On the Structure and Applications of the STEENROD Algebra.

On the Structure and Applications of the STEENROD Algebra

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§ 1. Introduction

This paper contains a proof of the following theorem in homotopy-theory.

Theorem 1.1. *If* $\Pi_{2n-1}(S^n)$ *and* $\Pi_{4n-1}(S^{2n})$ *both contain elements of* HOPF *invariant one, then* $n \leqslant 4$.

This theorem, of course, is only significant if n is of the form 2^m. We note that it yields an independent proof of the following theorem of H. TODA [5]; there is no element of HOPF invariant one in $\Pi_{31}(S^{16})$.

The author conjectures that this theorem can be improved; it is included mainly to motivate and illustrate the methods here introduced.

These methods depend on a certain spectral sequence. It leads, roughly speaking, from the cohomology of the (mod p) STEENROD algebra[1]) to the p-components of the stable homotopy groups of spheres. This spectral sequence may by regarded, on the one hand, as an extension of ADEM's method of studying homotopy groups by considering cohomology operations of the second and higher kinds. On the other hand, it may be regarded as a reformulation of the method of killing homotopy groups.

Theorem 1.1 follows from a superficial study of this spectral sequence. It requires, however, some knowledge (though very little) of the cohomology of the STEENROD algebra. Our methods for studying the cohomology of the STEENROD algebra depend on a thorough knowledge of the structure of the STEENROD algebra. This is obtained by classical methods (cf. [2])[1a]).

[1]) The (mod p) STEENROD algebra, where p is a prime, has as generators the symbols β_p, P_p^k if $p > 2$, Sq^k if $p = 2$. The relations are those which are universally satisfied by the BOCKSTEIN boundary β_p and the STEENROD operations P_p^k or Sq^k in the cohomology of topological spaces. See [2].

The cohomology of the STEENROD algebra is defined below.

[1a]) Note added in proof. I learn that J. W. MILNOR has made an elegant study of the structure of the STEENROD algebra, which overlaps in content with § 5.

§ 2. Summary of Results and Methods

We will next summarise these methods in more detail. We will consider the spectral sequence, and its multiplicative structure; we will give some data on the cohomology of the STEENROD algebra, and show how to deduce Theorem 1.1. We begin with the spectral sequence.

Let X be a space, let $S^n X$ be the iterated suspension of X, and let $\Pi_m^S(X)$ be the stable or S-homotopy group $\operatorname{Dir Lim} \Pi_{m+n}(S^n X)$. Let p be a prime, and let K^m be the subgroup of $\Pi_m^S(X)$ which consists of elements whose order is finite and prime to p. Let $H_*(X)$ be the (augmented) singular homology group of X; we suppose $H_t(X)$ finitely generated for each t. We make the convention that when we write "cohomology", it means "cohomology with Z_p coefficients", and when we write "$H^*(X)$" it means "$H^*(X; Z_p)$", the (augmented) singular cohomology group of X with Z_p coefficients. The group $H^*(X)$, then, has the structure of a left module over A, the (mod p) STEENROD algebra. We give Z_p the trivial A-module structure. That is, the unit in A acts as a unit, while $a(Z_p) = 0$ if $a \in A_q$ with $q > 0$. (Here the grading q of the STEENROD algebra $A = \sum_q A_q$ is defined by $\deg \beta_p = 1$, $\deg P_p^k = 2k(p-1)$, $\deg Sq^k = k$.) The group $\operatorname{Ext}_A^*(H^*(X), Z_p)$ is now defined[2]). It is bigraded; the grading s is the grading of Ext_A^s, while the grading t arises from the grading of $H^t(X)$ and that of the STEENROD algebra A[3]). With these notations, we have the following theorem.

Theorem 2.1. *There is a spectral sequence, with terms $E_r^{s,t} = E_r^{s,t}(X)$ which are zero if $s < 0$ or if $t < s$, and with differentials*

$$d_r : E_r^{s,t} \to E_r^{s+r, t+r-1}$$

satisfying the following conditions.

(i) *There is a canonical isomorphism*

$$E_2^{s,t} \cong \operatorname{Ext}_A^{s,t}(H^*(X), Z_p) .$$

(ii) *There is a canonical isomorphism*

$$E_{r+1}^{s,t} \cong H^{s,t}(E_r ; d_r) .$$

(iii) *There is a canonical monomorphism from $E_R^{s,t}$ to $E_r^{s,t}$ for $s < r < R \leqslant \infty$.*

(iv) *If (using (iii)) we regard $E_r^{s,t}$ as a subgroup of $E_{s+1}^{s,t}$ for $s < r \leqslant \infty$, we have*

$$E_\infty^{s,t} = \bigcap_{s<r<\infty} E_r^{s,t} .$$

[2]) See [4].
[3]) See § 3.

(v) *There exist groups* $B^{s,t}$ *such that*

$$B^{s,t} \subset B^{s-1,t-1} \subset \cdots \subset B^{0,t-s}, \quad B^{0,m} = \Pi_m^S(X)$$

and

$$E_\infty^{s,t} \cong B^{s,t}/B^{s+1,t+1} .$$

(vi) $\bigcap_{t-s=m} B^{s,t} = K^m$.

This is the spectral sequence referred to in § 1. The convergence statements (iv) and (vi) are not needed for the proof of Theorem 1.1.

If we take $X = S^0$, then $\Pi_m^S(X)$ becomes the stable group of the m-stem, and the term E_2 becomes $\operatorname{Ext}_A^*(Z_p, Z_p)$. We shall write this $H^*(A)$, and refer to it as the cohomology of the Steenrod algebra.

We turn next to the products in this spectral sequence.

Theorem 2.2. *If* $X = S^0$, *then it is possible to define products*

$$E_r^{s,t} \otimes E_r^{s',t'} \to E_r^{s+s',t+t'}$$

(in the spectral sequence of Theorem 2.1) with the following properties.

(i) *The products are associative, and anticommutative for the degree* $t - s$.

(ii) *The product* $E_2^{s,t} \otimes E_2^{s',t'} \to E_2^{s+s',t+t'}$ *coincides, except for a sign* $(-1)^{ts'}$, *with the cup-product*[4])

$$H^{s,t}(A) \otimes H^{s',t'}(A) \to H^{s+s',t+t'}(A) .$$

(iii) $d_r(uv) = (d_r u)v + (-1)^{(t-s)} u(d_r v)$.

(iv) *The products commute with the isomorphisms* $E_{r+1}^{s,t} \cong H^{s,t}(E_r; d_r)$ *and with the monomorphisms from* $E_R^{s,t}$ *to* $E_r^{s,t}$ (*if* $s < r < R \leqslant \infty$).

(v) *The products in* E_∞ *may be obtained by passing to quotients from the composition product*

$$\Pi_m^S(S^0) \otimes \Pi_{m'}^S(S^0) \to \Pi_{m+m'}^S(S^0) .$$

We offer next some remarks on the interpretation of these theorems. We should explain that it is possible to define a filtration F_s of $\Pi_m^S(S^0)$ by considering cohomology operations of higher kinds. We consider only those operations which act on cohomology with Z_p coefficients. Let $\alpha : S^{n+m} \to S^n$ be a map. Form a complex $K = S^n \cup E^{n+m+1}$ by using α as an attaching map. Then $H^n(K) = Z_p$ and $H^{n+m+1}(K) = Z_p$, at least so long as α induces the zero map of cohomology. Suppose that the following condition holds: if Φ is any non-trivial stable cohomology operation of the r^{th} kind, with $r < s$, and of degree $(m + 1)$, then $\Phi : H^n(K) \to H^{n+m+1}(K)$ is defined and zero. Then we set $\alpha \in F_s$; this defines $F_s \subset \Pi_m^S(S^0)$. It is a subgroup, and $F_s \supset F_{s+1}$. We

[4]) This cup-product will be defined in § 4.

complete the definition by setting $\alpha \in F_1$ if α induces the zero map of cohomology.

The author supposes that this filtration coincides with that given by Theorem 2.1 (in case $X = S^0$); that is, $F_s = B_{s,s+m}$. However, he has not tried to prove this proposition, which is not material to this paper.

We have next to compare the classical method of killing homotopy groups with the method of calculation provided by Theorem 2.1. It is clear that both rely on the information contained in CARTAN's calculation of $H^*(\Pi;n)$. However, from accounts of the classical method, one obtains the impression that it enables one to calculate a great deal, but that one cannot guarantee in advance exactly how much. With the formalism of Theorem 2.1 the situation is more clear; we can effectively compute the term E_2 (to any finite dimension); we cannot, at present, give a convenient method for effective computation of the differentials d_r, or of the group extensions involved.

In the case $X = S^0$, it is possible to obtain information about the group extensions involved in $\Pi_m^S(S^0)$ from Theorem 2.2; for this theorem will in particular inform us about the composite of an element in $\Pi_m^S(S^0)$ and the element of degree p in $\Pi_0^S(S^0)$.

The author's interest is particularly attracted to the phenomena which arise because the differentials d_r may not be zero; it will appear that Theorem 1.1 is a case in point.

It is clear that in order to study the spectral sequence of Theorems 2.1, 2.2 we shall need some information at least about the term E_2. Although $H^*(A)$ is defined by means of resolutions, to study it in this way seems unrewarding. We therefore employ the spectral sequence which relates the cohomology rings $H^*(\Lambda)$, $H^*(\Gamma)$ and $H^*(\Omega)$ of an algebra Γ, a normal subalgebra Λ and the corresponding quotient algebra Ω [5]). To make use of this spectral sequence, we prove results on the structure of the STEENROD algebra. Some of these concern a descending sequence of subalgebras A^r of the STEENROD algebra $A = A^1$. These are such that

(i) $\bigcap_r A^r$ has the unit as a base.

(ii) A^r is a normal subalgebra of A^s if $r > s$; and A^s is then a free module over A^r.

(iii) The quotient $A^r//A^{r+1}$ is an algebra whose cohomology is known.

Such results enable one to apply the method of calculation indicated. They form a large part of the technical labour of this paper.

We will show next how Theorem 1.1 can be deduced (using Theorems 2.1 and 2.2) from a very superficial knowledge of $H^*(A)$ (in the case $p = 2$). We

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[5]) See [4], p. 349.

write $H^s(A) = \sum_t H^{s,t}(A)$, and catalogue various facts we suppose known. First, we suppose it known that $H^0(A)$ has as a base the element 1 in dimension $t = 0$, and that $H^1(A)$ has as a base elements h_m in dimension $t = 2^m$, $m = 0, 1, 2, \dots$ [6]).

We further suppose known the following lemma.

Lemma 2.3. *If there is an element of Hopf invariant one in $\Pi_{2n-1}(S^n)$, whose S-class is h'', then $n = 2^m$, and the class h' of h'' in $E_\infty^{1,n}$ passes by the canonical monomorphism to the class h_m in $E_2^{1,n}$.*

Conversely, if h_m lies in the image of $E_\infty^{1,2^m}$, then there is an element of Hopf invariant one in $\Pi_{2n-1}(S^n)$ for $n = 2^m$.

This lemma will follow as soon as the spectral sequence is set up. In case $m = 0$, the lemma is still true with a suitable interpretation; we may take (and now define) a "map of Hopf invariant one in $\Pi_1(S^1)$" to be the map of degree two [7]).

Lastly, we will assume the following theorem on $H^*(A)$ (in case $p = 2$).

Theorem 2.4. *The products $h_i h_j$ in $H^2(A)$ are subject to the following relation only: –*

$$h_i h_{i+1} = 0 .$$

The products $h_i h_j h_k$ in $H^3(A)$ are subject to the following three relations only: –

$$h_i h_{i+1} h_j = 0 \ , \quad (h_i)^2 h_{i+2} = (h_{i+1})^3 \ , \quad h_i (h_{i+2})^2 = 0 \ .$$

We will now deduce Theorem 1.1. Let us suppose (for a contradiction) that h''_m, h''_{m+1} are S-classes containing maps of Hopf invariant one, and that $m \geqslant 3$. Consider the element $h_0(h_m)^2$ in $E_2^{3,2^{m+1}+1}$. By Theorem 2.4 it is non-zero. It is a cycle for d_2 by Theorem 2.2. It is not a boundary for d_2, because $d_2 E_2^{1,2^{m+1}}$ is generated by $d_2 h_{m+1}$, and this is zero because h_{m+1} is in the image of $E_\infty^{1,2^{m+1}}$ (Lemma 2.3). Therefore $h_0(h_m)^2$ yields a non-zero element in $E_3^{3,2^{m+1}+1}$. This implies that, in $\Pi^S_{2^{m+1}-2}$, the element $h''_0(h''_m)^2$ is non-zero, that is, $2(h''_m)^2$ is non-zero. But since composition in stable homotopy groups is anti-commutative and the dimension of h''_m is odd, we have $2(h''_m)^2 = 0$. This contradiction proves Theorem 1.1.

We note as a corollary of the proof, that the differential $d^2 \colon E_2^{1,16} \to E_2^{3,17}$ maps h_4 to $h_0(h_3)^2$, and is thus non-zero. This remark may be paraphrased as follows.

Corollary 2.5. *If Sq^{16} is considered as a cohomology operation of the second kind, it has a non-trivial decomposition.*

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[6]) These are related to the elements Sq^{2^m} in A, and will be defined at the beginning of § 6.

