Minimal Surfaces in the Large.

Minimal Surfaces in the Large[1])

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1. Introduction

The purpose of this paper is to give what seem to be the "correct" definitions for the global study of minimal surfaces, and to derive a number of consequences. We obtain, in particular, the following result:

A complete minimal surface whose normals omit a neighborhood of some direction must be a plane.

This theorem, with the additional hypothesis that the surface be simply connected, was conjectured by Nirenberg, and was proved in [3]. The proof given there, however, is quite different from the present one. In fact the main result of this paper is, in a sense, Lemma 5, in which we show that a certain geometric problem concerning minimal surfaces is precisely equivalent to a purely analytic problem concerning analytic functions in the unit circle. Once this equivalence is established, the above result follows almost trivially. Furthermore, we use it to disprove a second conjecture of Nirenberg, in which he suggests that one should be able to replace the assumption that the normals omit a full neighborhood, by the weaker one that the normals omit three distinct directions. The use of Liouville's Theorem in proving the first conjecture, and a possibly analogous use of Picard's Theorem, made the second conjecture quite plausible, but we shall in fact prove:

There exists a complete, simply-connected minimal surface whose normals omit precisely four distinct directions.

This result, in combination with the previous one, leads one immediately to ask if there is a best possible theorem of this type. For reasons which will become apparent in the course of the proofs given below, we are led to make the following guess.

Conjecture[2]). *Given a closed point set E on the unit sphere, necessary and sufficient that there exist a complete minimal surface whose normal map omits precisely the points of E, is that E have logarithmic capacity zero.*

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[1]) This paper was presented as a talk to the International Colloquium on Differential Geometry and Topology at Zurich, June 1960.

[2]) After this paper was written, L. Ahlfors pointed out that half of the above conjecture (the necessity) follows easily from Lemmas 5 and 6 below. We have added a brief section at the end of this paper giving the reasoning.

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2. Simply-connected minimal surfaces

Throughout this paper we shall use the following notation:

D will denote either the unit disk, $|\zeta| < 1$, or the entire complex ζ-plane, $|\zeta| < \infty$.

E^3 will denote EUCLIDEAN 3 space.

$\mathfrak{x}$ will denote a point of E^3 with coordinates (x_1, x_2, x_3), or the corresponding vector.

Definition. A *simply-connected minimal surface* S is a mapping $x_k = h_k(\zeta)$, $k = 1, 2, 3$, of D into E^3, such that the functions $h_k(\zeta)$ are harmonic and satisfy the further conditions that if we set

$$\varphi_k(\zeta) = \frac{\partial h_k}{\partial \xi} - i\frac{\partial h_k}{\partial \eta}, \qquad \zeta = \xi + i\eta, \tag{1}$$

then

$$\begin{aligned} &\text{a)} \quad \sum_{k=1}^{3} \varphi_k^2(\zeta) = 0, \\ &\text{b)} \quad \sum_{k=1}^{3} |\varphi_k^2(\zeta)| \neq 0, \end{aligned} \tag{2}$$

for all ζ in D.

Remark. From the fact that the functions $h_k(\zeta)$ are harmonic, it follows that the functions $\varphi_k(\zeta)$ are analytic. Thus, to every simply-connected minimal surface there corresponds by (1) a set of analytic functions $\varphi_k(\zeta)$ satisfying (2). Conversely, if we are given any three analytic functions $\varphi_k(\zeta)$ satisfying (2), and ask for harmonic functions $h_k(\zeta)$ related to $\varphi_k(\zeta)$ by (1), we have

$$h_k(\zeta) = \mathrm{Re} \int \varphi_k(\zeta) d\zeta$$

which means that the functions $h_k(\zeta)$ are determined up to an additive constant, or more geometrically, that the analytic functions $\varphi_k(\zeta)$ determine S up to translation.

