of the normal bundle of $M$ in $M'$ is zero.

*Proof.* Let $B$ be a small tube around $M$. $B$ is then the normal bundle of $M$. We will show that no fiber $S$ of $B$ bounds in $B$. Assume that $S = \partial C$ in $B$ mod $p$ for some $p$. Let $D$ be the set of normal vectors of length $\leqq \epsilon$ having $S$ as boundary. Then $C - D$ is a cycle mod $p$ in $M'$ intersecting $M$ in exactly one point. This is impossible because $M \sim O$ in $M'$.

The above theorem also follows easily from results of Thom[5]).

3. *Plücker coordinates in $G$.* Let $e_1, e_2, e_3, e_4$ be an orthonormal base for $E$ such that $e_1 \wedge e_2 \wedge e_3 \wedge e_4$[6]) is the orientation of $E$. If $R$ is any oriented plane of $E$, let $f_1, f_2$ be an orthonormal base in $R$ such that $f_1 \wedge f_2$ is the orientation of $R$. Then

$$f_1 \wedge f_2 = a_{12}\, e_1 \wedge e_2 + a_{23}\, e_2 \wedge e_3 + a_{31}\, e_3 \wedge e_1 + a_{34}\, e_3 \wedge e_4 + a_{14}\, e_1 \wedge e_4 + a_{24}\, e_2 \wedge e_4 \,.$$

These "Plücker coordinates" $a_{ij}$ of $R$ are independent of the choice of $f_1, f_2$ and satisfy the two relations

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3) *Seifert, H.*, Math. Zeitschr. 41 (1936) 1—17. — *Whitney, H.*, Lectures in Topology, Univ. of Mich. Press (1941) 101—141.

4) We owe this simple description of the proof to Professor H. Hopf, who also called our attention to the problem settled in this paper.

5) *Thom, R.*, C. R. Paris 230, 507—508 (1950).

6) The wedge denotes Grassmann multiplication as in Bourbaki, N., Algèbre Multilinéare, Hermann, Paris (1948).

$$a_{12} a_{34} + a_{23} a_{14} + a_{31} a_{24} = 0 \quad (1)$$

$$\Sigma a_{ij}^2 = 1 \ . \quad (2)$$

Conversely, any set of six real numbers satisfying (1) and (2) are the Plücker coordinates of some oriented plane in $E$; hence, $G$ is homeomorphic to the subset of six space consisting of $a_{ij}$ such that (1) and (2) hold. We introduce a linear change of coordinates by

$$x_1 = a_{12} + a_{34} \qquad x_2 = a_{23} + a_{14} \qquad x_3 = a_{31} + a_{24}$$

$$y_1 = a_{12} - a_{34} \qquad y_2 = a_{23} - a_{14} \qquad y_3 = a_{31} - a_{24} \ .$$

Then $G$ is homeomorphic to the subset of six space consisting of $(x_i, y_j)$ such that $\Sigma x_i^2 = \Sigma y_j^2 = 1$.

Let $S_1$, $S_2$ be the unit spheres in the $x$-space and $y$-space respectively. We orient $S_1$ and $S_2$ by the orientations $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ of the $x$-space and $y$-space. Let $h$: $G \to S_1 \times S_2$ be the homeomorphism defined above using the Plücker coordinates.

Let $\alpha$; $G \to G$ map each oriented plane $R$ into its normal plane $R'$, oriented so that $R$, $R'$ determine the given orientation of $E$. We want to determine the mapping $h \alpha h^{-1}$: $S_1 \times S_2 \to S_1 \times S_2$. If $R$ has Plücker coordinates $a_{ij}$ and $R'$ has Plücker coordinates $b_{ij}$, it is easy to see that the following equations are satisfied

$$\sum_k a_{ik} b_{jk} = 0 \qquad (i \neq j)$$

$$\Sigma a_{ij} b_{kl} = 1 \ ,$$

the last summation being taken over all even permutations of 1, 2, 3, 4.

It follows from these that $b_{ij} = a_{kl}$, where $i, j, k, l$ is an even permutation of 1, 2, 3, 4. Therefore, we see that

$$h \alpha h^{-1}(x, y) = (x, -y)$$

where $-y$ denotes the antipodal point to $y$.

4. *The Theorem.* Let $M$ be a closed oriented surface in $E$. Let $t$: $M \to G$ and $n$: $M \to G$ be the maps defined by taking tangent planes and normal planes respectively. It is clear that $t = \alpha n$ and $n = \alpha t$.