[7]) When we use this map as an attaching map, Sq^1 is non-zero in the resulting complex.

The decomposition asserted is of the form

$$Sq^{16}u = \sum_i a_i \Phi_i(u) \quad (\mathrm{mod}\, Q) \quad (\text{if } u \in K).$$

Here $a_i \in \sum_{1 \leqslant j \leqslant 15} A_j$, Φ_i is of the second kind, and for the space X concerned we have

$$H^*(X) \supset K \supset \bigcap_{\substack{a \in A_j \\ 1 \leqslant j \leqslant 15}} \mathrm{Ker}\, a\,, \quad H^*(X) \supset Q \subset \sum_{\substack{a \in A_j \\ 1 \leqslant j \leqslant 15}} \mathrm{Im}\, a\,.$$

Such a decomposition evidently shows that it is impossible to form a complex $S^n \cup E^{n+16}$ in which Sq^{16} is non-zero. It would be interesting to know whether such decompositions can be proved directly by ADEM's method, or by any other method.

Theorem 2.4 has the following obvious corollary.

Corollary 2.6. *If h''_i, h''_j, h''_k are S-classes of dimensions $2^i - 1$, $2^j - 1$, $2^k - 1$ containing elements of HOPF invariant one, then the S-classes $h''_i h''_j$ and $h''_i h''_j h''_k$ are non-zero except perhaps in the following cases (where h''_0 is to be interpreted as the class 2ι of dimension 0).*

$$h''_i h''_{i+1}\,, \quad h''_i h''_{i+1} h''_j\,, \quad h''_i (h''_{i+2})^2\,, \quad h''_0 (h''_j)^2\,.$$

The case which concerns $(h''_i)^2$ is due to ADEM [1].

This concludes our summary of results and methods.

§ 3. The Spectral Sequence

In this section we prove Theorem 2.1 by constructing the spectral sequence. We do this, roughly speaking, by taking the homotopy exact couple of a sequence $Y_0 \supset Y_1 \supset Y_2 \ldots$ of spaces. These are such that Y_0 is equivalent to an iterated suspension $S^n X$, and $\sum_s H^*(Y_s, Y_{s+1})$ (with the cohomology boundary) is an A-free resolution of $H^*(X)$. Actually we only obtain this property for a finite number of dimensions at one time, as we have to keep to a stable range. We shall therefore consider a finite sequence $Y_0 \supset Y_1 \supset \cdots \supset Y_k$ of spaces which have the required properties in a finite range of dimensions, specified by a parameter l. By increasing k and l we obtain increasing portions of the spectral sequence. The reader will lose little (except the details needed for rigour) if he replaces formulae containing k and l by suitable phrases containing the words "sufficiently large". We conclude by proving the convergence of the spectral sequence.

We proceed to give the details. Let X be a space and C a chain complex of left A-modules, with augmentation onto $H^*(X)$: –

$$\sum_t H^t(X) \overset{\varepsilon}{\leftarrow} \sum_t C_{0,t} \overset{d}{\leftarrow} \sum_t C_{1,t} \leftarrow \cdots \leftarrow \sum_t C_{s,t} \leftarrow \cdots$$

Here, for example, C may be an acyclic resolution of $H^*(X)$ by free left A-modules: it is understood that a free bigraded module has a base whose elements are bihomogeneous. The algebra A is still the (mod p) STEENROD algebra. The second grading t is to be preserved by d and ε and to become the topological dimension in $H^t(X)$; the operations of A on C satisfy

$$A_q \cdot C_{s,t} \subset C_{s,t+q} .$$

We suppose that $C_{s,t} = 0$ if $t < s$ and that each $C_{s,t}$ is finitely generated; it is always possible to find resolutions satisfying these conditions (recall that $H^t(X)$ is finitely generated).

Suppose given also integers k, l. By a *realisation* of the resolution C, we understand an integer n ($n \geqslant l + 1$) and a sequence $Y_0 \supset Y_1 \supset \cdots \supset Y_k$ of CW-complexes and subcomplexes with the following properties.

(1) Y_0 and $S^n X$ are of the same singular homotopy type. (This induces isomorphisms [8]) $i: H^t(X) \cong_A H^{n+t}(Y_0)$).

Y_s is $(n-1)$-connected (for $0 \leqslant s \leqslant k$); $\Pi_r(Y_s, Y_{s+1})$ is finite and p-primary (for all r, $0 \leqslant s < k$).

(2) There are isomorphisms φ: $C_{s,t} \cong_A H^{n+t-s}(Y_s, Y_{s+1})$ for $0 \leqslant s < k$, $t \leqslant l$.

(3) The following diagrams are commutative (for $t \leqslant l$ and for $s + 1 < k$, $t \leqslant l$ respectively).

$$\begin{array}{ccc} C_{s,t} & \xleftarrow{\quad d \quad} & C_{s+1,t} \\ \downarrow \varphi & & \downarrow \varphi \\ H^{n+t-s}(Y_s, Y_{s+1}) & \xleftarrow{(-1)^n d} & H^{n+t-s-1}(Y_{s+1}, Y_{s+2}) \end{array} \qquad \begin{array}{ccc} H^t(X) & \xleftarrow{\varepsilon} & C_{0,t} \\ \downarrow i & & \downarrow \varphi \\ H^{n+t}(Y_0) & \longleftarrow & H^{n+t}(Y_0, Y_1) \end{array}$$

We note that a realisation for some l is also a realisation for any less l. Similarly, from a realisation we can obtain realisations with less k, by ignoring some subspaces Y_s, or with greater n, by suspension. This is the reason for the sign $(-1)^n d$; it is inserted so that the diagram is preserved on suspension. (The suspension isomorphism is defined using a coboundary map, and it anticommutes with other coboundary maps.)

We know that resolutions of $H^*(X)$ exist; it is necessary to prove that realisations of them exist.

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[8]) The symbol $\cong_A$ indicates an isomorphism commuting with the operations from A.

Lemma 3.1. *Let C be an acyclic resolution of $H^*(X)$ by free left A-modules (as above); let k, l be integers. Then X, C, k, l have a realisation.*

Proof. Suppose given X, C, k and l, as above. We take $n = l + 1$. Let $Z_{s,t}$ be the subgroup of cycles in $C_{s,t}$. Let $\mathrm{Hom}_A(C_{s,t}, Z_p)$ stand by convention for the "component" of $\mathrm{Hom}_A(C, Z_p)$ in dimension (s,t), that is, the image of $\mathrm{Hom}_A(\sum_u C_{s,u}, Z_p)$ in $\mathrm{Hom}(C_{s,t}, Z_p)$. Suppose, as an inductive hypothesis, that we have defined a space F_s such that

$$H^{n+t-s-1}(F_s) \cong_A Z_{s,t} \quad \text{for} \quad t<n \ .$$

Take a space B_{s+1} with

$$\Pi_{n+t-s-1}(B_{s+1}) \cong \mathrm{Hom}_A(C_{s+1,t}, Z_p) \ , \qquad k^{n+t-s}(B_{s+1}) = 0 \ .$$

Since $\sum_t C_{s+1,t}$ is a free left A-module and $C_{s+1,t}$ is finitely generated, we deduce that

$$H^{n+t-s-1}(B_{s+1}) \cong_A C_{s+1,t} \quad (\text{for } t - s - 1 < n) \ .$$

Take also a CW-complex F'_s of the same singular homotopy type as F_s, and take a (singular) equivalence[9]) $F'_s \to F_s$. We may now take a map $f_{s+1} : F'_s \to B_{s+1}$ such that the following diagram is commutative (for $t<n$).

$$\begin{array}{ccc} H^{n+t-s-1}(F'_s) & \xleftarrow[f^*_{s+1}]{} & H^{n+t-s-1}(B_{s+1}) \\ \downarrow \cong_A & & \downarrow \cong_A \\ Z_{s,t} & \xleftarrow{(-1)^n d} & C_{s+1,t} \end{array}$$

Factor the map f_{s+1} through an equivalence and a fibration; we obtain a fibre-space $F_{s+1} \to E_{s+1} \to B_{s+1}$ and a (singular) equivalence e_s: $E_{s+1} \to F_s$. The spectral sequence of the fibre-space reduces to an exact sequence in the low dimensions, and we easily show that

$$H^{n+t-s-2}(F_{s+1}) \cong_A Z_{s+1,t} \quad \text{for} \quad t<n \ .$$

This induction is started (with $s = -1$) by interpreting F_{-1} as $S^n X$ and $(-1)^n d$: $C_{0,t} \to Z_{-1,t}$ as ε: $C_{0,t} \to H^t(X)$. We use it to define pairs E_s, F_s for $s \leqslant k - 1$, with singular equivalences e_s: $E_s \to F_{s-1}$.

Let Y'_s be the total mapping cylinder of the maps e_v for $v \geqslant s$; it is obtained from

$$F_{s-1} \cup \bigcup_{s \leqslant v \leqslant k-1} I \times E_v$$

by identifying $0 \times x$ with $1 \times e_v(x)$ if $x \in E_v$ for $v>s$, and $0 \times x$ with $e_s(x)$ if $x \in E_s$. We have embeddings $Y'_0 \supset Y'_1 \supset \cdots \supset Y'_k$.

[9]) A singular equivalence is a map inducing isomorphisms of all homotopy groups.

It is clear that Y'_0 is equivalent to $S^n X$, and that $\Pi_r(Y'_s, Y'_{s+1})$ is finite and p-primary. Applying the Hurewicz isomorphism mod p to F_s, we see that Y'_s is $(n-1)$-connected. With a routine use of spectral theory, we see that

$$H^{n+t-s}(Y'_s, Y'_{s+1}) \cong_A C_{s,t} \quad (\text{for } t-s<n) .$$

We have also two commutative diagrams, namely those set out in the definition of a realisation as condition (3). These follow on setting up the appropriate inclusive diagrams, and hold for $t<n$, $t-s<n$ respectively.

It remains only to replace these spaces Y'_s by CW-complexes Y_s. The proof of Lemma 3.1 is complete.

It is clear that a resolution admits more than one realisation. Therefore, if we construct anything from a realisation, we must prove a uniqueness theorem. Such a theorem will follow by standard naturality arguments from two naturality lemmas, which we give as lemmas 3.4, 3.5. We preface them with some remarks on realisations.

Let C be an acyclic left A-complex, and let $\{Y_s\}$ be a realisation of it; we have the following lemma.

Lemma 3.2. (i) *For* $s<k$, $t\leqslant l$ *we have the following commutative diagram, in which the columns are isomorphisms.*

$$\begin{array}{ccc} H^{n+t-s}(Y_s, Y_{s+1}) & \xleftarrow{d} & H^{n+t-s-1}(Y_{s+1}) \\ \downarrow \cong_A & & \downarrow \cong_A \\ C_{s,t} & \longleftarrow & Z_{s,t} \end{array}$$

ii) *For* $s<k$, $t\leqslant l$ *the map* $H^{n+t-s}(Y_s) \to H^{n+t-s}(Y_{s+1})$ *is zero.*

This lemma is proved by a trivial induction over s, using the exact cohomology sequence of the pair (Y_s, Y_{s+1}).

Now let C be a complex of free left A-modules, and let $\{Y_s\}$ be a realisation of it; we have the following lemma.

Lemma 3.3. (i) *For complexes W of dimension* $<n+l$, *the compression of a map* $f: W\to Y_s$ *into* Y_{s+1} *is equivalent to its compression to a point in* Y_s/Y_{s+1}.

(ii) *If* $t<l$, *then*

$$\Pi_{n+t-s}(Y_s, Y_{s+1}) \cong \Pi_{n+t-s}(Y_s/Y_{s+1}) \cong \mathrm{Hom}_A(C_{s,t}, Z_p)$$

$$k^{n+t-s+1}(Y_s/Y_{s+1}) = 0 .$$

Proof. The first part depends only on the data that Y_s and Y_{s+1} are $(n-1)$-connected and (by the definition of a realisation) $n\geqslant l+1$. As for the second, the projection

$$\Pi_r(Y_s, Y_{s+1}) \to \Pi_r(Y_s/Y_{s+1})$$

is isomorphic for $r \leqslant 2n - 2$, hence for $r < n + l$. For a prime p' distinct from p the p'-component of $\Pi_r(Y_s/Y_{s+1})$ is zero, by inspecting homology. Since $\sum_t C_{s,t}$ is A-free, there is a map from Y_s/Y_{s+1} to

$$\times_t K(\mathrm{Hom}_A(C_{s,t}, Z_p), n + t - s)$$

which induces isomorphisms of cohomology in dimensions $\leqslant n + l - s$.

Next suppose that we have the following data.

(1) $f: X \to Z$ is a map.

(2) C, D are left A-complexes, with augmentations onto $H^*(X)$, $H^*(Z)$; C is acyclic and D is A-free.

(3) $\{Y_s\}$, $\{W_s\}$ are realisations of C, D.

We will suppose that these realisations have the same n, k and l; this will be sufficient for our purposes, by remarks above. Then we have the following lemma.

Lemma 3.4. *There is a map $g: Y_0 \to W_0$ equivalent to $S^n f$ with*

$$g(Y_s^{n+l-s}) \subset W_s^{n+l-s} \quad (\text{for } s \leqslant k) .$$

We postpone the proof of this lemma until we have stated Lemma 3.5.

