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On excision and principal fibrations[1])

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The object of this note is to show how the homotopy exact sequences of [2] and the operator theory of [4] may be used to study the excision maps which arise in a fibre or cofibre triple[2]) (see [3]). In the case of the cofibre triple the conclusion of Theorems 1′ and 2′ are well-known; in the dual case, on which we put the greater weight, they appear not to be so familiar. But in any case we wish to stress that our interest lies in exhibiting the elementary and automatic procedure whereby Theorems 1 and 2 follow from Propositions 2 and 3 rather than in the actual conclusions of those theorems. The following theorem then expresses the main content of this note.

Theorem. *Let* $F \xrightarrow{j} X \xrightarrow{p} Y$ *be a fibre triple over a connected base such that*

$$p^{**}: H^m(Y) \to H^m(X, F)$$

is an isomorphism (with any coefficient group) if $m \leq k$ *and a monomorphism if* $m = k+1$. *Then if* B *is 1-connected*[3]) *the excision mapping*

$$\varepsilon: \Pi(Y, B) \to \Pi_1(j, B)$$

is onto if $\pi_i(B) = 0,\ i \geq k+1$, *and has zero kernel if* $\pi_i(B) = 0,\ i \geq k+2$.

As a consequence we derive a theorem on principal fibrations which has been announced by Ganea (Lemma 2.1 of [5]).

We study the category of spaces[4]) of the homotopy type of CW-complexes. Let

$$F \xrightarrow{j} X \xrightarrow{p} Y \qquad (1)$$

be a differential triple and let

$$\begin{array}{ccc} \Pi_n(p, B) & \xrightarrow{\partial} & \Pi_{n-1}(Y, B) \\ \downarrow \varepsilon_1 & & \downarrow \varepsilon_2 \\ \Pi_n(F, B) & \xrightarrow{J} & \Pi_n(j, B) \end{array} \qquad (2)$$

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1) The content of this note was presented as a talk to the International Colloquium on Differential Geometry and Topology, Zürich, June 1960.

2) The question was suggested in a letter of F. P. Peterson.

3) It is sufficient that B be n-simple for all n.

4) With given base-point.

be the associated *transgression square*, where $\varepsilon_1, \varepsilon_2$ are the (generalized) excision maps (see [3]). Then it is known that the square (2) anticommutes if $n > 1$. By embedding (2) in a map of exact sequences with the excisions flanked by identity maps of $\Pi_*(X, B)$ it is easy to deduce that

Proposition 1. *If* $n \geq 2$, *then*

(a) *If* $\varepsilon_1 : \Pi_n(p, B) \to \Pi_n(F, B)$ *is onto and* $\varepsilon_1 : \Pi_{n-1}(p, B) \to \Pi_{n-1}(F, B)$ *has zero kernel, then* $\varepsilon_2 : \Pi_{n-1}(Y, B) \to \Pi_n(j, B)$ *is onto;*

(b) *if* $\varepsilon_2 : \Pi_{n-1}(Y, B) \to \Pi_n(j, B)$ *is onto, then* $\varepsilon_1 : \Pi_n(p, B) \to \Pi_n(F, B)$ *is onto;*

(c) *if* $\varepsilon_1 : \Pi_n(p, B) \to \Pi_n(F, B)$ *is* $(1, 1)$, *then* $\varepsilon_2 : \Pi_{n-1}(Y, B) \to \Pi_n(j, B)$ *is* $(1, 1)$;

(d) *if* $\varepsilon_3 : \Pi_{n-1}(Y, B) \to \Pi_n(j, B)$ *is* $(1, 1)$ *and* $\varepsilon_2 : \Pi_n(Y, B) \to \Pi_{n+1}(j, B)$ *is onto; then* $\varepsilon_1 : \Pi_n(p, B) \to \Pi_n(F, B)$ *is* $(1, 1)$.