By way of justification of our definition, we note that in terms of the classical notation

$$E = \mathfrak{x}_\xi \cdot \mathfrak{x}_\xi, \qquad F = \mathfrak{x}_\xi \cdot \mathfrak{x}_\eta, \qquad G = \mathfrak{x}_\eta \cdot \mathfrak{x}_\eta \tag{3}$$

we have the formulae

$$\begin{aligned} &\sum_{k=1}^{3} \varphi_k^2 = (E - G) - 2iF, \\ &\sum_{k=1}^{3} |\varphi_k|^2 = E + G. \end{aligned} \tag{4}$$

Hence condition (2a) simply says that $E = G$, $F = 0$, which means that (ξ, η) are isothermal coordinates. Condition (2b), in the presence of a), precisely guarantees that the JACOBian matrix of the map of D into E^3 has everywhere (maximum possible) rank two. This latter condition implies that each point has a neighborhood which is mapped one-to-one onto a piece of surface S' in E^3, and the fact that the coordinate functions x_k are harmonic in terms of isothermal parameters is a well-known condition for S' to be a minimal surface in the classical sense.

Conversely, if we start with a "general surface" S, defined by a mapping of an abstract differentiable surface S_0 into E^3, where we assume that the JACOBian matrix of the mapping has everywhere rank two, then the metric in E^3 induces a differential metric on S_0, and a corresponding complex-analytic structure (provided that S_0 is orientable, which it will be, in particular, if it is simply-connected). Namely, the admissible coordinates are those which represent isothermal parameters with a given orientation. The condition that the image surface in E^3 be locally a minimal surface is, as mentioned above, that the coordinate functions be harmonic in terms of isothermal parameters. We are thus led back to conditions (1) and (2) in terms of arbitrary isothermal parameters ξ, η. Finally S is said to be simply-connected if S_0 is, and we can then, by the KOEBE uniformization theorem, choose a single coordinate system on all of S_0, and in fact choose this to be either the unit disk or the plane. (The alternative of the sphere cannot occur here, for if S_0 were compact, then all the coordinate functions, being harmonic, would be constant.)

For all the classical local properties of minimal surfaces used here and in the following, we refer to Chapter II, "Minimal surfaces in the small," of the book of RADO [5].

We proceed further, as in the chapter just cited, by replacing the triple of functions φ_k satisfying relation (2a) by a pair of essentially arbitrary functions f, g.

Lemma 1. *Given an arbitrary simply-connected minimal surface, and the associated analytic functions $\varphi_k(\zeta)$ defined by (1), if we set*

$$f = \varphi_2 - i\varphi_2 \tag{5}$$

$$g = \varphi_3/(\varphi_2 - i\varphi_2)\,, \tag{6}$$

then f and g have the properties:

A. *f is analytic throughout D; the zeros of f, if any, are of even order.*

B. *g is meromorphic in D; poles of g correspond precisely to zeros of f,* *and more specifically, g has a pole of order m at a point where f has a zero of order $2m$.*

The functions φ_k are expressed in terms of f and g by

$$\begin{aligned}\varphi_1 &= \tfrac{1}{2}f(1-g^2)\\ \varphi_2 &= \frac{i}{2}f(1+g^2)\\ \varphi_3 &= fg\,.\end{aligned}\tag{7}$$

Proof. We note first that condition (2a) can be rewritten in the form

$$(\varphi_1 - i\varphi_2)(\varphi_1 + i\varphi_2) = -\varphi_3^2\,.\tag{8}$$

Thus when $f=0$ we have $\varphi_3 = 0$, and the left-hand side of (8) must have a zero of even order. But the two factors on the left cannot vanish simultaneously, since this would imply $\varphi_1 = \varphi_2 = \varphi_3 = 0$, contradicting (2b). Thus a zero of $\varphi_1 - i\varphi_2$ is always of the same order as the zero of φ_3^2 at the same point. This establishes A and B. Equations (7) follow immediately from (5), (6), and (8).

Lemma 2. *Given any two functions f and g in the domain D, satisfying conditions* A *and* B *of lemma 1, the functions φ_k defined by* (7) *will satisfy* (2), *and thus define a unique minimal surface up to translation.*

Proof. Conditions A and B guarantee that the functions φ_k will be analytic throughout D. From (7) we have

$$\sum_{k=1}^{3} |\varphi_k|^2 = \tfrac{1}{2}|f|^2(1+|g|^2)^2\tag{9}$$

and this cannot vanish because of B. Condition (2a) is a direct verification.

