Projectively Flat Spaces with Recurrent Curvature.

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Introduction

Let A_N be an affinely connected N-dimensional space with a symmetric connection (i.e. a connection without torsion). A_N is a *projectively flat space*, or simply, a P_N if there exists a coordinate system in terms of which the finite equations of the paths are linear. A_N is of *recurrent curvature*[1]) if the covariant derivative of its curvature tensor is the tensor product of a non-zero covariant vector and the curvature tensor itself.

The purpose of this paper is to determine all the projectively flat spaces with recurrent curvature. For convenience we shall denote such a space by P_N^*. It is found that the space P_N^* (or rather its connection) depends on 2 arbitrary functions of one variable or on 3 arbitrary functions of one variable according as its Ricci tensor is symmetric or non-symmetric. The actual construction of the connection of the P_N^* depends on the solution of a differential equation of the Riccati type and on the solution of a completely integrable system of differential equations.

Projectively flat space with covariantly constant curvature tensor (i.e. projectively flat symmetric space) is also considered. We prove that it is a well-known type of projectively flat space characterized by its Ricci tensor being symmetric and covariantly constant.

1. Preliminaries[2])

Throughout this paper, each of the indices $a, h, i, j, \ldots$ runs through the range $1, \ldots, N$; ∂_k denotes partial differentiation with respect to the kth coordinate; and a repeated index implies summation.

Let A_N be a linearly connected N-dimensional space with a symmetric connection Γ_{ji}^h (i.e. $\Gamma_{ji}^h = \Gamma_{ij}^h$). The curvature tensor, the Ricci tensor, and the tensor P_{ji} are defined respectively by:

$$R_{kji}{}^h = \partial_k \Gamma_{ji}^h - \partial_j \Gamma_{ki}^h + \Gamma_{ka}^h \Gamma_{ji}^a - \Gamma_{ja}^h \Gamma_{ki}^a \,, \tag{1.1}$$

$$R_{ji} = R_{aji}^a \,, \tag{1.2}$$

[1]) Riemannian spaces with recurrent curvature have been studied in great detail by H. S. Ruse [2] and A. G. Walker [3]. Certain classes of non-Riemannian spaces with recurrent curvature have been studied by Y. C. Wong [4].

[2]) See Eisenhart [1], but our notation is slightly different from his.

$$P_{ji} = -\frac{N}{N^2-1} R_{ji} - \frac{1}{N^2-1} R_{ij} \,. \tag{1.3}$$

It follows from (1.3) that P_{ji} is symmetric iff the RICCI tensor R_{ji} is symmetric.

By definition, A_N is of *recurrent curvature* if its curvature tensor satisfies the condition

$$\nabla_l R_{kji}{}^h = r_l R_{kji}{}^h \,, \tag{1.4}$$

where ∇ denotes the covariant differentiation and r_l a non-zero covariant vector. If

$$\nabla_l R_{kji}{}^h = 0 \,, \tag{1.5}$$

the space A_N is said to be *symmetric.*

A_N is *projectively flat* iff there exists a coordinate system in terms of which the components Γ^h_{ji} have the form

$$\Gamma^h_{ji} = \varnothing_j A^h_i + \varnothing_i A^h_j \,, \tag{1.6}$$

where A^h_i is the KRONECKER delta and $\varnothing_j$ a set of N functions. The special form (1.6) for Γ^h_{ji} is preserved by affine transformations of coordinates. Expressed in terms of P_{ji}, a well-known necessary and sufficient condition for A_N to be projectively flat is

$$\left.\begin{aligned} & R_{kji}{}^h + A^h_k P_{ji} - A^h_j P_{ki} - (P_{kj} - P_{jk}) A^h_i = 0 \,, \\ & \nabla_k P_{ji} - \nabla_j P_{ki} = 0 \,. \end{aligned}\right\} \tag{1.7}$$