Notice that in case (a) above we ask that ε_1 have zero kernel; if $n = 2$, $\Pi_{n-1}(p, B)$ does not admit, in general, a natural group structure. We will henceforth call a transformation with zero kernel a monomorphism, so that a monomorphism need not be $(1, 1)$.

We now specialize (1) to a *fibre triple* and will study conditions under which ε_1 and ε_2 are onto or monomorphic. We will consider particularly the case $n = 1$. Let B be the Eilenberg-MacLane complex $K(G, m)$. Then $\varepsilon_2 : \Pi(Y, B) \to \Pi_1(j, B)$ may be identified with

$$p^{**} : H^m(Y; G) \to H^m(X, F; G) .$$

We thus have Serre's theorem [6]

Proposition 2. *If* Y *is* $(p-1)$*-connected and* F *is* $(q-1)$*-connected,* $p \geq 2, q \geq 1$, *then* $\varepsilon_2 : \Pi(Y, K(G, m)) \to \Pi_1(j, K(G, m))$, $m \geq 1$, *is an isomorphism if* $m \leq p + q - 1$ *and a monomorphism if* $m = p + q$.

We may immediately infer from Proposition 1 (a) and (c)

Proposition 3. *If* Y *is* $(p-1)$*-connected and* F *is* $(q-1)$*-connected,* $p \geq 2, q \geq 1$, *then* $\varepsilon_1 : \Pi_1(p, K(G, m)) \to \Pi_1(F, K(G, m))$, $m \geq 1$, *is an isomorphism if* $m \leq p + q - 1$ *and a monomorphism if* $m = p + q$.

Let Z be the space $Y \cup_p CX$ obtained by attaching CX to Y by means of $p : X \to Y$. Then ΣF, the suspension of F, is embedded in Z by the obvious map $i : \Sigma F \to Z$ and it is easy to see that $\varepsilon_1 : \Pi_1(p, B) \to \Pi_1(F, B)$ may be identified with the restriction map

$$i^* : \Pi(Z, B) \to \Pi(\Sigma F, B) .$$

Similarly, let $D = X \cup_j CF$ and extend p to $q: D \to Y$ by mapping CF to the base-point. Then it is easy to see that $\varepsilon_2: \Pi(Y, B) \to \Pi_1(j, B)$ may be identified with

$$q^*: \Pi(Y, B) \to \Pi(D, B) .$$

Our main result is the following

Theorem 1. *If* Y *is* $(p-1)$*-connected and* F *is* $(q-1)$*-connected,* $p \geq 2$, $q \geq 1$, *and if* $\pi_i(B) = 0, i \geq p + q$, *then*

$$\varepsilon_1: \Pi_1(p, B) \to \Pi_1(F, B)$$

$$\text{and} \quad \varepsilon_2: \Pi(Y, B) \to \Pi_1(j, B)$$

are onto.

We deal in detail with the assertion about ε_1 and prove a crucial lemma.

Lemma 1. *Let* $u: U \to V$ *be the fibration induced by a map* $g: V \to K(G, m)$, $m \leq p + q$ *and suppose we have a diagram*

$$\begin{array}{ccccc} \Sigma F & \xrightarrow{h} & U & & \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle u} & & \\ Z & \xrightarrow{k} & V & \xrightarrow{g} & K(G, m) \end{array}$$

with $ki \simeq uh$. *Then we may find* $l: Z \to U$ *with* $li \simeq h, ul \simeq k$.

Proof of Lemma. Now $gki \simeq guh \simeq 0$. Thus by Proposition 3, $gk \simeq 0$ so that k may be lifted to $s: Z \to U$ with $us = k$. Then $usi \simeq uh$. Thus by Proposition 4.6 of [4], $(si)^t \simeq h$ for some map $t: \Sigma F \to K(G, m-1)$; here we use the operation of $K(G, m-1) = \Omega K(G, m)$ on U to induce an operation of Map $(A, K(G, m-1))$ on Map (A, U) as in [4]. But, again by Proposition 3, $t \simeq di$ for some $d: Z \to K(G, m-1)$. Put $l = s^d$. Then $li = (si)^{di} \simeq (si)^t \simeq h$ and $ul \simeq us = k$.