By virtue of lemmas 1 and 2, we see that the study of simply-connected minimal surfaces reduces to the study of the two, essentially arbitrary, functions f and g. The only restriction is on their zeros and poles, as given in A and B.

The reason why this particular pair of functions is better suited to studying the surface than certain other pairs which might serve equally for its representation, is that the functions f and g have important geometric interpretations. A special case is the following.

Lemma 3. *The condition $f=0$, or equivalently, $g=\infty$, occurs precisely at points where the normal to the surface S points in the direction of the positive x_3-axis.*

Proof. Using $\varphi_1 = i\varphi_2$, we find that

$$\mathfrak{x}_\xi \times \mathfrak{x}_\eta = (0, 0, |\varphi_1|^2) \qquad \text{if } f = 0\,.\tag{10}$$

Lemma 4. *If we let* $g = u + iv$, *we have for the unit normal* $\mathfrak{n}$,

$$\mathfrak{n} = \left(\frac{2u}{|g|^2+1}, \frac{2v}{|g|^2+1}, \frac{|g|^2-1}{|g|^2+1}\right), \quad g \neq \infty \tag{11}$$

$$\mathfrak{n} = (0, 0, 1), \quad \text{if} \quad g = \infty. \tag{12}$$

Proof. (12) follows directly from (10). (11) follows from (7). More specifically, we have

$$\mathfrak{x}_\xi = \tfrac{1}{2}(s(1 - u^2 + v^2) + 2tuv, \ -t(1 + u^2 - v^2) - 2suv, \ 2su - 2tv)$$
$$\mathfrak{x}_\eta = \tfrac{1}{2}(-t(1 - u^2 + v^2) + 2suv, \ -s(1 + u^2 - v^2) + 2tuv, \ -2sv - 2tu)$$

and

$$\begin{aligned}\mathfrak{x}_\xi \times \mathfrak{x}_\eta &= \tfrac{1}{2}(s^2 + t^2)(1 + u^2 + v^2)(u, v, \tfrac{1}{2}(u^2 + v^2 - 1)) \\ &= \tfrac{1}{2}|f|^2(1 + |g|^2)(u, v, \tfrac{1}{2}(|g|^2 - 1))\end{aligned} \tag{13}$$

from which (11) follows.

Lemma 4 plays a key role in all that follows, the reason being that (11) is precisely the formula for stereographic projection from the u, v-plane onto the unit sphere. We thus have the following crucial geometric interpretation of the function $g(\zeta)$:

The function $g(\zeta)$ *defined by* (6) *may be obtained as the composition of three mappings: the defining map of* S *from* D *into* E^3, *followed by the Gauss spherical map (by parallel normals) into the unit sphere, followed by stereographic projection from the point* (0, 0, 1).

In particular, the geometric statement that the normals to S omit certain directions is equivalent to the analytic one, that $g(\zeta)$ omits certain values.

We turn next to the Riemannian metric induced on D by the surface S. We have for the element of arc length ds, the formula

$$ds = \tfrac{1}{2}|f|(1 + |g|^2)|d\zeta| \tag{14}$$

which follows either from (13), or from (4) and (9), in view of the fact that for isothermal coordinates we have $ds^2 = E|d\zeta|^2$, and $E = G = (E + G)/2$.

We recall some standard definitions.

Definition. A *path going to the boundary* of a region is the continuous image of the half-line $t \geqslant 0$ such that for any compact subset K of the region there exists t_0 such that the image of $t \geqslant t_0$ lies outside of K.

Definition. A surface with a Riemannian metric is *complete* if every path going to the boundary has infinite length.

For a discussion of this notion of completeness, and its relation to non-continuability, we refer to the classical paper of Hopf-Rinow [1].