Next suppose that we have the following data.

(1) $\{Y_s\}$, $\{W_s\}$ are realisations, as above.

(2) g_0, g_1 are homotopic maps, with

$$g_\varepsilon(Y_s^{n+l-s}) \subset W_s^{n+l-s} \quad (\text{for } s \leqslant k,\ \varepsilon = 0, 1) .$$

Then we have the following lemma.

Lemma 3.5. *There is a homotopy $h: g_0 \sim g_1$ with*

$$h(I \times Y_s^{n+l-s}) \subset W_{s-1}^{n+l-s+1} \quad (0 < s \leqslant k) .$$

It is clear that the map g constructed by Lemma 3.4 will yield a map $g^*: D_{s,t} \to C_{s,t}$ of resolutions (at least for $s < k$, $t < l - 1$). Similarly, the homotopy constructed by Lemma 3.5 will yield a homotopy $h^*: D_{s-1,t} \to C_{s,t}$ between two such maps.

Proof of Lemma 3.4. Let us assume that $f: X \to Z$, C, D, $\{Y_s\}$ and $\{W_s\}$ are as given in the data. There is some map $g: Y_0 \to W_0$ equivalent to $S^n f: S^n X \to S^n Z$; we have to examine the obstruction to compressing it so that Y_s^{n+l-s} maps into W_s^{n+l-s}. Suppose we have compressed it so that

$$g(Y_u^{n+l-u}) \subset W_u^{n+l-u} \quad \text{for} \quad u \leqslant s.$$

By Lemma 3.3 we can deform $g \mid Y_{s+1}^{n+l-s-1}$ through W_s into W_{s+1} if (and only if) the map

$$H^{n+t-s}(W_s/W_{s+1}) \to H^{n+t-s}(Y_{s+1})$$

is zero (for $t<l$). But this map can be factored through

$$H^{n+t-s}(Y_s) \to H^{n+t-s}(Y_{s+1}) ,$$

which is zero by Lemma 3.2. This completes the proof of Lemma 3.4 by induction over s.

The proof of Lemma 3.5 is analogous to that of Lemma 3.4; the obstruction is the composite map

$$H^{n+t-s+1}(W_{s-1}/W_s) \to H^{n+t-s+1}(I \times Y_s, \dot{I} \times Y_s)$$
$$\to H^{n+t-s+1}(I \times Y_{s+1}, \dot{I} \times Y_{s+1}) .$$

We next proceed to obtain the spectral sequence of Theorem 2.1. As indicated above, this is that determined by the homotopy exact couple of the complexes Y_s. Each particular group or homomorphism in it may be obtained from some realisation with finite k and l. Corresponding terms obtained from different realisations may be identified, using homomorphisms constructed with the use of Lemma 3.4.

The details are as follows. Let X be a space, C an acyclic A-free resolution of $H^*(X)$, and let $\{Y_s\}$ be a realisation of C with $k \geqslant s + r$, $l>r + t$. Let $G_r^{s,t}$, $D_r^{s,t}$ be the images by i, d of $\Pi_{n+t-s}(Y_s, Y_{s+r})$, $\Pi_{n+t-s+1}(Y_{s-r+1}, Y_s)$ in $\Pi_{n+t-s}(Y_s, Y_{s+1})$. (If $s - r + 1<0$, Y_{s-r+1} is to be interpreted as Y_0.) Then we may define $E_r^{s,t} = G_r^{s,t}/D_r^{s,t}$.

Similarly, let $\{Y_s\}$ be a realisation with $k \geqslant s + 1$, $l>t + 1$. Let $G_\infty^{s,t}$, $D_\infty^{s,t}$ be the images of $\Pi_{n+t-s}(Y_s)$, $\Pi_{n+t-s+1}(Y_0, Y_s)$ in $\Pi_{n+t-s}(Y_s, Y_{s+1})$. Then we may define $E_\infty^{s,t} = G_\infty^{s,t}/D_\infty^{s,t}$.

The map $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$ is obtained (if both groups are defined) by passing to the quotient from the homotopy boundary $(-1)^n d$. (The sign is introduced so that d_r is preserved on suspension.)

If we consider only the terms which can be obtained from a single realisation, the formal properties of a spectral sequence are easily verified. In particular, we have $E_{r+1}^{s,t} \cong H^{s,t}(E_r; d_r)$; there is a canonical monomorphism $E_R^{s,t} \to E_r^{s,t}$ for $s<r<R \leqslant \infty$; and $E_\infty^{s,t} \cong B^{s,t}/B^{s+1,t+1}$, where $B^{s,t}$ is the image of $\Pi_{n+t-s}(Y_s)$ in $\Pi_{n+t-s}(Y_0)$. Thus $E_\infty^{s,t}$, for $t - s = m$, is a quotient obtained by filtering $B^{0,m} = \Pi_m^S(X)$. We may also identify $E_2^{s,t}$. In fact, if $k \geqslant s + 2$, $l>t + 2$ we have

$$E_1^{s,t} = \Pi_{n+t-s}(Y_s, Y_{s+1}) \cong \mathrm{Hom}_A(C_{s,t}, Z_p)$$

and similarly for $(s+1)$. The map $d_1\colon E_1^{s,t} \to E_1^{s+1,t}$ is obtained by transposition from $d\colon C_{s+1,t} \to C_{s,t}$, as one sees by using the pairing of Π_* and H^*. Thus

$$E_2^{s,t} \cong \mathrm{Ext}^{s,t}(H^*(X), Z_p)\,.$$

The next point we should consider is the identification of corresponding terms obtained from different realisations. This follows by standard methods from Lemmas 3.4, 3.5, and is omitted.

It remains only to prove the convergence of the spectral sequence, by proving

$$\text{(vi)}\quad \bigcap_{t-s=m} B^{s,t} = K^m\,, \qquad \text{(iv)}\quad \bigcap_{r>s} E_r^{s,t} = E_\infty^{s,t}\,.$$

The inclusions

$$\bigcap_{t-s=m} B^{s,t} \supset K^m\,, \qquad \bigcap_{r>s} E_r^{s,t} \supset E_\infty^{s,t}$$

are elementary; we have to prove the opposite inclusions.

We begin with (vi), in the case when $H_t(X)$ is finite and p-primary for each t, so that the same is true of $\Pi^S_{m'}(X)$. Given m, we will construct a realisation $\{Y_s\}$ and an integer u such that $\Pi_{n+m'}(Y_u) = 0$ for $m' \leqslant m$. The corresponding complex C will be A-free but not necessarily acyclic. The construction is by induction, as for Lemma 3.1. Suppose constructed a space F_s with finite p-primary homotopy groups; let $\Pi_{n+m'}(F_s)$ $(= G$ say) be the first that is not zero. Let F'_s be an equivalent complex, and let

$$f_s\colon\ F'_s \to K(G/pG, n+m')$$

be a map inducing the projection $G \to G/pG$ of homotopy groups. Factor f_s through an equivalence and a fibration; let the fibre be F_{s+1}. The induction is started with $F_{-1} = S^n X$. If we form a mapping-cylinder and take equivalent complexes, as in the proof of Lemma 3.1, we obtain a realisation $\{Y_s\}$ with the required properties.

This construction gives the integer u; for if integers f_i are taken so that $p^{f_i}\Pi^S_i(X) = 0$ $(0 \leqslant i \leqslant m)$ it is sufficient to take $u = \sum_0^m f_i$. Thus u depends only on X. We see that n, k and l can be taken as large as required.

Now let $\{Y'_s\}$ be a realisation of a resolution C' of $H^*(X)$, for the same n. According to Lemmas 3.4, 3.5 there is a well-defined map from that part of a spectral sequence defined by $\{Y'_s\}$ to that part of a spectral sequence defined by $\{Y_s\}$. Now, in the latter we have

$$\bigcap_{\substack{t-s=m\\ s\leqslant u}} B^{s,t}(Y) = 0\,.$$

Therefore, if k and l are taken large enough, we have

$$\bigcap_{\substack{t-s=m\\ s\leqslant u}} B^{s,t}(Y') = 0$$

in the former. This concludes the proof in this case.

We next transfer this result to the case in which $H_t(X)$ (though finitely-generated, as always) is not necessarily finite or p-primary. In fact, let

$$x \in \Pi^S_m(X)$$

be an element not in K^m, that is, not of finite order prime to p. Let f be such an integer that the equation $x = p^f y$ has no roots y in $\Pi^S_m(X)$. Let Z be formed by attaching the cone on SX to SX by a map of degree p^f on the suspension coordinate. Then there is an inclusion map $SX \to Z$; this induces a map of spectral sequences. The image of x in $\Pi^S_{m+1}(Z)$ is non-zero; so it does not lie in

$$\bigcap_{\substack{t-s=m+1\\ s\leqslant u}} B^{s,t}(Z)$$

(for a certain u). Therefore x itself does not lie in

$$\bigcap_{\substack{t-s=m\\ s\leqslant u}} B^{s,t}(X) .$$

This argument gives a value for the integer u; for if $\Pi^S_i(X)$ is non-zero for just g values of i with $i \leqslant m$, then it is sufficient to take $u = 2fg$.

At this point we have completed the proof of (vi).

We turn next to the proof of (iv). Here we have to argue from the structure of $\Pi_{n+t-s}(Y_s)$. Now this structure is not invariant, presumably, unless we restrict $\{Y_s\}$. We therefore proceed as follows.

We call a resolution C *minimal* if the numer of A-free generators in $C_{s,t}$ is the least possible, given the structure of $C_{s,t'}$ for $t'<t$ and of $C_{s',t'}$ for $s'<s$. Since each $C_{s,t}$ is finitely-generated, one can prove by induction that if C, C' are minimal resolutions of $H^*(X)$, then any map $f\colon C \to C'$ (compatible with the identity map of $H^*(X)$) is an isomorphism. It follows from this (using the five-lemma) that if $\{Y_s\}$, $\{Y'_s\}$ are realisations of C, C', and if

$$f\colon \{Y_s\} \to \{Y'_s\}$$

is a map constructed by Lemma 3.4, then f_* maps $\Pi_{n+t-s}(Y_s)$ isomorphically (for $s \leqslant k$, $t<l-1$). In the same range, $\Pi_{n+t-s}(Y_s)$ is preserved on suspending $\{Y_s\}$.

We suppose, then, that C is a minimal resolution of $H^*(X)$. Let $\{Y_s\}$ be a realisation of C with $k \geqslant 2s+1$, $l>s+1+t$, so that $E^{s,t}_{s+1}$, $E^{s,t}_\infty$ are defined. Let $F_r^{s,t}$ be the p-component of the torsion subgroup of

$$\Pi_{n+t-s-1}(Y_{s+r}) ;$$

let $K_r^{s,t}$ be the image (by d) of $\Pi_{n+t-s}(Y_{s+1}, Y_{s+r})$ in $\Pi_{n+t-s-1}(Y_{s+r})$. Then $K_r^{s,t} \subset F_r^{s,t}$, $F_r^{s,t}$ is finite, and there is a monomorphism (induced by d) from $E_r^{s,t}/E_\infty^{s,t}$ to $F_r^{s,t}/K_r^{s,t}$ (if $r > s$). If we take this monomorphism for two values of r, it commutes with the maps induced by inclusions.

Next take $r = s + 1$, and take an element $e \neq 0$ in $E_{s+1}^{s,t}/E_\infty^{s,t}$. For each member x of the corresponding coset e' in $F_{s+1}^{s,t}/K_{s+1}^{s,t}$, the equation $x = p^f y$ is insoluble in $\Pi_{n+t-s-1}(Y_{2s+1})$, provided that f is suitably chosen; for example, let p^f be the order of $F_r^{s,t}$.

It is next necessary to suppose that the realisation $\{Y_s\}$ has n, k and l so large that $E_{s+1+2f(t-s+1)}^{s,t}$ is defined. This is possible, because by our supposition that C is minimal, we may replace $\{Y_s\}$ by another realisation with increased n, k and l, without changing f.

Next note that for suitable n', $\{S^{n'} Y_{s'}\}$ (over $s' \geqslant 2s + 1$) is a realisation of a resolution of $H^*(Y_{2s+1})$. Applying to this our results on (vi), we see that for each $x \in e'$, $S^{n'}x$ is not in the image of $\Pi_{n'+n+t-s-1}(S^{n'} Y_{2s+1+2f(t-s+1)})$. Desuspending, we see that for each $x \in e'$, x is not in the image of

$$\Pi_{n+t-s-1}(Y_{2s+1+2f(t-s+1)}) .$$

Therefore e is not in the image of $E_{s+1+2f(t-s+1)}^{s,t}$.

This concludes the proof of (iv), and of Theorem 2.1.