For $N > 2$, $(1.7)_2$ is a consequence of $(1.7)_1$; for $N = 2$, $(1.7)_1$ is an identity. There exists, for (1.6), a function $\varnothing$ such that $\varnothing_j = \partial_j \varnothing$ iff $P_{ji} = P_{ij}$. Since on account of (1.7) a projectively flat space with $P_{ji} = 0$ is flat, we shall always assume that

$$P_{ji} \neq 0 \,. \tag{1.8}$$

2. Some necessary conditions

By definition, a P^*_N is a non-flat projectively flat A_N with recurrent curvature and therefore it is characterized by the conditions (1.4), (1.7) and (1.8). To find all the P^*_N's, we first derive a set of necessary and sufficient conditions and then determine the functions $\varnothing_j$ in (1.6) which satisfy these conditions.

From (1.4) we obtain $\nabla_l R_{ji} = r_l R_{ji}$, and consequently

$$\nabla_k P_{ji} = r_k P_{ji} \qquad (r_k \neq 0) \,. \tag{2.1}$$

It is easy to see that conversely (1.4) is a consequence of (2.1) and $(1.7)_1$.

On account of (2.1), equation $(1.7)_2$ becomes

$$r_k P_{ji} = r_j P_{ki} ,$$

which is equivalent to

$$P_{ji} = r_j p_i , \tag{2.2}$$

where p_i is some non-zero vector (since $P_{ji} \neq 0$).

Using (2.2) in $(1.7)_1$, we see that the curvature tensor of a P_N^* has the form

$$R_{kji}{}^h = -(A_k^h r_j - A_j^h r_k) p_i + (r_k p_j - r_j p_k) A_i^h . \tag{2.3}$$

On account of (2.2), equation (2.1) becomes

$$(\nabla_k r_j) p_i + r_j (\nabla_k p_i) = r_k r_j p_i , \tag{2.4}$$

from which it follows that

$$\begin{aligned} \nabla_j r_i &= s_j r_i , \\ \nabla_j p_i &= t_j p_i , \end{aligned} \tag{2.5}$$

for some vectors s_j, t_j. These equations show that r_i and p_i are of the form

$$\begin{aligned} r_i &= \lambda \xi_i \qquad (\xi_i = \partial_i \xi \neq 0) , \\ p_i &= \mu \eta_i \qquad (\eta_i = \partial_i \eta \neq 0) , \end{aligned} \tag{2.6}$$

for some scalars ξ, η, λ, μ.

Substitution of (2.6) in (2.3) and (2.5) gives

$$R_{kji}{}^h = \psi[(A_k^h \xi_j - A_j^h \xi_k)\eta_i - (\xi_k \eta_j - \xi_j \eta_k) A_i^h] \tag{2.7}$$

$$\begin{aligned} \nabla_j \xi_i &= \alpha \xi_j \xi_i , \\ \nabla_j \eta_i &= \beta \eta_j \eta_i , \end{aligned} \tag{2.8}$$

where ψ, α, β are some scalars.

We now proceed to consider the integrability conditions of the differential equations (2.8). For any covariant vector u_i, we have the Ricci identity

$$\nabla_k \nabla_j u_i - \nabla_j \nabla_k u_i = - R_{kji}{}^h u_h. \tag{2.9}$$

Using (2.7) in (2.9) and putting $u_i = \xi_i$, we get, since $\xi_i \neq 0$,

$$(\xi_k \alpha_j - \xi_j \alpha_k) = -\psi(\xi_k \eta_j - \xi_j \eta_k) \quad (\alpha_j = \partial_j \alpha) ,$$

i.e.

$$\xi_k(\alpha_j + \psi \eta_j) = \xi_j(\alpha_k + \psi \eta_k)$$

which is equivalent to

$$\alpha_j + \psi \eta_j = \varrho \xi_j \tag{2.10$_1$}$$

for some scalar ϱ. Similarly, using (2.7) in (2.9) and putting $u_i = \eta_i$, we get

$$\beta_j - 2\psi\xi_j = \sigma\eta_j \qquad (\beta_j = \partial_i\beta) \tag{2.10$_2$}$$

for some scalar σ.