Essentially what we have done up to this point is to show that an arbitrary simply-connected minimal surface can be represented by a pair of functions f, g, and to express some of the basic surface quantities, such as normal and element of arc length, in terms of these functions. We are now in a position to give a purely analytic formulation of the kind of geometric theorem we wish to consider.

Lemma 5. *Let E be a point set on the unit sphere, containing at least three points, and in particular, the point $(0, 0, 1)$. Then the following two statements are equivalent.*

I. *A complete simply-connected minimal surface whose normal map omits the points of E must be a plane.*

II. *Let f and g be arbitrary analytic functions in the unit disk $|\zeta| < 1$, subject to the conditions that f is never zero, and g never takes on the values which correspond to E under stereographic projection from $(0, 0, 1)$. Then there is a path C going to the boundary of the unit disk for which*

$$\int_C |f|(1 + |g|^2)\,|d\zeta| < \infty. \tag{15}$$

Proof. Suppose first that for a given point set E, the statement II is false. Then there exists a particular pair of analytic functions f, g with $f \neq 0$, such that

$$\int_C |f|(1 + |g|^2)\,|d\zeta| = \infty \tag{16}$$

for *all* paths C going to the boundary. Since $f \neq 0$ and $g \neq \infty$, these functions automatically satisfy conditions A and B of lemma 1, and thus by lemma 2 can be used to define a simply-connected minimal surface S. By (14), we see that condition (16) means that S is complete. By lemma 4, the fact that g omits the values corresponding to E means that the normal map of S omits the points of E. Finally, as will be proved in lemma 6, g could not be constant, or else (16) would not hold for all paths C. The minimal surface S would therefore not be a plane, it would be complete and its normal map would omit the points of E. Thus if II is false, I is also false.

Suppose next that II is true for a particular set E. In order to show that I is true, we start with an arbitrary simply-connected minimal surface S whose normal map omits the points of E, and show that either it is a plane, or else it is *not* complete. We have two cases to consider, according as the domain D of definition of S is the whole plane or the unit disk.

Case 1. D is the whole plane, $|\zeta| < \infty$. Since the set E contains at least three points, we have by lemma 4 that the function $g(\zeta)$ associated with S is a meromorphic function in the whole plane which omits three values, hence is constant. Again by lemma 4, this means that S has a single normal direction, and if it is complete it must be a plane.

Case 2. D is the disk, $|\zeta| < 1$. In this case the functions f and g associated with S satisfy the hypotheses of statement II, and hence (15) must hold for some path C. But in view of (14), this says precisely that S is not complete.

Thus if II is true, I is also true, and the lemma is proved.

In the course of the above proof we made use of a purely function-theoretic lemma, which also plays a key role in the proof of the main theorem. It is the following.

Lemma 6. Let $f(\zeta)$ be an arbitrary function analytic in $|\zeta| < 1$ subject only to the condition that it is never zero. Then there exists a path C going to the boundary of $|\zeta| < 1$, such that

$$\int_C |f(\zeta)| \, |d\zeta| < \infty .$$

Proof. Let $w = F(z) = \int_0^z f(\zeta) d\zeta$. Then $F(z)$ maps $|z| < 1$ onto a RIEMANN surface which is not the whole plane and which has no algebraic branch points. Such a surface must have a boundary point at finite distance. More specifically, if we let $z = G(w)$ be that branch of the inverse function satisfying $G(0) = 0$, then since $|G(w)| < 1$, there is a largest disk $|w| < R < \infty$ in which $G(w)$ is defined. There must then be a point w_0 with $|w_0| = R$ such that $G(w)$ cannot be extended to a neighborhood of w_0. Let L be the line segment from 0 to w_0, and let C be the image of L under $G(w)$. Then C must go to the boundary of $|z| < 1$, since otherwise there would be a sequence $w_n \to w_0$ such that $z_n = G(w_n) \to z_0$, with $|z_0| < 1$. But then $F(z_0) = w_0$, and since $F'(z_0) = f(z_0) \neq 0$, $F(z)$ would map a neighborhood of z_0 one-to-one onto a neighborhood of w_0, so that $G(w)$ would be extendable to a neighborhood of w_0. Thus the path C goes to the boundary of $|z| < 1$, and we have

$$\int_C |f(z)| \, |dz| = \int_L |dw| = R < \infty .$$

Combining lemmas 5 and 6, we arrive easily at our first main result.