§ 4. Multiplicative Properties of the Spectral Sequence

In this section we prove Theorem 2.2, by establishing the multiplicative properties of the spectral sequence in case $X = S^0$. For clarity, we proceed in slightly greater generality. Let X, X' be spaces, and let $X'' = X \times X'/X \vee X'$. We will show that there is a pairing

$$E_r^{s,t}(X) \otimes E_r^{s',t'}(X') \to E_r^{s+s',t+t'}(X'') .$$

Our method is to take realisations $\{Y_s\}$, $\{Y'_{s'}\}$ corresponding to X, X' and form a realisation $\{Y''_{s''}\}$, using the join operation, so that Y''_0 is the join $Y_0 * Y'_0$. If $Y_0 \sim S^n X$, $Y'_0 \sim S^{n'} X'$, then $Y''_0 \sim S^{n+n'+1}(X'')$. We have a join operation in relative homotopy groups; it will appear that it gives a pairing

$$\Pi_*(Y_s, Y_u) \otimes \Pi_*(Y'_{s'}, Y'_{u'}) \to \Pi_*(Y''_{s''}, Y''_{u''})$$

for $s'' = s + s'$, $u'' = \mathrm{Min}(s + u', u + s')$. Such pairings yield a pairing of spectral sequences, by passing to quotients.

It is convenient to begin the detailed work by describing the cup-product

$$\mathrm{Ext}_A^{s,t}(H^*(X),Z_p)\otimes\mathrm{Ext}_A^{s',t'}(H^*(X'),Z_p)\to\mathrm{Ext}_A^{s+s',t+t'}(H^*(X''),Z_p)\ .$$

This depends on the diagonal map $\Delta: A\to A\otimes A$ of the STEENROD algebra, which is defined as follows. There is one and only one universal formula for expanding the result of an operation a applied to a cup-product uv; let it be

$$a(uv)=\sum_i(-1)^{jq}(a^{L,i}u)(a^{R,i}v)$$

where $a\in A$, $\deg(v)=q$, $a^{L,i}\in A_j$ $(j=j(i))$ and $a^{R,i}\in A$. Then we set

$$\Delta a=\sum_i a^{L,i}\otimes a^{R,i}\ .$$

One verifies that this defines a diagonal map[10]).

We next remark that since our cohomology groups are augmented, we have $H^*(X'')\cong H^*(X)\otimes H^*(X')$; the isomorphism is defined using the external cup-product. The operations of A on $H^*(X'')$ are given by

$$a(u\otimes v)=\sum_i(-1)^{jq}(a^{L,i}u\otimes a^{R,i}v)\ ;$$

here $v\in H^q(X')$, while a, $a^{L,i}$, $a^{R,i}$ and j are as above.

Next suppose that C, C' are resolutions of $H^*(X)$, $H^*(X')$. Then we may make $C\otimes C'$ into an acyclic A-complex, whose homology in dimension $s=0$, $s'=0$ is $H^*(X'')$. In fact, we set

$$d(c\otimes c')=dc\otimes c'+(-1)^s c\otimes dc' \qquad (c\in C_{s,t})$$
$$a(c\otimes c')=\sum_i(-1)^{jt'}(a^{L,i}c\otimes a^{R,i}c') \qquad (c'\in C'_{s',t'})\ .$$

It follows that if C'' is a resolution of $H^*(X'')$, there is a map $m: C''\to C\otimes C'$. On the other hand, there is a pairing

$$\mu:\ \mathrm{Hom}_A(C,Z_p)\otimes\mathrm{Hom}_A(C',Z_p)\to\mathrm{Hom}_A(C\otimes C',Z_p)$$

defined by $(\mu(h\otimes h'))(c\otimes c')=(hc)(h'c')$. The composite $m^*\mu^*$ yields the required cup-product

$$\mathrm{Ext}_A^{s,t}(H^*(X),Z_p)\otimes\mathrm{Ext}_A^{s',t'}(H^*(X'),Z_p)\to\mathrm{Ext}_A^{s+s',t+t'}(H^*(X''),Z_p)\ .$$

Now that we have considered the acyclic A-complex $C\otimes C'$, we will consider a realisation of it. Let $\{Y_s\}$, $\{Y'_{s'}\}$ be realisations of C, C'. We will define Y''_0 by setting $Y''_0=Y_0*Y'_0$, and subcomplexes of Y''_0 by setting

$$Y''_{s''}=\bigcup_{s+s'=s''}Y_s*Y'_{s'}\ .$$

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[10]) The products in $A\otimes A$ are defined by $(a\otimes b)(c\otimes d)=(-1)^{il}(ac\otimes bd)$ (for $a\in A_i$, $d\in A_l$).

We may suppose that Y_0, Y'_0 are enumerable CW-complexes; Y''_0 is then an enumerable CW-complex. We repeat that since $Y_0 \sim S^n X$ and $Y'_0 \sim S^{n'} X'$ we have $Y''_0 \sim S^{n+n'+1}(X'')$.

In order to show that $\{Y''_{s''}\}$ is a realisation of $C \otimes C'$, we must display $H^*(Y''_{s''}, Y''_{s''+1})$. Let

$$\begin{aligned} \varphi: \quad & C_{s,t} \to H^{n+t-s}(Y_s, Y_{s+1}) \,, \\ \varphi': \quad & C_{s',t'} \to H^{n'+t'-s'}(Y'_{s'}, Y'_{s'+1}) \end{aligned}$$

be the given isomorphisms[11]). Then we may define a map

$$\varphi'': \quad C_{s,t} \otimes C'_{s',t'} \to H^{n''+t''-s''}(Y''_{s''}, Y''_{s''+1})$$

(where $n'' = n + n' + 1$, $s'' = s + s'$, $t'' = t + t'$) by setting

$$\varphi(u \otimes v) = (-1)^{(t-s)n' + ts'} E(\varphi u \cdot \varphi' v) \ .$$

Here we employ the fact that $Y_0 * Y'_0 \sim S(Y_0 \times Y'_0 / Y_0 \vee Y'_0)$. The element $\varphi u \cdot \varphi' v$ is defined using the exterior cup-product; the map E is defined using excision and suspension. The map

$$\varphi'': \sum_{\substack{s+s'=s'' \\ t+t'=t''}} C_{s,t} \otimes C'_{s',t'} \to H^{n''+s''-t''}(Y''_{s''}, Y''_{s''+1})$$

is an isomorphism. One verifies that it commutes with the operations of A. One also verifies that it satisfies the third condition imposed on a realisation, by making the diagrams for d and ε commutative. (The sign in the definition of φ'' is essential at these points.)

We have now verified that $\{Y''_{s''}\}$ is a realisation of $C \otimes C'$. Maps or homotopies of $\{Y_s\}$ or $\{Y'_{s'}\}$ induce maps or homotopies of $\{Y''_{s''}\}$. It follows that the spectral sequence associated with $\{Y''_{s''}\}$ is well-defined. Since $C \otimes C'$ is acyclic, there is (by Lemmas 3.4, 3.5) a well-defined map from this spectral sequence to that associated with X''. It remains, therefore, to define a pairing

$$E_r^{s,t}(Y) \otimes E_r^{s',t'}(Y') \to E_r^{s+s',t+t'}(Y'') \ .$$

To do this, we now introduce the join operation in relative homotopy groups. This is defined by the join of maps of oriented cells[12]). We obtain a pairing

$$\Pi_m(K, L) \otimes \Pi_{m'}(M, N) \to \Pi_{m+m'+1}(K*M, K*N \cup L*M) \ .$$

This is natural with respect to maps of K, L and M, N. If α, β lie in the groups paired, then

$$jd(\alpha * \beta) = -i(d\alpha * \beta) + (-1)^{m+1} i'(\alpha * d\beta) \ .$$

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[11]) In this section we omit to make explicit the finite ranges of dimensions in which these isomorphisms are supposed to hold. The details are similar to those in § 3.

[12]) The join $K * L$ is oriented as $S(K \times L / K \vee L)$, and we suspend over the first coordinate.

Here j, i, i' are the canonical maps with values in $\Pi_{m+m'}(K*N \cup L*M, L*N)$.

The products we require, however, represent composition products, not join products; the two differ by a sign. With this in mind, we define the product

$$\Pi_{n+t-s}(Y_s, Y_u) \otimes \Pi_{n'+t'-s'}(Y'_{s'}, Y'_{u'}) \to \Pi_{n''+t''-s''}(Y''_{s''}, Y''_{u''})$$

by $\alpha \times \beta = (-1)^{(t-s)n'} \alpha * \beta$. (Here we have $n'' = n + n' + 1$, $s'' = s + s'$, $t'' = t + t'$, $u'' = \mathrm{Min}(s + u', u + s')$.) We now have the boundary formula

$$j(-1)^{n''} d(\alpha \times \beta) = i((-1)^n d\alpha \times \beta) + (-1)^{(t-s)} i'(\alpha \times (-1)^{n'} d\beta) \ .$$

The following statements are now open to verification. Firstly, the pairing of relative homotopy groups passes to quotients, and defines a pairing

$$E_r^{s,t}(Y) \otimes E_r^{s',t'}(Y') \to E_r^{s+s',t+t'}(Y'') \ .$$

Secondly, the boundary d_r satisfies

$$d_r(uv) = (d_r u)v + (-1)^{(t-s)} u(d_r v) \ .$$

Thirdly, the products are natural with respect to the isomorphisms

$$E_{r+1}^{s,t} \cong H^{s,t}(E_r; d_r)$$

and with respect to the monomorphisms $E_R^{s,t} \to E_r^{s,t}$ for $s < r < R \leqslant \infty$.

Fourthly, the composite map

$$E_2^{s,t}(X) \otimes E_2^{s',t'}(X') \to E_2^{s+s',t+t'}(X'')$$

coincides, except for the sign $(-1)^{ts'}$, with the pairing

$$\mathrm{Ext}_A^{s,t}(H^*(X), Z_p) \otimes \mathrm{Ext}_A^{s',t'}(H^*(X'), Z_p) \to \mathrm{Ext}_A^{s+s',t+t'}(H^*(X''), Z_p) \ .$$

(This follows from the description above, on considering the pairing of Π_* and H^*.)

Fifthly, the products are associative, and anticommutative for the grading $(t - s)$. This follows from analogous facts for the join operation, together with naturality arguments. We note that the products in $\mathrm{Ext}_A^*(Z_p, Z_p)$ are associative, and satisfy the anticommutative law

$$uv = (-1)^{ss'+tt'} vu \quad (\text{for } u \in \mathrm{Ext}^{s,t} \ , \quad v \in \mathrm{Ext}^{s',t'}) \ .$$

Sixthly, consider the case $X = X' = X'' = S^0$. Then a composition product is defined in $\Pi_*^S(S^0)$. This passes to the quotient and defines products in $E_\infty^{s,t}(S^0)$, which coincide with those considered above. (This follows from the known equivalence between composition products and join products.)

This concludes the proof of Theorem 2.2.

§ 5. The Structure of the STEENROD Algebra

We collect our results on the subalgebras of the (mod p) STEENROD algebra A in Theorem 5.1. This theorem is followed by explanatory definitions, and a great deal of proof. This is followed by two further theorems (5.12, 5.13) giving further information needed in considering $H^*(A)$. One concerns a self-map $A \to A$; the other concerns the commutators in A.

If $p > 2$ we have:

Theorem 5.1$_p$. (*a*) *Any finite set of elements of A generates a finite algebra.*

(*b*) *A contains subalgebras $A^{R,T}$ for each $1 \leqslant R < \infty$, $1 \leqslant T \leqslant \infty$ with the following properties.*

(*c*) *$A^{R,T} = A^{1,T} \cap A^{R,\infty}$. If $Q \leqslant T$, then $A^{1,Q} \subset A^{1,T}$. If $P \leqslant R$, then $A^{P,\infty} \supset A^{R,\infty}$. $A^{R,T}$ is the unit subalgebra if $R > T$. The subalgebra $A^{1,T}$ is that generated by β and P^k for $k < p^{T-1}$. We have $A^{1,\infty} = A$.*

(*d*) *If $R \leqslant T < \infty$, the rank of $A^{R,T}$ is $2^U p^V$, where $U = T - R + 1$, $V = \frac{1}{2}(T - R)(T - R + 1)$.*

(*e*) *If $P < Q$, $A^{Q,T}$ is normal in $A^{P,T}$, so that $A^{P,T}//A^{Q,T}$ exists; and $A^{P,T}$ is free, qua (left or right) module over $A^{Q,T}$.*

(*f*) *If $P < Q < R$, then $A^{Q,T}//A^{R,T}$ is embedded monomorphically in $A^{P,T}//A^{R,T}$; the former is normal in the latter, with quotient isomorphic to $A^{P,T}//A^{Q,T}$.*

(*g_p*) *Moreover, if $R \leqslant P + Q$, then $A^{Q,T}//A^{R,T}$ is central in $A^{P,T}//A^{R,T}$, in the sense that if $a_{i,j} \in A^{P,T}$, $b_{k,l} \in A^{Q,T}$, then in $A^{P,T}//A^{R,T}$ we have*

$$\{a_{i,j}\}\{b_{k,l}\} = (-1)^{jl}\{b_{k,l}\}\{a_{i,j}\} .$$

(*h_p*) *If $R \leqslant T$, we have*

$$A^{R,T}//A^{R+1,T} \cong E(\sum_{t=0}^{R-2} p^t, 1) \otimes P(\sum_{t=0}^{R-1} p^t, 0; p^{T-R}) .$$

If $p = 2$ we have:

Theorem 5.1$_2$. *The statements (a) to (f) hold word for word on interpreting "p" as 2, "β" as Sq^1 and "P^k" as Sq^{2k}. The statements (g_p), (h_p) become:*

(*g_2*) *Moreover, if $R \leqslant P + Q$, then $A^{Q,T}//A^{R,T}$ is central in $A^{P,T}//A^{R,T}$.*

(*h_2*) *If $R \leqslant T$, we have*

$$A^{R,T}//A^{R+1,T} \cong P(2^R - 1; 2^{T-R+1}) .$$

Explanatory definitions. All our algebras are algebras with unit and with diagonal [4, p. 211] over the field Z_p. They are graded if $p = 2$ and bigraded if $p > 2$. Their components in dimension 0 or (0,0) are their unit subalgebras (here the unit subalgebra has the unit as a Z_p base). All our maps of algebras preserve this structure; in particular, the injections of subalgebras and projections onto quotient algebras do so.