Equations (2.10) tell us that

The scalars $\alpha, \beta, \psi, \varrho, \sigma$ *can be expressed as functions of* ξ, η *alone.* (2.11)

On account of (2.11), equations (2.10) can be re-written as

$$\left.\begin{aligned} (\alpha_\xi - \varrho)\xi_j + (\alpha_\eta + \psi)\eta_j = 0 \quad ,\\ (\beta_\xi - 2\psi)\xi_j + (\beta_\eta - \sigma)\eta_j = 0 \ , \end{aligned}\right\} \tag{2.12}$$

where $\alpha_\xi = \dfrac{\partial\alpha}{\partial\xi}$, $\alpha_\eta = \dfrac{\partial\alpha}{\partial\eta}$, etc. Two cases arise according as whether or not η is functionally dependent on ξ, i.e. whether or not η_i and ξ_i are proportional. Now, by (2.2) and (2.6), η_i and ξ_i are proportional or not according as the tensor P_{ji} (or the RICCI tensor) is symmetric or not. We shall consider these two cases separately in § 3 and § 4 respectively.

3. P_N^* with symmetric RICCI tensor

In this case, $\eta = \eta(\xi)$ so that (2.12) reduce to

$$(\alpha_\xi - \varrho) + (\alpha_\eta + \psi)\eta' = 0 \ ,$$
$$(\beta_\xi - 2\psi) + (\beta_\eta - \sigma)\eta' = 0 \ ,$$

where $\eta' = d\eta/d\xi$. Since these equations just determine the unspecified functions ϱ, σ of ξ which appear in (2.10), the *integrability conditions of* (2.8) *reduce to the mere fact that* α, β *and* ψ *are functions of* ξ.

On account of $\eta = \eta(\xi)$, equations (2.7) and (2.8) become

$$R_{kji}{}^h = \psi\eta'(A_k^h\xi_j - A_j^h\xi_k)\xi_i \ ,$$
$$\nabla_j\xi_i = \alpha\xi_j\xi_i \ ,$$
$$\nabla_j\xi_i = (\beta\eta' - \eta''/\eta')\xi_j\xi_i \ .$$

Hence, we have the first part of the following

Theorem 3.1. *For a* P_N^* *with symmetric RICCI tensor, there exists a scalar* ξ *and functions* θ *and* α *of* ξ *such that*

$$R_{kji}{}^h = \theta(A_k^h\xi_j - A_j^h\xi_k)\xi_i \ , \tag{3.1}$$

$$\nabla_j\xi_i = \alpha\xi_j\xi_i \ , \tag{3.2}$$

where $\xi_i = \partial_i\xi$.

[*The integrability condition of* (3.2) *is identically satisfied.*]

Conversely, an A_N *satisfying this condition is a* P_N^* (*or a symmetric* P_N) *with symmetric* RICCI *tensor.*

Proof. We need only prove the last part of the theorem. It follows from (3.1) that

$$P_{ji} = -\theta\,\xi_j\xi_i\,, \tag{3.3}$$

and so P_{ji} is symmetric. On account of (3.3) and (3.1), condition $(1.7)_1$ is satisfied. Furthermore, a simple calculation involving (3.3) and (3.2) will show that $\nabla_k P_{ji} = r_k P_{ji}$, with

$$r_j = -(\theta_\xi/\theta + 2\alpha)\,\xi_j \quad \left(\theta_\xi = \frac{d\theta}{d\xi}\right). \tag{3.4}$$

It follows from this that $(1.7)_2$ is satisfied and that $\nabla_l R_{kji}{}^h = r_l R_{kji}{}^h$. The A_N is therefore a P_N^* or a symmetric P_N according as $\theta_\xi/\theta + 2\alpha \neq 0$ or $= 0$. Hence our theorem is completely proved.