Over $G$ there is a bundle of circles obtained by considering as the fiber over an oriented plane through $O$ the unit circle in that plane. Let $\Omega$ denote the characteristic class of this bundle and let $\Omega_t$, $\Omega_n$ denote the characteristic classes of the tangent and normal bundles of $M$. Then

$$t^* \Omega = \Omega_t \ , \qquad n^* \Omega = \Omega_n \ .$$

The bundle of circles over $G$ defined above is the Stiefel manifold $V$ of ordered pairs of orthogonal unit vectors through $O$ in $E$ and is easily seen to be homeomorphic to $S^2 \times S^3$. The following section of Gysin's sequence

$$H^1(V) \overset{\psi}{\to} H^0(G) \overset{\cup\Omega}{\to} H^2(G) \overset{f^*}{\to} H^2(V) \overset{\psi}{\to} H^1(G)$$

shows that $\Omega$ is a generator of the kernel of $f^*$ in $H^2(V)$, since $H^1(V)$ and $H^1(G)$ are trivial. To find the kernel of $f^*$ we determine the homomorphism

$$f_* : \; H_2(V) \to H_2(G)$$

of the second homology groups.

A generating 2-cycle in $V$ is $S^2 \times e_4$. The points $z$ of $S^2$ can be represented as vectors of the form $z_1 e_1 + z_2 e_2 + z_3 e_3$. Then

$$f\,(\sum_{i=1}^{3} z_i\, e_i,\, e_4) = \sum z_i (e_i \wedge e_4)$$

and so

$$h\, f\,(\Sigma\, z_i\, e_i,\, e_4) = (z,\, -z)\ .$$

Therefore, we see that $f_* (S^2 \times e_4) = S_1 - S_2$. If $S_1^*$, $S_2^*$ denote cohomology classes dual to the homology classes $S_1$, $S_2$, then the kernel of $f^*$ consists of all elements of the form $u\,(S_1^* + S_2^*)$ where $u$ is an integer. Orient $S_1$ and $S_2$ so that $\Omega = S_1^* + S_2^*$. Orient $M$ so that $\Omega_t \cdot M = \chi_M$ $=$ Euler characteristic of $M$. Then

$$\Omega_t = t^*\,(S_1^* + S_2^*) = t^*\, S_1^* + t^*\, S_2^*$$

and

$$\Omega_n = n^*\,(S_1^* + S_2^*) = t^*\, \alpha^*\,(S_1^* + S_2^*) = t^*\,(S_1^* - S_2^*) = t^*\, S_1^* - t^*\, S_2^*\ .$$

Since $\Omega_n = 0$, we see that

$$(t^*\, S_1^*) \cdot M = (t^*\, S_2^*) \cdot M = (\tfrac{1}{2})\, \chi_M\ .$$

We summarize the above results in the theorem:

*Let $M$ be a closed orientable surface in four space $E$. Let $G$ be the Grassmann manifold of oriented planes through $O$ in $E$ and let $t\colon\ M \to G$ be the map into oriented planes through $O$ parallel to the tangent planes of $M$. Since $G$ is homeomorphic to $S_1 \times S_2$, we have $t_*(M) = u_1\, S_1 + u_2\, S_2$. Then $S_1$, $S_2$ and $M$ can be oriented so that $u_1 = u_2 = (\tfrac{1}{2})\, \chi_M$* where $\chi_M$ *is the Euler characteristic of $M$.*

5. *Remarks.* The above theorem expresses relations between differential topological invariants of surfaces imbedded in Euclidean space

and suggests a more general problem. To describe the general situation let $M^k \subset E^{k+l}$ be a manifold of dimension $k$ differentiably imbedded in a Euclidean space of $k + l$ dimensions. Let $G(k, l)$ be the Grassmann manifold of $k$-dimensional linear spaces through a point $O$ and $G(l, k)$ that of $l$-dimensional linear spaces through $O$. There is a natural homeomorphism

$$\alpha : \; G(k, l) \to G(l, k) \; .$$

Using tangent planes and normal planes to $M$ we define mappings

$$t : \; M \to G(k, l) \; , \qquad n : \; M \to G(l, k)$$

such that

$$t = \alpha^{-1} n \; , \qquad n = \alpha t \; .$$

The general problem is to study the relation between the homomorphisms

$$\left.\begin{array}{l} t^* ; \; H^p(G(k, l)) \to H^p(M) \\ n^* ; \; H^p(G(l, k)) \to H^p(M) \end{array}\right\} p = 0, 1, \ldots$$

We hope to study this question on a later occasion.

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