Theorem 1. *If the normals to a complete simply-connected minimal surface S omit a full neighborhood of some direction, then S is a plane.*

Proof. We may choose axes in E^3 so that the normals omit a neighborhood of the positive x_3-axis. Then by lemma 5 we must show that (15) holds for some path C, where in this case $f \neq 0$ and $|g| < M < \infty$. But this is an immediate consequence of lemma 6.

Theorem 2. *There exists a complete simply-connected minimal surface S whose normals omit precisely 4 directions.*

Proof. Denote by R the region consisting of the complex z-plane minus the points $2n\pi i$, where n runs through all the integers. The function $G(z) = 1/(1 - e^z)$ never assumes the values $0, 1, \infty$ in R.

Denote by $\hat{R}$ the universal covering surface of R, which is known to be conformally equivalent to the unit disk. Let $\Phi(\zeta)$ be a conformal map of $|\zeta| < 1$ onto $\hat{R}$ and let Π be the projection map of $\hat{R}$ onto R. Then $F(\zeta) = \Pi(\Phi(\zeta))$ is a locally one-to-one analytic map, and hence $f(\zeta) = F'(\zeta) \neq 0$. We set $g(\zeta) = [G(F(\zeta))]^{\frac{1}{2}}$, where we may choose either branch of the square root. Then $g(\zeta)$ omits precisely the values $0, \pm 1, \infty$. If we construct the simply-connected minimal surface S associated with this particular choice of f, g, its normal map omits the four corresponding points of the sphere. We now show that this surface is complete.

Let C be an arbitrary curve going to the boundary of $|\zeta| < 1$ and let C' be the image of C under $z = F(\zeta)$. We then have

$$\int_C |f(\zeta)| (1 + |g(\zeta)|^2) |d\zeta| = \int_{C'} (1 + |G(z)|) |dz| . \tag{17}$$

We consider two cases.

Case 1. C' has infinite length. Then we have

$$\int_{C'} (1 + |G(z)|) |dz| \geqslant \int_{C'} |dz| = \infty .$$

Case 2. C' has finite length. Then C' must tend to a specific point z_0, and since C goes to the boundary of $|\zeta| < 1$, z_0 must be one of the points $2n\pi i$. In a suitable neighborhood of z_0 we therefore have

$$1 - e^z = (z - z_0)[1 + \tfrac{1}{2}(z - z_0) + \cdots]$$

and

$$|G(z)| \geqslant \frac{1}{2|z - z_0|} .$$

If we let C'' be the part of C' lying in this neighborhood, we have

$$\int_{C'} (1 + |G(z)|) |dz| \geqslant \int_{C''} \frac{|dz|}{2|z - z_0|} = \infty .$$

The integral in (17) is therefore divergent for every path C going to the boundary. Thus the surface is complete and the theorem is proved.

Before turning to the case of general minimal surfaces, there are several comments which are worth making in the simply-connected case.

Remark 1. If one observes closely the construction used in Theorem 2, it becomes clear that what was actually done was to construct a non-simply-connected minimal surface using the region R, and then use the somewhat artificial device of the universal covering surface to obtain a simply-connected surface with essentially the same properties. This is precisely the device which makes it possible to omit the hypothesis of simple connectivity in Theorem 1, as will be pointed out in the next section. However, one could contend that a surface obtained in such a manner was not "truly" simply-connected. For example, if we choose D to be the whole ζ-plane in the definition of a simply-connected minimal surface, and set

$$x_1 = \mathrm{Re}\,\{\cosh \zeta\}$$
$$x_2 = \mathrm{Im}\,\{\sinh \zeta\}$$
$$x_3 = \mathrm{Re}\,\{\zeta\}\,,$$

then we obtain the ordinary catenoid

$$x_1^2 + x_2^2 = \cosh^2 x_3$$

covered infinitely many times. The surface constructed in Theorem 2 can be shown to be of this nature. However, without giving details, we state the following result:

If we use the same function $g(\zeta)$ as in the proof of Theorem 2, but modify the function $f(\zeta)$ slightly, then we obtain a complete simply-connected minimal surface whose normals omit four directions, and which has the additional property that no two neighborhoods in D map onto the same neighborhood in space.