Proof of Theorem. Under the hypotheses of the theorem the spaces ΣF, Z, D, and Y are all 1-connected. Thus in proving the theorem we may suppose that B is 1-connected. Take a Postnikov decomposition of B. We then have the diagram

$$\begin{array}{ccl}
\Sigma F & \xrightarrow{\;h\;} & B \\
\Big\downarrow & & \downarrow \\
 & & B_{(p+q-2)} \to K(\pi_{p+q-1}, p+q) \\
 & & \vdots \\
i & & B_{(m-1)} \to K(\pi_m, m+1) \\
 & & \downarrow \\
 & & B_{(m-2)} \to K(\pi_{m-1}, m) \\
 & & \vdots \\
\Big\downarrow & & B_{(2)} \to K(\pi_3, 4) \\
 & & \downarrow \\
Z & \longrightarrow & * \to K(\pi_2, 3)\ .
\end{array}$$

We wish to define a map $l : Z \to B$ with $li \simeq h$. We apply the Lemma to lift the constant map $Z \to *$ successively through the terms of the Postnikov decomposition. This proves the assertion about ε_1. The assertion about ε_2 is proved in exactly the same way; we merely replace $i : \Sigma F \to Z$ by $q : D \to Y$ and invoke Proposition 2 in place of Proposition 3.

We now complete the generalization of Propositions 2 and 3 by proving

Theorem 2. *If* Y *is* $(p-1)$*-connected and* F *is* $(q-1)$*-connected,* $p \geq 2$, $q \geq 1$, *and if* $\pi_i(B) = 0$, $i \geq p + q + 1$, *then*

$$\varepsilon_1 : \Pi_1(p, B) \to \Pi_1(F, B)$$

$$\text{and} \quad \varepsilon_2 : \Pi(Y, B) \to \Pi_1(j, B)$$

are monomorphisms.

We again express the essential step in the proof in the form of a lemma.

Lemma 2. *Let* $u : U \to V$ *be the fibration induced by a map* $g : V \to K(G, m)$, $m \leq p + q + 1$, *and let* $\pi_i(V) = 0, i \geq p + q + 1$. *Then if* $\varepsilon_1 : \Pi_1(p, V) \to \Pi_1(F, V)$ *is a monomorphism, so is* $\varepsilon_1 : \Pi_1(p, U) \to \Pi_1(F, U)$.

Proof of Lemma. Let $l : Z \to U$ be a map. We must show that $li \simeq 0$ implies $l \simeq 0$. Now if $li \simeq 0$ then $uli \simeq 0 : \Sigma F \to V$. Thus by hypothesis $ul \simeq 0$, so that $l \simeq ks$, where $k : K(G, m-1) \to U$ is the embedding of the fibre and s is a map $s : Z \to K(G, m-1)$. Then $ksi \simeq 0$ so that $si \simeq (\Omega g)t'$ for some $t' : \Sigma F \to \Omega V$. Now by Theorem 1, $i^* : \Pi(Z, \Omega V) \to \Pi(\Sigma F, \Omega V)$ is onto so that $t' \simeq ti$ for some $t : Z \to \Omega V$. Then $si \simeq (\Omega g)ti : \Sigma F \to K(G, m-1)$ so that, by Proposition 3, $s \simeq (\Omega g)t$. But then $l \simeq ks \simeq k(\Omega g)t \simeq 0$ and the Lemma is proved.

Proof of Theorem. Again we may suppose B 1-connected and take a POSTNIKOV decomposition of B. We are given a map $l: Z \to B$ such that $li \simeq 0$ and the Lemma enables us to deduce successively that the projections $Z \to B_{(r)}$, $r = 2, 3, \ldots, p+q$, of l are nullhomotopic; the final „projection" is of course just l itself. The changes necessary to prove the result for ε_2 are obvious.