In order to construct the Γ_{ji}^h of a P_N^* with symmetric RICCI tensor, we choose a coordinate system in which Γ_{ji}^h have the form

$$\Gamma_{ji}^h = \phi_j A_i^h + \phi_i A_j^h\,, \tag{3.5}$$

where ϕ_j are N functions of the coordinates. With respect to the Γ_{ji}^h given by (3.5), the covariant derivation of a covariant vector u_i is

$$\nabla_j u_i = \partial_j u_i - \phi_j u_i - \phi_i u_j\,, \tag{3.6}$$

and the curvature tensor has the components

$$R_{kji}{}^h = -A_k^h \phi_{ji} + A_j^h \phi_{ki} + (\phi_{kj} - \phi_{jk}) A_i^h\,. \tag{3.7}$$

where

$$\phi_{ji} = \partial_j \phi_i - \phi_j \phi_i\,. \tag{3.8}$$

Now identifying (3.7) with (3.1) and rewriting (3.2) by means of (3.6), we get

$$\begin{aligned} \partial_j \phi_i &= \phi_j \phi_i - \theta(\xi)\,\xi_j \xi_i\,, \\ \partial_j \xi_i &= \phi_j \xi_i + \phi_i \xi_j + \alpha(\xi)\,\xi_j \xi_i\,, \end{aligned} \tag{3.9}$$

where $\xi_i = \partial_i \xi$.

It is easy to verify that for any functions $\theta(\xi)$ and $\alpha(\xi)$ of ξ, the integrability conditions of the differential equations (3.9) in the $N+1$ unknown functions ϕ_i, ξ are satisfied, on account of the equations (3.9) themselves. Now the solution of a completely integrable system of differential equations contains a finite number of arbitrary constants, while the special form (3.5) for the connection of a P_N is preserved by affine transformations of coordinates which depend on a finite number of constants. But neither of these would alter the fact that Γ_{ji}^h as given by (3.5) depends on two arbitrary functions of one variable, namely, the functions θ, α of ξ. Hence (cf. (3.4)) we have

Theorem 3.2. *If we take any functions θ, $\alpha (\alpha \neq -\theta_\xi/2\theta)$ of one variable ξ, and any solution $\varnothing_i$, ξ of the completely integrable system* (3.9), *then the connection $\Gamma_{ji}^h = \varnothing_j A_i^h + \varnothing_i A_j^h$ defines a P_N^* with symmetric* RICCI *tensor; and any P_N^* with symmetric* RICCI *tensor can be constructed in the way. Thus the most general P_N^* with symmetric* RICCI *tensor depends on 2 arbitrary functions of one variable, and the actual construction of the connection of such P_N^* depends on the solution of a completely integrable system of differential equations.*

4. P_N^* with non-symmetric RICCI tensors

Let us now return to (2.12). Since η_i, ξ_i are not proportional, (2.12) are equivalent to

$$\alpha_\xi = \varrho\,, \qquad \beta_\eta = \sigma\,,$$
$$\alpha_\eta + \psi = 0\,, \qquad \beta_\xi - 2\psi = 0\,.$$

The first two equations merely determine the unspecified scalars ϱ, σ, which appear in (2.10). Therefore,

$$\alpha_\eta + \psi = 0\,, \qquad \beta_\xi - 2\psi = 0$$

are the integrability conditions of equations (2.8).

From this and (2.7), (2.8) and (2.11), we have the first part of the following

Theorem 4.1. *For a P_N^* with non-symmetric* RICCI *tensor, there exist functionally independent scalars ξ and η and functions ψ, α, β of ξ* and *η such that*

$$R_{kji}{}^h = \psi[(A_k^h \xi_j - A_j^h \xi_k)\eta_i - (\xi_k \eta_j - \xi_j \eta_k) A_i^h] \tag{4.1}$$

$$\left.\begin{aligned} \nabla_j \xi_i &= \alpha \xi_j \xi_i\,, \\ \nabla_j \eta_i &= \beta \eta_j \eta_i\,, \end{aligned}\right\} \tag{4.2}$$

where $\xi_i = \partial_i \xi$, $\eta_i = \partial_i \eta$.