Remark 2. By virtue of lemma 5 the geometric problems we have been considering are reduced to the following analytic problem: *given two analytic functions f, g in $|\zeta| < 1$ such that $f \neq 0$, what condition on g implies the existence of a path C to the boundary such that $\int_C |f|(1+|g|^2)|d\zeta| < \infty$?*

What we have shown in Theorems 1 and 2 is that g bounded is such a condition, while g omitting four values is not. A reasonable guess is that g should be of bounded characteristic *(beschränktartig)*. From a theorem of NEVANLINNA ([2], p. 201) it follows that if E is a closed set of points in the plane, then the necessary and sufficient condition that every meromorphic function $g(\zeta)$ in $|\zeta| < 1$ whose values omit the set E must be of bounded characteristic is that E have positive capacity. It was this theorem that led to the conjecture stated in the introduction.

3. General minimal surfaces

Definition. A *minimal surface* S is a mapping of an abstract differentiable surface S_0 into E^3, such that each point of S_0 has a neighborhood in which the complex parameter ζ, with $|\zeta| < 1$, is an admissible parameter, and the induced map of $|\zeta| < 1$ into E^3 defines a simply-connected minimal surface.

We may note briefly that in the case that S_0 is orientable we have an induced conformal structure. The functions φ_k as well as f defined by (1) and (5) depend on the local parameter and in fact, have the character of analytic differentials. On the other hand, the function g defined by (6) is invariant, and represents a meromorphic function on the whole surface, with the same geometric interpretation as before.

In any case, even where S_0 is not orientable, it will have an induced RIEMANNian metric ds.

Definition. If a point p of S corresponds to a point p_0 of S_0, then we define the *distance from* p *to the boundary* to be the number

$$d = \inf_C \int ds$$

where C is an arbitrary path from p_0 to the boundary of S_0.

In particular, S is complete if $d = \infty$.

We also have a universal covering surface $\hat{S}_0$ of S_0 which will be simply-connected, and an induced metric $d\hat{s}$ on $\hat{S}_0$.

Lemma 7. *If the point* $\hat{p}_0$ *of* $\hat{S}_0$ *projects onto the point* p_0 *of* S_0, *then the distance* $\hat{d}$ *from* $\hat{p}_0$ *to the boundary equals the distance* d *from* p_0 *to the boundary.*

Proof. Let $\hat{C}$ be any curve of finite length on $\hat{S}_0$, and let C be its projection on S_0. Then C has (the same) finite length, and either C goes to the boundary of S_0 or else tends to a point p_1 of S_0. In the latter case the curve $\hat{C}$ tends to a point $\hat{p}_1$ and cannot go to the boundary of $\hat{S}_0$. Thus if a curve $\hat{C}$ goes to the boundary of $\hat{S}_0$, then either it has infinite length, or else its projection goes to the boundary of S_0. In the definition of $\hat{d}$ we need therefore consider only those curves $\hat{C}$ whose projection goes to the boundary of S_0. Conversely, if C goes to the boundary of S_0, it can be "lifted" to a curve $\hat{C}$ which must then go to the boundary of $\hat{S}_0$. Thus the sets of numbers over which we form the infimum are identical, and we have $\hat{d} = d$.