We may apply Theorem 1 to obtain conditions under which a fibration with ΩQ in the fibre is induced by a map of the base into Q. The result may be expressed by

Theorem 3. *Let* $\Omega Q \overset{j}{\to} X \overset{p}{\to} Y$ *be a fibre triple where* Y *is* $(p-1)$*-connected,* $p \geq 2$, *and* Q *is* 1*-connected with the homotopy groups of* Q *zero outside a band of width* $p-1$. *Then the fibre triple is equivalent to one induced by a map* $Y \to Q$.

Proof. Suppose that the groups $\pi_i(Q)$ are zero except perhaps if $q+1 \leq i \leq q+p-1$. Then ΩQ is $(q-1)$-connected, $q \geq 1$, and we may apply Theorem 1 to deduce that $\varepsilon_1 : \Pi_1(p, Q) \to \Pi_1(\Omega Q, Q)$ is onto. Thus, there exists a commutative diagram

$$\begin{array}{ccccc} \Omega Q & \longrightarrow & X & \longrightarrow & Y \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ \Omega Q & \longrightarrow & EQ & \longrightarrow & Q \end{array}$$

with $u \simeq 1$. It is now a standard result that the given fibre triple is equivalent to that induced by $w: Y \to Q$, in the sense that we have a commutative diagram

$$\begin{array}{ccccc} \Omega Q & \longrightarrow & X & \longrightarrow & Y \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle 1} \\ \Omega Q & \longrightarrow & E_w & \longrightarrow & Y, \end{array}$$

where the lower triple is that induced by w.

The duals of these results appear to have an independent interest; we state the results in a form which emphasizes the duality, but perhaps disguises their familiarity. Let

$$F \overset{q}{\to} X \overset{i}{\to} Y \tag{1'}$$

be a differential triple and

$$\begin{array}{ccc} \Pi_n(A, i) & \xrightarrow{\partial} & \Pi_{n-1}(A, Y) \\ \downarrow{\scriptstyle \varepsilon_1} & & \downarrow{\scriptstyle \varepsilon_2} \\ \Pi_n(A, F) & \xrightarrow{J} & \Pi_n(A, q) \end{array} \tag{2'}$$

the associated (covariant) transgression square. There is then the obvious dual of Proposition 1. We specialize (1′) to a *cofibre triple*. Now if A is the MOORE complex $K'(G, m)$ then $\varepsilon_1 : \Pi_1(A, i) \to \Pi_1(A, F)$ may be identified with

$$q_{**} : \pi_{m+1}(G; X, Y) \to \pi_{m+1}(G; F)$$

so that, by an evident extension of the BLAKERS-MASSEY theorem [1], we have

Proposition 3′. *If Y is $(p-1)$-connected and F is $(q-1)$-connected, $p \geq 2, q \geq 2$, then $\varepsilon_1 : \Pi_1(K'(G, m), i) \to \Pi_1(K'(G, m), F)$ is an isomorphism if $m \leq p + q - 4$ and onto if $m = p + q - 3$. If G is free then ε_1 is an isomorphism if $m = p + q - 3$ and onto if $m = p + q - 2$.*

We infer (from Proposition 1′, which we leave implicit)

Proposition 2′. *If Y is $(p-1)$-connected and F is $(q-1)$-connected, $p \geq 2, q \geq 2$, then $\varepsilon_2 : \Pi(K'(G, m), Y) \to \Pi_1(K'(G, m), q)$ is an isomorphism if $m \leq p + q - 4$ and onto if $m = p + q - 3$. If G is free then ε_2 is an isomorphism if $m = p + q - 3$ and onto if $m = p + q - 2$.*