The STEENROD algebra A (over Z_p) is defined as above. If $p>2$ it may be bigraded by setting $\mathrm{Deg}(P_p^k) = (k,0)$, $\mathrm{Deg}(\beta_p) = (0,1)$[13]. The single grading q corresponding to the bigrading (i,j) is given by $q = 2(p-1)i + j$.

The other algebras introduced are as follows. The exterior algebra $E(i,j)$ (over Z_p) has a Z_p-base $\{1, f'\}$. The element 1 is the unit; the element f' has bidegree (i,j). The product is given by $(f')^2 = 0$; the diagonal is given by $\Delta(f') = f'\otimes 1 + 1\otimes f'$. The integer j must be odd (if $p>2$).

The truncated divided polynomial algebra $P(i,j;k)$ (over Z_p) has a Z_p-base containing one element f_l of bidegree (li, lj) for each l such that $0 \leqslant l < k$. The product is given by $f_l \cdot f_m = (l,m) f_{l+m}$; the diagonal is given by

$$\Delta f_l = \sum_{m+n=l} f_m \otimes f_n \ .$$

(Here the binomial coefficients mod p are defined by $(l,m) = \dfrac{(l+m)!}{l!\, m!}$). The integer j must be even (if $p>2$); and k must be a power of p, or else ∞. If $k=\infty$, the algebra is not truncated. The algebra $P(i;k)$ is similarly defined, but graded instead of bigraded. For the tensor product of algebras, see [4].

If A is a bigraded algebra, as above, we define $I(A) = \sum_{(i,j)\neq(0,0)} A_{i,j}$; similarly for a graded algebra. If A is an algebra containing B as a subalgebra, we call B *normal* in A (cf. [4] p. 349) if $A\cdot I(B) = I(B)\cdot A$; we then define $A//B = A/I(B)\cdot A$.

Since the word "dimension" is already in use for the grading, we speak of the *rank* of a subalgebra, meaning its dimension when considered as a vector space over Z_p.

This concludes the explanatory definitions.

Proof. The proof will proceed in several stages. Following SERRE, THOM and CARTAN, we shall make use of a faithful representation of the STEENROD algebra, obtained by allowing it to operate on a Cartesian product of spaces of type $(Z_p, 1)$. We take first the case $p>2$.

Let X be the Cartesian product of $n+n'$ spaces, each of type $(Z_p,1)$; let their fundamental classes be $x_1, \ldots, x_n, x'_1, \ldots, x'_{n'}$. Set

$$y_i = \beta_p(x_i), \quad y'_i = \beta_p(x'_i) \ .$$

Let $u \in H^{2n+n'}(X)$ be the cup-product $y_1 \ldots y_n x'_1 \ldots x'_{n'}$.

$H^*(X)$ is the tensor product of exterior algebras generated by the x_i, x'_i and polynomial algebras generated by the y_i, y'_i. We shall need a notation

[13]) In [2] the second grading is called the "type".

for certain polynomials $D^I_{n,n'}$ lying in $H^*(X)$. Let

$$I = \{\varepsilon_1, \lambda_1, \ldots, \varepsilon_i, \lambda_i, \ldots\}$$

be a sequence of integers with $\lambda_i \geqslant 0$, $\varepsilon_i = 0$ or 1; only a finite number are to be non-zero. Following CARTAN [2], we define the polynomial $D^I_{n,n'}$ as follows. Among the monomials $y_1^{h_1} \ldots y_n^{h_n}$, consider those in which just λ_i exponents are p^i (for each i), the remainder being 1. Let the sum of such monomials be s[14]). Next consider the monomials obtained from $x'_1 \ldots x'_{n'}$ by replacing x'_j by $(y'_j)^{p^i}$ for just ε_{i+1} values of j (this for each i). Each such monomial is to be taken with a sign, namely the signature of a certain permutation ϱ of $1, \ldots, n'$. Here ϱ brings the factors x'_k in the monomial to the left (arranged from left to right in increasing order of k) and the factors $(y'_j)^{p^i}$ to the right (arranged in increasing order of i). Let the sum of such signed monomials be t. Then we define $D^I_{n,n'} = st$.

The elements $D^I_{n,n'}$ in $H^{2n+n'+q}(X)$ generate a submodule $D^{2n+n'+q}$; they are linearly independent if n, n' are sufficiently large (depending on q).

Evidently a Z_p-linear function $\theta\colon A \to H^*(X)$ is defined by $\theta(a) = a(u)$. This is the representation used.

Theorem 5.2. *If n, n' are sufficiently large (depending on q) then $\theta_q\colon A_q \to H^{2n+n'+q}(X)$ has kernel zero and image $D^{2n+n'+q}$.*

This theorem is due to CARTAN [2]. His proof shows also that θ preserves the bigrading, if the second degree of polynomials $D^I_{n,n'}$ is defined by $j = \sum_i \varepsilon_i$.

We next note that this representation has a convenient relation to the diagonal in A. In fact, if the space Y is defined using $m + m'$ factors of type $(Z_p, 1)$, then $X \times Y$ is homeomorphic to the space Z defined using $(n + m) + (n' + m')$ factors of type $(Z_p, 1)$. If v and w are the analogues of u for Y and Z, then uv corresponds to w in this homeomorphism. Let θ_X, θ_Y, θ_Z be the functions θ for the three spaces. By evaluating $a(uv)$ and $a(w)$ we have the following obvious lemma.

Lemma 5.3. *If $a \in A$ and $\theta_Z a = D^I_{n+m,n'+m'}$ then*

$$(\theta_X \otimes \theta_Y)(\Delta a) = \sum_{J+K=I} (-1)^{\varepsilon} D^J_{n,n'} \otimes D^K_{m,m'} .$$

Here, for sequences $I = \{\varepsilon_i, \lambda_i\}$, $J = \{\eta_i, \mu_i\}$ and $K = \{\zeta_i, \nu_i\}$, the equation $J + K = I$ means that $\eta_i + \zeta_i = \varepsilon_i$ and $\mu_i + \nu_i = \lambda_i$ for each $i \geqslant 1$. The sign is given by

$$\varepsilon = \sum_{j>k} \eta_j \zeta_k .$$

[14]) Thus, s will be zero if $\sum_i \lambda_i > n$. Similarly, later, for t.

Since Δa has among its components the term $a \otimes 1$, Lemma 5.3 has the following corollary: if $\theta_Z a = D^I_{n+m,n'+m'}$, then $\theta_X a = D^I_{n,n'}$. It follows that we may write $\theta a = D^I$ to mean that $\theta_X a = D^I_{n,n'}$ for all n, n'; similarly for the equation $\theta a = \sum_I \lambda_I D^I$.

Theorem 5.2 allows us to exhibit certain distinguished elements in the STEENROD algebra. In fact, let us define $I(r,k)$ $(r \geqslant 1, k \geqslant 0)$ by $\lambda_r = k$, $\lambda_i = 0$ for $i \neq r$, $\varepsilon_i = 0$ for all i. Define $e_{r,k} \in A$ by $\theta(e_{r,k}) = D^{I(r,k)}$. Define I'_r $(r \geqslant 1)$ by $\lambda_i = 0$ for all i, $\varepsilon_r = 1$, $\varepsilon_i = 0$ for $i \neq r$. Define $e'_r \in A$ by $\theta(e'_r) = D^{I'(r)}$.

These elements have the following properties, which indeed characterise them.

(1) $e_{r,0}$ is 1, the unit.

(2) $$\operatorname{Deg} e_{r,k} = (k \sum_{t=0}^{r-1} p^t, 0) .$$

$$\operatorname{Deg} e'_r = (\sum_{t=0}^{r-2} p^t, 1) .$$

(3) $$\Delta e_{r,k} = \sum_{i+j=k} e_{r,i} \otimes e_{r,j}$$

$$\Delta e'_r = e'_r \otimes 1 + 1 \otimes e'_r .$$

(4) If x, y are the generators of $H^*(Z_p, 1; Z_p)$, then

$$e_{r,k}(x) = 0 \qquad\qquad e'_r(x) = y^{p^r - 1}$$

$$e_{r,k}(y) = \begin{cases} y & (k = 0) \\ y^{p^r} & (k = 1) \\ 0 & \text{otherwise} \end{cases} \qquad\qquad e'_r(y) = 0 .$$

For example, (3) follows immediately from Lemma 5.3 by using Theorem 5.2. From (3) and (4) we deduce the following property by induction:

(5) $$e_{r,k}(y^{p^s}) = \begin{cases} y^{p^s} & (k = 0) \\ y^{p^{r+s}} & (k = p^s) \\ 0 & \text{otherwise} \end{cases} \qquad\qquad e'_r(y^{p^s}) = 0 .$$

Our next theorem will show in what sense the elements $e_{r,k}$, e'_r are generators for A. In order to state it, let us regard e_{r,k_r}, for each $r \geqslant 1$, as an expression in the variable k_r. Let us order together in some fixed order the expressions e_{r,k_r} and e'_r. Let us form monomials by omitting from this ordering all but a finite number of terms, and then inserting integer values $k_r \geqslant 1$ for the remaining variables k_r. These monomials then represent elements of A. The identity element is included, as the empty product. We then have:

Theorem 5.4. *For each fixed ordering, such monomials form a base for A.*

Proof. Let M be a typical monomial, of single grading q. By Theorem 5.2, it is sufficient to show that the elements $M(u)$ form a base for $D^{2n+n'+q}$ (for n, n' sufficiently large). To this end, we order the base D^I of $D^{2n+n'+q}$ by ordering the sequences I. Following CARTAN [2], we order them lexicographically from the right[15]).

Using (3) and (5), we deduce:

$$(6)\quad \begin{aligned} e_{r,k} D^I &= (\lambda_r, k) D^J + \sum_{K>J} \mu_K D^K \\ e'_r D^I &= (-1)^\varepsilon D^L \qquad (\text{if } \varepsilon_r = 0) \\ e'_r D^I &= 0 \qquad\qquad (\text{if } \varepsilon_r = 1)\ . \end{aligned}$$

Here $I = \{\varepsilon_i, \lambda_i\}$, $J = I + I(r,k)$, and $L = I + I'(r)$ if $\varepsilon_r = 0$; the sign is given by $\varepsilon = \sum_{i<r} \varepsilon_i$.

Let $I' = \{\varepsilon'_i, \lambda'_i\}$ be another sequence, $J' = I' + I(r,k)$, and $L' = I' + I'(r)$ if $\varepsilon'_r = 0$. Then we have, trivially:

(7) If $I < I'$, then $J < J'$, and $L < L'$ if $\varepsilon_r = \varepsilon'_r = 0$.

From (6) and (7) we deduce, by induction, that:

(8) If M is a typical monomial, then

$$M(u) = \eta D^I + \sum_{J>I} \mu_J D^J \ ,$$

where $\eta = \pm 1$, and the sequence I is determined by M as follows. ε_r is 1 or 0 according as e'_r is in M or omitted; λ_r is 0 if e_{r,k_r} is omitted from M, and otherwise it is the integer substituted for k_r.

We see that there is a $(1-1)$ correspondence between M and I. Therefore the elements $M(u)$ in $D^{2n+n'+q}$ form a base for it. This concludes the proof of Theorem 5.4.

The statement and proof of Theorem 5.4 remain valid if, instead of using expressions e_{r,k_r}, one for each r, we use expressions $(e_{r,p^i})^{d_{r,i}}$, one for each r and each $i \geqslant 0$. We have then to substitute, for the variables $d_{r,i}$, integer values such that $0 < d_{r,i} < p$.

We may now obtain the subalgebras $A^{R,T}$. Given a fixed ordering, as in Theorem 5.4, we may consider the monomials M in which the factors $e_{r,k}$ have $r \geqslant R$, $k < p^{T-r}$ and the factors e'_r have $R \leqslant r \leqslant T$. These form a base for

[15]) However, our argument differs from his in that our inductions (if stated) would proceed in the opposite direction along the ordering. Since the bases ordered are finite (for each q) this is immaterial.

a submodule $A^{R,T}$ of A. This is so whenever $1 \leqslant R < \infty$, $1 \leqslant T \leqslant \infty$; but we note that the base reduces to the unit element if $T < R$. The inclusion and intersection properties in Theorem 5.1 (c) are trivial.

Theorem 5.5. *$A^{R,T}$ is independent of the ordering chosen. It is a subalgebra of A and closed for the diagonal.*

Proof. Consider the polynomials D^I for which the sequence $I = \{\varepsilon_i, \lambda_i\}$ satisfies $\lambda_i < p^{T-i}$ for each i, $\lambda_i = 0$ for $i < R$, and $\varepsilon_i = 0$ unless $R \leqslant i \leqslant T$. These generate a submodule $D^{R,T}$ of $H^*(X)$.

