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An Entire Function of Restricted Growth

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W. KAPLAN [1] has used an approximation theorem of KELDYS and MERGELYAN to show that there exists an entire function f such that each half-line in the z-plane contains infinitely many disjoint segments of length 1 on which $|f(z)| < 1$. In an oral communication, he has asked whether there exists such a function of finite order. I shall construct an appropriate example.

Theorem. *There exists a sequence* $\{(t_n, r_n)\}$ *such that the function*

$$f(z) = \prod_{n=1}^{\infty} \{1 - (z/r_n)^n\}^{t_n} \tag{1}$$

is entire and has the property that each half-line contains infinitely many disjoint segments of length 1 *on which* $|f(z)| < 1$. *Corresponding to each real-valued function* $h(r)$ *satisfying the condition* $h(r)/\log r \to \infty$ *as* $r \to \infty$, *the sequence* $\{(t_n, r_n)\}$ *can be chosen in such a way that the inequality*

$$|f(re^{i\theta})| < r^{h(r)}$$

holds for all r $(r > r_0)$ *and all* θ.

The proof will be carried out by choosing the pairs (t_n, r_n) in such a way that $|f(z)| < 1/n$ throughout each of the annular sectors S_{nh} $(h = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ determined by the conditions

$$z = (1 + s)r_n e^{i(\theta + 2\pi h/n)} \quad (s \text{ and } \theta \text{ real}, |s| < 1/n^2, |\theta| < \pi/n^2).$$

We consider first the product of the first m factors in (1), for values of z in one of the sectors S_{mh}. The restrictions on s and θ imply that

$$|1 - (z/r_m)^m| = |1 - (1 + s)^m e^{im\theta}| < C_1/m$$

(throughout the proof, we shall use symbols C_j to denote constants independent of m). This in turn implies that

$$|1 - (z/r_m)^m|^{t_m} < (C_1/m)^{t_m}. \tag{2}$$

Also, if $2 < r_1 < r_2 < \ldots$, then, for each index k $(k = 1, 2, \ldots, m - 1)$ and for all z in S_{mh}, we may write

$$|1 - (z/r_k)^k| < (C_2 r_m)^k.$$

Therefore, if $r_m > 1/C_2$ and

$$t_n > t_1 + 2t_2 + \dots + (n-1)\,t_{n-1}$$

for $n = 2, 3, \dots,$ we conclude that

$$\prod_{k=1}^{m-1} |\, 1 - (z/r_k)^k \,|^{t_k} < (C_2 r_m)^{t_1 + 2t_2 + \dots + (m-1)t_{m-1}} < (C_2 r_m)^{m\,t_{m-1}}$$

in S_{mh}. Together with (2), this yields the inequality

$$\prod_{k=1}^{m} |\, 1 - (z/r_k)^k \,|^{t_k} < (C_1/m)^{t_m} (C_2\, r_m)^{m\,t_{m-1}}\,.$$

For an appropriate constant C_3, the right member is less than $1/2m$ if

$$r_m = \exp \frac{C_3 t_m \log m}{m\,t_{m-1}}\,. \tag{3}$$

We note that r_m, determined as a function of t_m by (3), tends to ∞ as $t_m \to \infty$.

Suppose that the constants t_n have been chosen for $n = 1, 2, \dots, m,$ and that each corresponding constant r_n satisfies (3) (with $t_0 = 1$). Since the product of the first m factors in (1) has modulus less than $1/2m$ in each of the m sectors S_{mh} $(h = 1, 2, \dots, m)$, the condition $|\, f(z)\,| < 1/m$ holds in each of the sectors provided the product of the remaining factors has modulus less than 2. Now

$$\begin{aligned}\prod_{m+1}^{\infty} |\, 1 - (z/r_n)^n \,|^{t_n} &\le \prod_{m+1}^{\infty} \{1 + (\,|\,z\,|/r_n)^n\}^{t_n} \\ &= \prod_{m+1}^{\infty} \exp\{t_n \log[1 + (|z|/r_n)^n]\} \\ &< \prod_{m+1}^{\infty} \exp\{t_n(|z|/r_n)^n\} \\ &= \exp \sum_{m+1}^{\infty} t_n \,|\, z\,|^n \exp\left\{-\frac{C_3 t_n \log n}{t_{n-1}}\right\}.\end{aligned}$$

Throughout each fixed disk $|\,z\,| < \varrho$, and for any fixed value of t_{n-1}, the term with index n in the infinite series in the last expression tends to 0 uniformly as $t_n \to \infty$. Therefore $|f(z)\,| < 1/n$ throughout each of the sectors S_{nh} $(h = 1, 2, \dots, n;\ n = 1, 2, \dots)$ provided $t_n \to \infty$ rapidly enough.

If $r_n > n^3$ for each n, then, by a familiar theorem on approximation by rational numbers [2, p. 48, Theorem 14], each half-line in the plane meets infinitely many sectors S_{nh} in a segment of length greater than 1. This proves the first part of the theorem.

The second part of the theorem follows from the facts that the logarithm of the maximum modulus function $M(r\,;\,f)$ is a convex function of $\log r$; that

$$|\,f(r_n e^{i\theta})\,| < (C_4 r_n)^{n t_n}\ ;$$

and that by (3),

$$t_n = \frac{n\, t_{n-1}}{C_3 \log n} \log r_n\ .$$

REFERENCES

[1] W. KAPLAN, *Paths of rapid growth of entire functions.* Comment. Math. Helv. 34/1 (1959).
[2] O. PERRON, *Die Lehre von den Kettenbrüchen.* Leipzig-Berlin, 1929.

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