[*The integrability conditions of* (4.2) *are*

$$\alpha_\eta + \psi = 0\,, \qquad \beta_\xi - 2\psi = 0.] \tag{4.3}$$

Conversely, any A_N which satisfies the above condition is a P_N^* with non-symmetric RICCI *tensor.*

Proof. We need only prove the last part of this theorem. It follows from (4.1) that

$$P_{ji} = -\psi \xi_j \eta_i\,, \tag{4.4}$$

and so P_{ji} is not symmetric. Now (4.4), together with (4.1) and (4.2) show that condition (1.7) for a P_N is satisfied. Furthermore, a simple calculation involving (4.4) and (4.2) will show that $\nabla_k P_{ji} = r_k P_{ji}$, and hence $\nabla_l R_{kji}{}^h = r_l R_{kji}{}^h$, with

$$r_k = \left(\frac{\psi_\xi}{\psi} + \alpha\right)\xi_k + \left(\frac{\psi_\eta}{\psi} + \beta\right)\eta_k .$$

This vector r_k is not zero. In fact, since η_k, ξ_k are not proportional, $r_k = 0$ would imply that

$$(\log \psi)_\xi + \alpha = 0 , \qquad (\log \psi)_\eta + \beta = 0 .$$

Differentiations of these give

$$\alpha_\eta = \beta_\xi ,$$

from which and (4.3) it follows that $\psi = 0$. But this gives a flat space. Hence our theorem is completely proved.

In order to construct the Γ_{ji}^h of a P_N^* with non-symmetric Ricci tensor, we proceed as in the case of symmetric Ricci tensor, and have (3.5), (3.6), (3.7) and (3.8). Identifying (3.7) with (4.1) and rewriting (4.2) by means of (3.6), we get

$$\left.\begin{aligned} \partial_j \varnothing_i &= \varnothing_j \varnothing_i + \psi(\xi, \eta)\,\xi_j \eta_i , \\ \partial_j \xi_i &= \varnothing_j \xi_i + \varnothing_i \xi_j + \alpha(\xi, \eta)\,\xi_j \eta_i , \\ \partial_j \eta_i &= \varnothing_j \eta_i + \varnothing_i \eta_j + \beta(\xi, \eta)\,\xi_j \eta_i , \end{aligned}\right\} \tag{4.4}$$

where $\xi_i = \partial_i \xi$, $\eta_i = \partial_i \eta$. From Theorem 4.1 it follows that an A_N with connection (3.5) is a P_N^* with non-symmetric Ricci tensor if the functions $\varnothing_j$ together with two functionally independent scalars ξ, η and some functions ψ, α, β of ξ, η, satisfy the differential equations (4.4). We know already (from Theorem 4.1) that the integrability conditions of $(4.4)_{2,3}$ are (4.3). It is easy to verify that on account of (4.4), the integrability condition of $(4.4)_1$ reduces to

$$(\psi_\eta + \psi\beta)(\xi_k \eta_j - \xi_j \eta_k) = 0 .$$

Since $\eta \neq \eta(\xi)$, this is equivalent to

$$\psi_\eta + \psi\beta = 0 .$$

Hence combining this with (4.3), we obtain the following complete set of integrability conditions of (4.4):

$$\alpha_\eta = -\psi , \qquad \beta_\xi = 2\psi , \qquad \psi_\eta + \psi\beta = 0 .$$

Rewrite these as

$$\frac{\partial \alpha}{\partial \eta} = -\psi, \quad \frac{\partial \beta}{\partial \xi} = 2\psi, \quad \frac{\partial \psi}{\partial \eta} = -\psi\beta. \tag{4.5}$$

From the last two equations, we obtain

$$\frac{\partial^2 \beta}{\partial \xi \, \partial \eta} = 2\frac{\partial \psi}{\partial \eta} = -2\psi\beta = -\beta\frac{\partial \beta}{\partial \xi},$$

i.e.