Lemma 5.6. $e_{r,k} D^{R,T} \subset D^{R,T}$ *if* $r \geqslant R$, $k < p^{T-r}$,

$$e'_r D^{R,T} \subset D^{R,T} \quad \textit{if} \quad R \leqslant r \leqslant T.$$

We will defer the proof of this lemma in order to show how the theorem follows from it. In fact, the lemma clearly implies that $\theta(A^{R,T}) \subset D^{R,T}$. But further, if T is finite, $A^{R,T}$ and $D^{R,T}$ have the same rank, namely $2^U p^V$ where $U = T - R + 1$, $V = \frac{1}{2}(T - R)(T - R + 1)$. Therefore $\theta(A^{R,T}) = D^{R,T}$ if T is finite; this implies the same equation with T infinite. The equation $\theta(A^{R,T}) = D^{R,T}$ shows that $A^{R,T}$ is independent of the ordering chosen; and with Lemma 5.6, it implies that $A^{R,T}$ is a subring. Lastly, it is clear from Lemma 5.3 and the definition of $D^{R,T}$ that $\theta^{-1}(D^{R,T})$ is closed for the diagonal. This concludes the deduction of the theorem from the lemma. We note that we have proved Theorem 5.1 (d).

Proof of Lemma 5.6. The property (6) above shows trivially that

$$e'_r D^{R,T} \subset D^{R,T} \quad \textit{if} \quad R \leqslant r \leqslant T .$$

Let us take D^I, where $I = \{\varepsilon_i, \lambda_i\}$, and form the expansion

$$e_{r,k} D^I = \sum_K \lambda_K D^K .$$

The sum may be given explicitly as follows, by using (3) and (5). It extends over sequences $J = \{\mu_0, \eta_1, \mu_1, \ldots, \eta_i, \mu_i, \ldots\}$ with $\sum_i (\mu_i + \eta_{i+1}) p^i = k$. The term given by J will correspond to the summands in which, for each i, just μ_i of the factors $y_j^{p^i}$ $(1 \leqslant j \leqslant n)$ and just η_{i+1} of the factors $(y'_j)^{p^i}$ $(1 \leqslant j \leqslant n')$ are operated on by e_{r,p^i}. The sequence $K = \{\zeta_i, \nu_i\}$ is given by

$$\nu_i = \lambda_i - \mu_i + \mu_{i-r} ,$$

$$\zeta_i = \varepsilon_i - \eta_i + \eta_{i-r} .$$

If $\lambda_i - \mu_i < 0$, $\varepsilon_i - \eta_i < 0$ or $\varepsilon_i - \eta_i + \eta_{i-r} = 2$, then the term given by

J is zero. The coefficient λ_K is given as follows.

$$\lambda_K = (-1)^{\varepsilon} \prod_i (\lambda_i - \mu_i, \mu_{i-r})$$

where $\varepsilon = \sum_{i<j<i+r} (\varepsilon_j - \eta_j)\eta_i$.

We now introduce the assumption that $D^I \epsilon D^{R,T}$, $r \geqslant R$, and $k < p^{T-r}$.

In fact, since $D^I \epsilon D^{R,T}$, we have $\lambda_i = 0$ and $\varepsilon_i = 0$ for $i < R$; we also have $r \geqslant R$. Using the formulae for ν_i, ζ_i we see that $\nu_i = 0$ and $\zeta_i = 0$ for $i < R$.

Similarly, since

$$\sum_i (\mu_i + \eta_{i+1}) p^i = k < p^{T-r} ,$$

we have $\eta_{i+1} = 0$ unless $i < T - r$. Thus $\eta_{i-r} = 0$ unless $i \leqslant T$. We also have $\varepsilon_i = 0$ unless $i \leqslant T$; thus $\zeta_i = 0$ unless $i \leqslant T$.

Lastly, we have $\lambda_i < p^{T-i}$, so $\lambda_i - \mu_i < p^{T-i}$. We also have

$$\sum_i (\mu_i + \eta_{i+1}) p^i = k < p^{T-r} ,$$

so $\mu_i < p^{T-r-i}$ and $\mu_{i-r} < p^{T-i}$. It follows that $(\lambda_i - \mu_i, \mu_{i-r}) = 0$ whenever $\lambda_i - \mu_i + \mu_{i-r} \geqslant p^{T-i}$.

These remarks in combination show that if $D^I \epsilon D^{R,T}$, $r \geqslant R$, and $k < p^{T-r}$, then the non-zero terms of the expansion

$$e_{r,k} D^I = \sum_K \lambda_K D^K$$

lie in $D^{R,T}$. The proof of the lemma is complete.

The statement and proof of Theorem 5.5 remain valid if, in defining $A^{R,T}$, we use, instead of the expressions e_{r,k_r}, the expressions

$$(e_{r,p^i})^{d_{r,i}} \quad (0 < d_{r,i} < p) .$$

We have to take those for which $r \geqslant R$ and $i < T - r$.

Corollary 5.7. *$A^{1,T}$ is the subalgebra generated by β and by P^k for $k < p^{T-1}$.*

Proof. Since $\beta = e_1'$, it lies in $A^{1,T}$ for $T \geqslant 1$. Since $P^k = e_{1,k}$, it lies in $A^{1,T}$ for $k < p^{T-1}$. On the other hand, by the remarks immediately above, $A^{1,T}$ admits a system of multiplicative generators in bidegrees $(p^i \sum_0^{r-1} p^t, 0)$ (where $i < T - r$) and $(\sum_0^{r-2} p^t, 1)$ (where $r \leqslant T$). These can be written in terms of the elements β and P^k, and by their dimensions we shall have $k < p^{T-1}$, each k. This concludes the proof of the corollary.

Corollary 5.8. *Any finite set F of elements of A generates a finite subalgebra.*

Proof. The elements of F may be expressed in terms of β and P^k for $k<p^{T-1}$, some finite T. Thus the subalgebra generated by F is contained in the finite subalgebra $A^{1,T}$.

In order to obtain our next result we will consider the representations of A by its operations on a different space. Let Y_Q be the Cartesian product of n lens spaces $Y_1, \ldots, Y_n$ of dimension $2p^Q-1$ and n' lens spaces $Y'_1, \ldots, Y'_{n'}$ of dimension $2p^{Q-1}-1$, so that we have

$$H^*(Y_i) = E(1)\otimes T(2\,;p^Q) \quad ,$$
$$H^*(Y'_j) = E(1)\otimes T(2\,;p^{Q-1}) \ ,$$

where T is a truncated polynomial algebra. We introduce notations for the elements of $H^*(Y_Q)$ exactly as before, writing v instead of u, and E^I instead of D^I for the distinguished polynomials. The submodule $E^{R,T}$ of $H^*(Y_Q)$ is defined word for word as $D^{R,T}$ is. Consider the sequences $I = \{\varepsilon_i, \lambda_i\}$ of grading q which satisfy $\varepsilon_i = 0$ and $\lambda_i = 0$ if $i \geqslant Q$; the corresponding polynomials E^I are linearly independent and form a base of $E^{2n+n'+q}$ (at least if n, n' are sufficiently large, depending on q). All other polynomials E^I of grading q are zero.

Theorem 5.9. *$A^{Q,T}$ is normal in $A^{P,T}$ if $P<Q$. The rule $\theta\{M\} = M(v)$ defines a Z_p-linear function*

$$\theta_q : (A^{P,T}//A^{Q,T})_q \to H^{2n+n'+q}(Y_Q\,;Z_p) \ .$$

If n, n' are sufficiently large (depending on q) then θ_q has kernel zero and image $E^{P,T} \cap E^{2n+n'+q}$.

Proof. The proof that $A^{Q,T}$ is normal in $A^{P,T}$ will be as follows. Each element of $A^{P,T}$ gives a Z_p-linear map from $H^*(Y_Q)$ to itself; these maps constitute a quotient ring $A^{P,T}/K$ of $A^{P,T}$. We will show that the kernel K is both $I(A^{Q,T})\cdot A^{P,T}$ and $A^{P,T}\cdot I(A^{Q,T})$.

Let us use the representation of Theorem 5.4, with the ordering e'_1, e_{1,k_1}, e'_2, $e_{2,k_2}, \ldots$ Let us divide the monomials of $A^{P,T}$ onto two classes; one, say B, shall consist of monomials formed from factors $e_{r,k}$ with $P\leqslant r<Q$, $k<p^{T-r}$ and e'_r with $P\leqslant r<Q$; the other, say C, shall consist of the remaining monomials. C is a base for a submodule L. Then, by the choice of ordering, $L\subset A^{P,T}\cdot I(A^{Q,T})$. We also note that if $r\geqslant Q$, then

$$e_{r,k}H^*(Y_Q) = 0 \ , \quad e'_r H^*(Y_Q) = 0 \ .$$

This shows that $A^{P,T}\cdot I(A^{Q,T})\subset K$. On the other hand, let M run through B; then, by property (8) (which remains valid for Y_Q) the elements $M(v)$ are linearly independent elements of $H^*(Y_Q)$. Hence the classes $\{M\}$ are linearly independent elements of $A^{P,T}/K$, and $K = L$. We conclude that

$$A^{P,T} \cdot I(A^{Q,T}) = K \;;$$

by using the opposite order in the monomials, we see that

$$I(A^{Q,T}) \cdot A^{P,T} = K \;.$$

We have shown that $A^{Q,T}$ is normal in $A^{P,T}$, and incidentally established the representation θ of $A^{P,T}//A^{Q,T}$. The proof of Theorem 5.9 is complete.

Corollary 5.10. *If* $P<Q<R$, *the injection* $A^{Q,T}//A^{R,T} \to A^{P,T}//A^{R,T}$ *is monomorphic.*

Proof. Both quotient algebras are represented by operations on the same module $H^*(Y_R)$.

This corollary, with Theorem 5.9, implies Theorem 5.1 (f).

For Theorem 5.1 (g), it is sufficient to prove the anticommutativity relation when $a_{i,j}$ and $b_{k,l}$ are generators $e_{k,r}$ or e'_r. This is easily done by expanding abv and bav.

It remains to obtain the structure of $A^{R,T}//A^{R+1,T}$ (for $R \leqslant T$). We have $e_{R,k} \in A^{R,T}$ if $k<p^{T-R}$, $e'_R \in A^{R,T}$; let their images in $A^{R,T}//A^{R+1,T}$ be f_k, f'.

Lemma 5.11. *The elements* $f_k(f')^\varepsilon$ ($\varepsilon = 0$ or 1, $0 \leqslant k<p^{T-R}$) *form a base for* $A^{R,T}//A^{R+1,T}$. *The product is given by*

$$f_k(f')^\varepsilon \cdot f_l(f')^\eta = \begin{cases} (k,l) f_{k+l}(f')^{\varepsilon+\eta} & (\varepsilon+\eta = 0 \text{ or } 1) \\ 0 & (\varepsilon+\eta = 2) \end{cases}$$

The diagonal is given by

$$\Delta f_k(f')^\varepsilon = \sum_{\substack{i+j=k \\ \eta+\zeta=\varepsilon}} f_i(f')^\eta \otimes f_j(f')^\zeta \;.$$

Proof. According to Theorem 5.9, the elements $\{M\}$ of $A^{R,T}//A^{R+1,T}$ are faithfully represented by the corresponding elements $M(v)$ of $H^*(Y_{R+1})$. The image module $E^{R,T}$ has as a base the polynomials E^I, where I runs over the sequences $\{\varepsilon_i, \lambda_i\}$ for which $0 \leqslant \lambda_R<p^{T-R}$, $\varepsilon_R = 0$ or 1, and $\lambda_i = 0$, $\varepsilon_i = 0$ if $i \neq R$. By (6), such a polynomial E^I is exactly $e_{R,k}(e'_r)^\varepsilon v$ with $k = \lambda_R$, $\varepsilon = \varepsilon_R$. Therefore the elements $f_k(f')^\varepsilon$ form a base of $A^{R,T}//A^{R+1,T}$.

The product formula for $M_1 M_2$ now follows by expanding $M_1(M_2 v)$, using (6). The diagonal formula follows similarly from Lemma 5.3.

The proof of Theorem 5.1_p is now complete.

The proof of Theorem 5.1_2 is analogous, but somewhat simpler. As already remarked, we interpret "p" as 2, "β_p" as Sq^1 and "P_p^k" as Sq^{2k}. We replace X by the CARTESIAN product of n spaces, each of type $(Z_2, 1)$, and delete all reference to the other n' factors. $H^*(X; Z_2)$ is now a polynomial algebra. In the arguments, we replace each y_i $(1 \leqslant i \leqslant n)$ by x_i, and delete all references to x_i. Thus, we take $u = x_1 x_2 \ldots x_n$, the product of the fundamental classes. Our indices I become sequences $\{\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots\}$; α_i replaces λ_i and ε_i is deleted. D^I is the sum of all monomials (in the x_i) in which exactly α_i exponents are 2^i, the rest being 1. We define $I(r,k)$ by $\alpha_r = k$, $\alpha_i = 0$ for $i \neq r$; we define $e_{r,k} \in A$ by $e_{r,k}(u) = D^{I(r,k)}$. Thus $\operatorname{Deg} e_{r,k} = k(2^r - 1)$. These $e_{r,k}$ replace those defined for $p > 2$; we do not define any e'_r, and delete all references to them. In the proof of Theorem 5.4, we order the sequences $\{\alpha_i\}$ lexicographically from the right. In defining $A^{R,T}$ and $D^{R,T}$, and in all subsequent arguments, we replace T by $T+1$ in each inequality restricting the choice of generators $e_{r,k}$ or of entries α_i in sequences. The rank of $A^{R,T}$ is thus 2^V where $V = \frac{1}{2}(T - R + 1)(T - R + 2)$. We represent $A^{P,T}//A^{Q,T}$ on a CARTESIAN product Y_Q of n real projective spaces of dimension $2^Q - 1$. In Theorem 5.11, we obtain base elements $f_k = \{e_{R,k}\}$ without elements f'.