Corollary. $\hat{S}_0$ *is complete if and only if* S_0 *is complete.*

We may now consider the simply-connected surface $\hat{S}_0$ as a RIEMANN surface which by the KOEBE uniformization theorem is conformally equivalent to either $|\zeta| < 1$ or the whole ζ-plane, and the induced map into E^3 defines a simply-connected minimal surface. We may summarize as follows:

If S is an arbitrary minimal surface and p is a point of S whose distance to the boundary is d, then there exists a simply-connected minimal surface $\hat{S}$ and a point $\hat{p}$ such that the distance from $\hat{p}$ to the boundary is d, the normal direction at $\hat{p}$ is the same as at p, and the normal map of $\hat{S}$ covers the same part of the sphere as the normal map of S.

This is the result, referred to in a footnote in [4] which allows us to remove the hypothesis of simple connectivity from all the theorems on GAUSS curvature given in [4].

Finally, if we apply it to the case $d = \infty$, we can remove the hypothesis of simple connectivity in theorem 1, and we obtain the result stated in the introduction:

A complete minimal surface whose normals omit a neighborhood of some direction must be a plane.

4. Concluding remarks

If we choose D to be the entire plane and let $f(\zeta) \equiv 1, g(\zeta) \equiv \zeta$, we obtain the classical surface of ENNEPER which is a complete minimal surface whose normals omit exactly one direction.

The catenoid, which may be parametrized as in the remark following Theorem 2, is a complete minimal surface whose normals omit two directions.

By choosing $g(\zeta) = G(F(\zeta))$, in the notation of the proof of Theorem 2, we obtain a complete minimal surface whose normals omit precisely three directions.

The difficulty in constructing further examples may be summarized as follows.

In the case where D is the whole plane, we may for example choose $f(\zeta) \equiv 1$ and $g(\zeta)$ to be an arbitrary entire function. All the minimal surfaces so constructed will be complete, but by PICARD's Theorem, the normals to the surface can omit at most two directions.

In the case where D is the unit disk, we can construct a minimal surface whose normals omit an arbitrary closed set E provided only that E contains at least three points, and that its complement R is connected. Namely, we choose for $g(\zeta)$ the function which maps $|\zeta| < 1$ onto the universal covering surface of R, and for $f(\zeta)$ an arbitrary function which is analytic and different from zero in $|\zeta| < 1$. These surfaces, however, will in general not be complete, and the question then is to find a way to construct a suitable function $f(\zeta)$ which will make the surface complete, or to show that none can exist.

5. On the conjecture in the introduction

As remarked in a footnote at the beginning of this paper, we have the following result, the reasoning being due to AHLFORS.

Theorem. *A complete minimal surface whose normal map omits a set of positive capacity is a plane.*

Proof. As in section 3, we may construct a complete simply-connected minimal surface S whose normal map omits the same set of positive capacity. By Lemma 5 we must show that whenever f and g are analytic functions in the unit disk such that $f \neq 0$ and g omits a set of positive capacity, then (15) holds for some curve C. But as remarked at the end of section 2, such a function g is of bounded characteristic, and hence ([2], p. 178) can be written as $g = g_1/g_2$, where $|g_1| < 1$, $|g_2| < 1$, and $g_2 \neq 0$. We then have

$$\int_C (1 + |g|^2)|f||d\zeta| = \int_C (|g_2|^2 + |g_1|^2)\left|\frac{f}{g_2^2}\right||d\zeta| < 2\int_C |f_1||d\zeta|$$

where we set $f_1 = f/g_2^2$. The result follows immediately by applying Lemma 6 to the function f_1.

BIBLIOGRAPHY

[1] H. HOPF and W. RINOW, *Über den Begriff der vollständigen differentialgeometrischen Fläche.* Comment. Math. Helv. *3*, 209–225 (1931).

[2] R. NEVANLINNA, *Eindeutige analytische Funktionen.* Berlin 1936.

[3] R. OSSERMAN, *Proof of a conjecture of Nirenberg.* Comm. Pure Appl. Math. *12*, 229–232 (1959).

[4] R. OSSERMAN, *On the* GAUSS *curvature of minimal surfaces.* Trans. Amer. Math. Soc. *96*, 115–128 (1960).

[5] T. RADO, *On the Problem of Plateau.* Ergebn. d. Math. II, *2* (1933). (Also Chelsea, New York 1951.)

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