Let Z be the space $Y \frown_i EX$, the fibre space over Y induced by the map i. There is then an evident projection p of Z onto ΩF, and $\varepsilon_1 : \Pi_1(A, i) \to \Pi_1(A, F)$ may be identified with

$$p_* : \Pi(A, Z) \to \Pi(A, \Omega F) .$$

Similarly if $D = X \frown_q EF$, then i lifts canonically into $j : Y \to D$ and $\varepsilon_2 : \Pi(A, Y) \to \Pi_1(A, q)$ may be identified with

$$j_* : \Pi(A, Y) \to \Pi(A, D) .$$

Then we may prove

Theorem 1′. *If Y is $(p-1)$-connected and F is $(q-1)$-connected, $p \geq 2$, $q \geq 2$, and if A is 1-connected and $H^i(A) = 0$, $i \geq p + q - 1$, then*

$$\varepsilon_1 : \Pi_1(A, i) \to \Pi_1(A, F)$$

$$\text{and} \quad \varepsilon_2 : \Pi(A, Y) \to \Pi_1(A, q)$$

are onto.

We must explain that we are demanding the vanishing of the cohomology groups of A in dimensions $\geq p+q-1$ *with any coefficients*. This is, of course, equivalent to the requirement that the integral homology groups of A should vanish in dimensions $\geq p+q-1$ and that $H_{p+q-2}(A)$ should be free. A similar explanation was, of course, superfluous in the case of Theorem 1 since the homotopy groups of B vanish in dimensions $\geq p+q$ with any coefficients if and only if they vanish in dimensions $\geq p+q$ with integer coefficients.

The proof of Theorem 1′ proceeds by way of a homology decomposition[5]) of A and employs the following lemma.

Lemma 1′. *Let* $u : V \to U$ *be the cofibration induced by a map* $g : K'(G, m) \to V$, $m \leq p+q-4$ *or* $m = p+q-3$ *and* G *free; and suppose we have a diagram*

$$\begin{array}{ccccc} \Omega F & \xleftarrow{\;h\;} & U & & \\ {\scriptstyle p}\big\uparrow & & \big\uparrow{\scriptstyle u} & & \\ Z & \xleftarrow{\;k\;} & V & \xleftarrow{\;g\;} & K'(G, m) \end{array}$$

with $pk \simeq hu$. *Then we may find* $l : U \to Z$ *with* $pl \simeq h$, $lu \simeq k$.

Similarly we prove

Theorem 2′. *If* Y *is* $(p-1)$*-connected and* F *is* $(q-1)$*-connected,* $p, q \geq 2$, *and if* A *is 1-connected and* $H^i(A) = 0$, $i \geq p+q-2$, *then*

$$\varepsilon_1 : \Pi_1(A, i) \to \Pi_1(A, F)$$

$$\text{and} \quad \varepsilon_2 : \Pi(A, Y) \to \Pi_1(A, q)$$

are monomorphisms.

Now let $\Sigma Q \overset{q}{\leftarrow} X \overset{i}{\leftarrow} Y$ be a cofibre triple; we obtain from Theorem 1′ conditions under which the cofibration is induced by a map $Q \to Y$. Precisely

Theorem 3′. *Let* $\Sigma Q \overset{q}{\leftarrow} X \overset{i}{\leftarrow} Y$ *be a cofibre triple where* Y *is* $(p-1)$*-connected and* Q *is 1-connected with the homology groups of* Q *zero outside a band of width* $p-1$. *Then the cofibre triple is equivalent to one induced by a map* $Q \to Y$.

Notice that we can tolerate the homology groups of Q occupying a band

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[5]) If A is actually $(p+q-2)$-dimensional we may replace the homology decomposition by a cellular decomposition and dispense with the condition that A be 1-connected. In any case the argument in this dual situation closely resembles a traditional obstruction argument.

of width p provided that the top homology group is free and $p \geqslant 2$; with this improvement Theorem 3′ takes account of the important case of a space Q with a single non-vanishing cohomology group[6]) G when G is finitely generated.

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[6]) With integer coefficients.