With these alterations and interpretations, all our intermediate theorems remain valid.

We next pass on to the last theorems of this section. If $p > 2$, we have

Theorem 5.12$_p$. *There is a homomorphism $h: A \to A$ of algebras with diagonal such that*

$$h(e_{r,k}) = \begin{cases} e_{r,k/p} & \text{if } k \equiv 0 \bmod p \\ 0 & \text{otherwise} \end{cases} \qquad h(e'_r) = 0 \ .$$

If $p = 2$, the theorem remains valid on interpreting "p" as 2 and omitting all reference to e'_r.

Proof. Let us take the space X as before, but with $n' = 0$; and in what follows, let us suppose as necessary that n is sufficiently large.

Let I be a sequence. Let the polynomial E^I be formed from D^I by substituting $(y_i)^p$ for y_i $(1 \leqslant i \leqslant n)$; similarly, let $v = y_1^p y_2^p \ldots y_n^p$. Let E be the submodule of $H^*(X)$ generated by the E^I; it is clearly closed under A.

Each element of A induces a linear map of E, and these constitute a quotient ring R of A. We will next show that the linear map $\{a\}$ is determined by $a(v)$. In fact, we have an analogue of Lemma 5.3; if $\Delta a = \sum_i a^{L,i} \otimes a^{R,i}$, then $a(v)$ determines $a^{L,i}(v)$, $a^{R,i}(v)$. This implies the following statement. Suppose that $a(v)$ determines $a(w)$ and $a(z)$ for all $a \in A$ and certain w, z. Then

$a(v)$ determines $a(wz)$ for all $a \in A$. But $a(v)$ determines $a(y_i^p)$ for all a and each i; therefore $a(v)$ determines $a(E^I)$.

This argument shows that in the epimorphism $A \to R$, the diagonal map passes to the quotient.

Next let S be the subalgebra $\sum_i A_{i,0}$ of A. We will identify R with S. In fact, let J run over the sequences in which $\varepsilon_i = 0$ for all i; and define a function $g: R \to S$ by $g\{a\} = b$ where

$$a(v) = \sum_J \lambda_J E^J \quad \text{and} \quad b(u) = \sum_J \lambda_J D^J .$$

It is clear that g commutes with the diagonal. We will show that g is a homomorphism. In fact, by the definition of g, we have the following statement, in which y stands for a fundamental class y_i:

$$\text{If } a \in A, \text{ and } ay^p = \sum_j \lambda_j y^{pj}, \text{ then } (g\{a\})y = \sum_j \lambda_j y^j .$$

From this we deduce, by passing to products, that:

$$\text{If } a \in A, \text{ and } ay^{pk} = \sum_l \mu_l y^{pl}, \text{ then } (g\{a\})y^k = \sum_l \mu_l y^l .$$

It follows that $g\{a\}((g\{b\})y) = (g\{ab\})y$.

Again, since g commutes with the diagonal, the statements

$$g\{a\}g\{b\}w = g\{ab\}w \;, \quad g\{a\}g\{b\}z = g\{ab\}z \quad (\text{for all a}, b)$$

imply $g\{a\}g\{b\}wz = g\{ab\}wz$. Therefore, for the fundamental class u, we have $g\{a\}g\{b\}u = g\{ab\}u$. Thus $g\{a\}g\{b\} = g\{ab\}$, and g is homomorphic.

It is also clear that g is monomorphic.

Lastly, the composite h: $A \to R \xrightarrow{g} S$ satisfies

$$h(e_{r,k}) = \begin{cases} e_{r,k/p} & \text{if } \; k \equiv 0 \pmod p \\ 0 \text{ otherwise} \end{cases}$$

$$h(e_r') \;=\; 0$$

Thus g is epimorphic. The existence and properties of h are established.

We state explicitly that h does not preserve the grading. We have

$$h(A_{i,j}) = \begin{cases} A_{i/p,0} & \text{if } \; i \equiv 0 \pmod p) \quad \text{and} \quad j = 0 \\ 0 \quad \text{otherwise} \end{cases}$$

The proof remains valid for $p = 2$ on interpreting "y_i" as x_i and "S" as A.

For the next theorem, we will fix on an ordering of the generators e_r' and $(e_{r,p^i})^{d_{r,i}}$. For definiteness, we take the ordering e_1', $(e_{1,1})^{d_{1,0}}$, $(e_{1,p})^{d_{1,1}}, \ldots,$ e_2', $(e_{2,1})^{d_{2,0}}$, $(e_{2,p})^{d_{2,1}}, \ldots, e_3', \ldots$

Next consider the anticommutator $[e,f] = ef - (-1)^{\varepsilon} fe$. Here e, f are distinct generators, with indices r, s say: $\varepsilon = 1$ if $e = e'_r$, $f = e'_s$ and $\varepsilon = 0$ otherwise. One of ef, fe is a permitted monomial for the ordering above; the other is not. The anticommutator can be expanded as a sum of permitted monomials. The dimension concerned contains at most one generator g, if any; let its coefficient in this expansion be λ, or if there is no generator in this dimension, set $\lambda = 0$. Since the anticommutator maps to zero in $A^{1,\infty}//A^{r+s,\infty}$ (by Theorem 5.1) its expansion takes the following form:

$$[e,f] = \lambda g + \sum_i M_i N_i \ .$$

Here $M_i \in I(A)$, $g, N_i \in I(A^{r+s,\infty})$, and g is a generator. If $p > 2$ we have the following theorem:

Theorem 5.13$_p$. *$\lambda \neq 0$ (mod p) in the following cases, and in these only.*
(a) *The pair e, f is e_{r,p^i}, $e_{s,p^{r+i}}$ in either order.*
(b) *The pair e, f is e'_r , $e_{s,p^{r-1}}$ in either order.*
If $p = 2$ the theorem remains valid on interpreting "p" as 2 and omitting all references to e'_r, including case (b).

Proof. We may first eliminate the case $e = e'_r$, $f = e'_s$, since there is then no g.

We take next the case $e = e'_r$, $f = e_{s,p^j}$, $g = e'_t$. The dimensions must satisfy

$$\sum_{u=0}^{r-2} p^u + p^j \sum_{u=0}^{s-1} p^u = \sum_{u=0}^{t-2} p^u \ .$$

The only solution is $j = r - 1$, $t = r + s$. We then have $[e,f] = -g$ by direct use of the Cartan representation.

We take third the case $e = e_{r,p^i}$, $f = e_{s,p^j}$, $g = e_{t,p^k}$. The dimensions must satisfy

$$p^i \sum_{u=0}^{r-1} p^u + p^j \sum_{u=0}^{s-1} p^u = p^k \sum_{u=0}^{t-1} p^u \ .$$

There are only two solutions; one is $k = i$, $j = r + i$, $t = r + s$; the other is obtained by interchanging e and f. It is sufficient to consider the first.

Consider the case $i = 0$. Here $[e,f] = -g$ by direct use of the Cartan representation.

Consider the case $i > 0$. Form the expansion

$$[e,f] = \lambda g + \sum_l M_l N_l \ .$$

Apply, i times over, the homomorphism h of Theorem 5.12$_p$. We obtain (say)

$$[e',f'] = \lambda g' + \sum_l M'_l N'_l \ .$$

Here the monomials $M'_l N'_l$ are still permitted (by the special choice of ordering); while $e' = e_{r,1}$, $f' = e_{s,p^r}$, $g' = e_{r+s,1}$. We have $\lambda = 1$ by the case $i = 0$.

This concludes the proof in case $p > 2$.

In case $p = 2$, we omit all references to e'_r. We have also to consider two further solutions of the equation for the dimensions.

Case (i) $i = j$, and either $r = 1$ or $s = 1$; say $s = 1$. Then $t = 1$ and $k = r + i$.

Case (ii) $i = j$, but neither $r = 1$ nor $s = 1$. We then have $k = i + 1 = j + 1$, $t = r = s$.

Both cases may be eliminated, since $g \in A^{r+s,\infty}$, so that $t \geqslant r + s$.

This concludes the proof in case $p = 2$. We have now obtained sufficient data on the structure of the STEENROD algebra.

§ 6. The Cohomology of the STEENROD Algebra

In this article we prove Theorem 2.4. The results of this section, therefore, are very far from complete, compared with those of § 5.

We take $p = 2$ throughout this article, and will be free to use the letter p for other purposes. Tensor products are taken over Z_2 unless otherwise stated.

When we wish to display specific elements in $H^*(A)$, we use the notation of the bar construction (see [3] p. 3–09). Thus we define

$$I(A) = \sum_{q>0} A_q\,, \quad (I(A))^0 = Z_2 \quad \text{and} \quad (I(A))^s = I(A) \otimes (I(A))^{s-1}\,.$$

We define $\bar{B}(A) = \sum_{s \geqslant 0} (I(A))^s$, and write a typical element as $[a_1|a_2|\ldots|a_s]$. The second grading t of $\bar{B}(A)$ is defined by $t = \sum_1^s q_i$ when $a_i \in A_{q_i}$. A boundary in $\bar{B}(A)$ is defined by

$$d[a_1|\ldots|a_s] = \sum_{1 \leqslant i \leqslant s} (-1)^i [a_1|\ldots|a_i a_{i+1}|\ldots|a_s]\,;$$

d is of degree $(-1, 0)$. The cohomology group $H^{s,t}(\bar{B}(A), Z_2)$ is $\mathrm{Ext}_A^{s,t}(Z_2, Z_2)$, that is $H^{s,t}(A)$.

For example, let us calculate $H^1(A)$. (Here, by convention, $H^s(A) = \sum_t H^{s,t}(A)$.) There are no coboundaries (except the zero cochain); while a cocycle is a Z_2-linear function f, with values in Z_2, defined for arguments $[a]$ $(a \in I(A))$, and such that $f[a_1 a_2] = 0$ (each $a_1, a_2 \in I(A)$.) By the known structure of the STEENROD algebra, there are unique cocycles f_m of dimension $t = 2^m$

$(m \geqslant 0)$ such that $f_m[Sq^{2^m}] = 1$; and these form a base for the cocycles. Define $h_m = \{f_m\}$; we have a base for $H^1(A)$.

The cup-product of two cocycles is obtained by transposition from the diagonal map

$$\Delta : \bar{B}(A) \to \bar{B}(A) \otimes \bar{B}(A) .$$

This is defined by

$$\Delta [a_1|a_2|\ldots|a_n] = \sum_{0 \leqslant p \leqslant n} [a_1|\ldots|a_p] \otimes [a_{p+1}|\ldots|a_n] .$$

As stated in § 2, our methods depend on a spectral sequence ([4], p. 349) relating the cohomology rings $H^*(\Lambda)$, $H^*(\Gamma)$ and $H^*(\Omega)$ of an algebra Γ, a normal subalgebra Λ and the corresponding quotient algebra $\Omega = \Gamma//\Lambda$. It is sufficient to take Λ central in Γ; this has the result that the operations of Ω on $H^*(\Lambda)$ become trivial, and we have $E_2^* \cong H^*(\Lambda) \otimes H^*(\Omega)$.

It is convenient to have a specific construction for the spectral sequence. We may obtain it by filtering $\bar{B}(\Gamma)$. A chain $[a_1|\ldots|a_s]$ is of filtration p if $a_i \in I(\Lambda)$ for $(s-p)$ values of i; we thus obtain homology and cohomology spectral sequences, in good duality. The isomorphisms

$$H^q(\Lambda) \leftarrow E_2^{0,q}, \quad E_2^{p,0} \leftarrow H^p(\Omega)$$

are induced by the natural maps $\bar{B}(\Lambda) \to \bar{B}(\Gamma)$, $\bar{B}(\Gamma) \to \bar{B}(\Omega)$. The (cohomology) spectral sequence has good products; these induce the isomorphism

$$H^p(\Omega) \otimes H^q(\Lambda) \to E_2^{p,0} \otimes E_2^{0,q} \to E_2^{p,q} .$$

Since this section is not a final treatment, the reader will perhaps excuse it if we do not give the details more fully.

We next proceed to the details of the calculation. We will write A^r for $A^{r,\infty}$, so that $A^1 = A$. We recall that $A^r//A^{r+1}$ is a divided polynomial algebra, with a Z_2-base $\{f_k\}$. Take a cocycle $f'_{r,i}$ in dimension $(1, (2^r-1)2^i)$ such that $f'_{r,i}[f_{2^i}] = 1$; let $h_{r,i}$ be its cohomology class. Then $H^*(A^r//A^{r+1})$ is a polynomial algebra with generators $h_{r,i}$ (for $i \geqslant 0$).

Let h_m be the image in $H^*(A^1//A^r)$ of $h_{1,m}$ in $H^*(A^1//A^2)$. This is consistent, for the original h_m in $H^*(A^1)$ is the image of $h_{1,m}$. By the same argument as before, $H^1(A^1//A^r)$ has as a base the elements h_m in it.

Now take $\Lambda = A^{r+1}//A^{r+2}$, $\Gamma = A^1//A^{r+2}$, $\Omega = A^1//A^{r+1}$. Let $d^2 : E_2^{0,1} \to E_2^{2,0}$ be the differential in the spectral sequence. Since

$$H^1(A^1//A^{r+2}) \to H^1(A^{r+1}//A^{r+2})$$

is zero (for $r \geqslant 1$), the classes $d^2 h_{r+1,i}$ must be non-zero elements of $H^2(A^1//A^{r+1})$. Let us write

$$d^2 h_{r+1,i} = g_{r,i} \in H^2(A^1//A^{r+1}) .$$

In case $r=1$, we have $g_{1,i}=h_i h_{i+1}$. This is proved as follows. In $H_2(A^1//A^2)$ we have one non-zero class in dimension $t = 3.2^i$; it is dual to $h_i h_{i+1}$, and may be represented by the cycle

$$[e_{1,2^i} | e_{1,2^{i+1}}] + [e_{1,2^{i+1}} | e_{1,2^i}] \quad (= z, \text{ say}).$$

By Theorem 5.13_2, we have in A a relation

$$e_{1,2^i} e_{1,2^{i+1}} + e_{1,2^{i+1}} e_{1,2^i} = \sum_j M_j N_j + e_{2,2^i}$$

where $M_j \in I(A^1)$, $N_j \in I(A^2)$. The chain

$$[e_{1,2^i} | e_{1,2^{i+1}}] + [e_{1,2^{i+1}} | e_{1,2^i}] + \sum_j [M_j | N_j]$$

has as boundary $[e_{2,2^i}]$; it is thus a cycle in $E^2_{2,0}$ and by the map $A^1 \to A^1//A^2$ it passes to z. Thus $d_2\{z\} = \{[e_{2,2^i}]\}$. Transposing into cohomology we have $d^2 h_{2,i} = h_i h_{i+1}$, as asserted.

In case $r > 1$, we must consider the behaviour of the class $g_{r,i}$ in the spectral sequence which arises when $\Lambda = A^r//A^{r+1}$, $\Gamma = A^1//A^{r+1}$, $\Omega = A^1//A^r$. We know that $H^2(A^1//A^{r+1})$ is filtered. The class $g_{r,i}$ will have an image in the first quotient, which is isomorphic to a subgroup of $E_2^{0,2}$; and if this image is zero, then $g_{r,i}$ will have an image in the second quotient, which is isomorphic to a subgroup of $E_2^{1,1}$. Now $E_2^{0,2} \cong H^2(A^r//A^{r+1})$, which we know; the image of $g_{r,i}$ is necessarily zero, by considering the grading t.

We seek, therefore, the image of $g_{r,i}$ in a subgroup of $E_2^{1,1}$. We will show it is exactly

$$h_{r,i+1} h_i + h_{r,i} h_{r+i} \;;$$

here the products are formed by considering h_j as an element of $E_2^{1,0}$ and $h_{r,j}$ as an element of $E_2^{0,1}$. In fact, by transposition, it is sufficient to determine the pairing of $g_{r,i}$ with a certain quotient module of $E^2_{1,1}$. We will construct representative cycles for $E^2_{1,1}$. By Theorem 5.13_2 we have relations

$$e_{r,2^{i+1}} e_{1,2^i} + e_{1,2^i} e_{r,2^{i+1}} = \sum_j M_j N_j + e_{r+1,2^i}$$

$$e_{r,2^i} e_{1,2^{r+1}} + e_{1,2^{r+1}} e_{r,2^i} = \sum_j P_j Q_j + e_{r+1,2^i}$$

where $M_j, P_j \in I(A^1)$, $N_j, Q_j \in I(A^{r+1})$. Thus the chains

$$[e_{r,2^{i+1}} | e_{1,2^i}] + [e_{1,2^i} | e_{r,2^{i+1}}] \quad (= w, \text{ say})$$

$$[e_{r,2^i} | e_{1,2^{r+i}}] + [e_{1,2^{r+i}} | e_{r,2^i}] \quad (= z, \text{ say})$$

give classes in $H_2(A^1//A^{r+1})$. These chains are of filtration 1, and yield a base for $E^2_{1,1}$ in dimension $t = (2^{r+1} - 1)2^i$ (as one verifies, knowing the structure of $E_2^{1,1}$). On applying the map $d_2 \colon H_2(A^1//A^{r+1}) \to H_1(A^{r+1}//A^{r+2})$, we have $d_2\{w\} = d_2\{z\} = \{[e_{r+1,2^i}]\}$. (This is proved by exactly the same argument as was used in the case $r = 1$; note that this d_2 lies in the spectral sequence for $\Lambda = A^{r+1}//A^{r+2}$, etc.) We have, then,

$$\{w\}g_{r,i} = \{z\}g_{r,i} = 1 \ .$$

On transposing, we have in $E_2^{1,1}$ the relation

$$\{g_{r,i}\} = h_{r,i+1}\, h_i + h_{r,i}\, h_{r+i} \ ,$$

as asserted.

Lemma 6.1. *In $H^2(A^1//A^{r+1})$ the elements $g_{r,i}$, with the elements $h_j h_k$ for which $j \leqslant k - 2$ or $j = k$, are linearly independent. The elements $h_j h_{j+1}$ are zero.*

In fact, if $r > 1$, then $H^2(A^1//A^{r+1})$ is filtered; the elements $g_{r,i}$ map to a linearly independent set in $E_2^{1,1}$, while the elements $h_j h_k$ map to zero. It remains to prove (by induction over r) that the elements $h_j h_k$ ($j \leqslant k - 2$ or $j = k$) are linearly independent. If they are so in $H^2(A^1//A^r)$, then they are so in $H^2(A^1//A^{r+1})$, unless in the spectral sequence concerned some linear combination of elements $h_j h_k$ is equal to $d^2(h_{r,l})$. This is impossible, by considering the grading t. To begin the induction, the elements named are linearly independent in $H^2(A^1//A^2)$.

The elements $h_j h_{j+1}$ are zero in $H^2(A^1//A^3)$, being $d^2(h_{2,j})$.

We next consider again the spectral sequence which arises when

$$\Lambda = A^r//A^{r+1} \ , \quad \Gamma = A^1//A^{r+1} \ , \quad \Omega = A^1//A^r \ .$$

Since $g_{r,i}$ lies in $H^2(A^1//A^{r+1})$, the elements $h_{r,i+1}h_i + h_{r,i}h_{r+i}$ in $E_2^{1,1}$ are cocycles for d^2. Thus the elements $(h_{r,i+1}h_i + h_{r,i}h_{r+i})h_j$ ($= z_{r,i,j}$ say) in $E_2^{2,1}$ are cocycles for d^2.

Lemma 6.2. *If $r \geqslant 2$, the classes $\{z_{r,i,j}\}$ in $E_3^{2,1}$ satisfy the following relations only:*

$$\{z_{r,i,r+i+1} + z_{r,i+1,i}\} = 0 \quad (r \geqslant 2) \ ,$$
$$\{z_{2,i,i+1}\} = 0 \quad (r = 2) \ .$$

If $r \geqslant 2$, $E_3^{0,2}$ has as a base the classes $\{(h_{r,i})^2\}$.

Proof. We have to examine $d^2 \colon E_2^{0,2} \to E_2^{2,1}$. Here $E_2^{0,2}$ has as a base the elements $h_{r,i}h_{r,j}$. In $E_2^{2,1}$ (if $r > 2$) the elements

$$h_{r,i}g_{r-1,j}, \quad h_{r,i}h_j h_k \quad (j \leqslant k - 2 \text{ or } j = k)$$

are linearly independent, while the elements $h_{r,i} h_j h_{j+1}$ are zero; this is by Lemma 6.1. The boundaries are given by

$$d^2(h_{r,i} h_{r,j}) = h_{r,i} g_{r-1,j} + h_{r,j} g_{r-1,i} \ .$$

From this, the conclusion follows.

In case $r = 2$, $E_2^{2,1}$ has as a base the elements $h_{2,i} h_j h_k$, while the boundaries are given by

$$d^2(h_{2,i} h_{2,j}) = h_{2,i} h_j h_{j+1} + h_{2,j} h_i h_{i+1} \ .$$

The conclusion is again elementary.

Lemma 6.3. *If* $r \geqslant 2$, *the classes* $\{g_{r,i} h_j\}$ *in* $H^3(A^1//A^{r+1})/p^* H^3(A^1//A^r)$ *satisfy the following relations only:*

(i) $g_{r,i} h_{r+i+1} + g_{r,i+1} h_i = 0 \quad (r \geqslant 2)$,
(ii) $\{g_{2,i} h_{i+1}\} = 0 \quad (r = 2)$.

Proof. The image of $g_{r,i} h_j$ in $E_3^{2,1}$ is $\{z_{r,i,j}\}$. On applying Lemma 6.2, we obtain the results of this lemma, except for the exact relation (i). This follows immediately from the remark (above) that $h_{r,i+1} h_i + h_{r,i} h_{r+i}$ is a cocycle for d^2.

Lemma 6.4. *In* $H^3(A^1//A^{r+1})$ $(r \geqslant 3)$ *the elements* $h_i h_j h_k$ *are subject to the following relations only.*

(i) $h_i h_{i+1} h_j = 0$.
(ii) $(h_i)^2 h_{i+2} = (h_{i+1})^3$.
(iii) $h_i (h_{i+2})^2 = 0$.

Proof. We know the structure of $H^3(A^1//A^2)$ and will prove the lemma by induction over r. We have to consider the spectral sequence in which $\Lambda = A^{r+1}//A^{r+2}$, $\Gamma = A^1//A^{r+2}$ and $\Omega = A^1//A^{r+1}$. In this we have to consider the differentials $d^2 : E_2^{1,1} \to E_2^{3,0}$ and $d^3 : E_3^{0,2} \to E_3^{3,0}$.

We have $d^2(h_{r+1,i} h_j) = g_{r,i} h_j$. By Lemma 6.3 this introduces no new relations into $p^* H^3(A^1//A^r)$ if $r > 2$. If $r = 2$, we have only to consider the case $j = i + 1$. We verify that

$$d^2(h_{3,0} h_1) = h_0 (h_2)^2$$

by a direct calculation (given below); the result

$$d^2(h_{3,i} h_{i+1}) = h_i (h_{i+2})^2$$

follows by using the homomorphism of Theorem 5.12_2. If $r = 1$, we have

$$d^2(h_{2,i} h_j) = h_i h_{i+1} h_j \ .$$

We next consider d_3. By Lemma 6.2, $E_3^{0,2}$ has as a base the classes

$$\{(h_{r+1,i})^2\} .$$

By considering the grading t, we see that d^3 introduces no new relations between the classes $h_i h_j h_k$ unless $r = 1$. We then verify

$$d^3\{(h_{2,0})^2\} = (h_0)^2 h_2 + (h_1)^3$$

by direct calculations (given below); the result

$$d^3\{(h_{2,i})^2\} = (h_i)^2 h_{i+2} + (h_{i+1})^3$$

follows by using the homomorphism of Theorem 5.12_2.

The direct calculations are similar to those above. They are carried out in homology, and are as follows (for brevity we have replaced the symbol $e_{r,2^i}$ by its dimension $(2^r - 1)2^i$.)

$$d\left\{\begin{matrix} [1|4|4] + [4|1|4] + [4|4|1] \\ + [1|2|6] + [2|1|6] + [2|6|1] \\ + [4|3|2] + [3|4|2] + [3|2|4] \end{matrix}\right\} = [2|7] + [7|2]$$

$$d\left\{\begin{matrix} [2|1|3] + [1|2|3] + [1|3|2] \\ + [2|2|2] \end{matrix}\right\} = [3|3]$$

$$d\left\{\begin{matrix} [4|1|1] + [1|4|1] + [1|1|4] \\ + [1|2|3] + [2|1|3] + [2|3|1] \end{matrix}\right\} = [3|3] .$$

This concludes the proof of Lemma 6.4, of Theorem 2.4, and of Theorem 1.1.

REFERENCES

[1] J. ADEM, *The iteration of the* STEENROD *squares in algebraic topology*, Proc. Nat. Acad. Sci. U. S. A. 38, 1952, p. 720–726.
[2] H. CARTAN, *Sur l'itération des opérations de* STEENROD. Comment. Math. Helv. 29, 1955, p. 40–58.
[3] H. CARTAN, *Séminaire* H. CARTAN *de l'Ecole Normale Supérieure*, 1954/55.
[4] H. CARTAN and S. EILENBERG, *Homological Algebra*. Princeton Mathematical Series.
[5] H. TODA, *Le produit de* WHITEHEAD *et l'invariant de* HOPF. C. R. Acad. Sci. (Paris) 241, 1955, p. 849–850.

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