$$\frac{\partial^2 \beta}{\partial \xi \, \partial \eta} + \tfrac{1}{2}\frac{\partial \beta^2}{\partial \xi} = 0.$$

Integration of this with respect to ξ gives

$$\frac{\partial \beta}{\partial \eta} + \tfrac{1}{2}\beta^2 = f(\eta), \tag{4.6$_1$}$$

where $f(\eta)$ is an arbitrary function of η. If β is any function of ξ, η satisfying $(4.6)_1$, then in virtue of (4.5), ψ and α can be expressed in terms of β as follows:

$$\psi = \tfrac{1}{2}\frac{\partial \beta}{\partial \xi}, \tag{4.6$_2$}$$

$$\alpha = -\tfrac{1}{2}\int \frac{\partial \beta}{\partial \xi} d\eta + h(\xi). \tag{4.6$_3$}$$

where $h(\xi)$ is an arbitrary function of ξ. Equations (4.5) are equivalent to equations (4.6).

Since equation $(4.6)_1$ is of the RICCATI type, the most general function $\beta(\xi, \eta)$ satisfying it is of the form

$$\beta(\xi, \eta) = \frac{\beta_1(\eta) + g(\xi)\beta_2(\eta)}{\beta_3(\eta) + g(\xi)\beta_4(\eta)}, \tag{4.7}$$

where $\beta_1(\eta), \ldots, \beta_4(\eta)$ are certain functions of η and $g(\xi)$ is an arbitrary function of ξ.

Hence, with an observation similar to that immediately before Theorem 3.2, we have

Theorem 4.2. *With any functions $f(\eta)$ and $h(\xi)$ of one variable η and ξ respectively, any solution $\beta(\xi, \eta)$ [cf. (4.7)] of equation $(4.6)_1$, and the functions $\psi(\xi, \eta)$, $\alpha(\xi, \eta)$ given by $(4.6)_2$, $(4.6)_3$, the system of differential equations (4.4) is completely integrable. If $\varnothing_i$, ξ, η are any solution of the completely* *integrable system* (4.4), *then the connection* $\Gamma_{ji}^h = \o_j A_i^h + \o_i A_j^h$ *defines a* P_N^* *with non-symmetric* RICCI *tensor; and any such* P_N^* *can be constructed in this way. Thus the most general* P_N^* *with non-symmetric* RICCI *tensor depends on* 3 *arbitrary functions of one variable, and the actual construction of the connection of such* P_N^* *depends on the solution of a differential equation of the* RICCATI *type and on the solution of a completely integrable system of differential equations.*

5. Projectively flat symmetric N-space

We now consider the A_N's characterized by

$$R_{kji}{}^h = -A_k^h P_{ji} + A_j^h P_{ki} + (P_{kj} - P_{jk}) A_i^h \tag{5.1}$$

$$\nabla_l R_{kji}{}^h = 0 \,. \tag{5.2}$$

On account of (5.1), the condition (5.2) is equivalent to

$$\nabla_k P_{ji} = 0 \,. \tag{5.2'}$$

Substituting (5.2′) into the RICCI identity

$$\nabla_l \nabla_k P_{ji} - \nabla_k \nabla_l P_{ji} = -R_{lkj}{}^a P_{ai} - R_{lki}{}^a P_{ja}$$

and taking account of (5.1), we obtain

$$\begin{aligned} 0 = {} & P_{li} P_{kj} - P_{ki} P_{lj} - (P_{lk} - P_{kl}) P_{ji} \\ & + P_{jl} P_{kj} - P_{jk} P_{li} - (P_{lk} - P_{kl}) P_{ji} \,, \end{aligned}$$

i.e. $$P_{li}(P_{kj} - P_{jk}) + P_{ki}(P_{jl} - P_{lj}) - 2(P_{lk} - P_{kl}) P_{ji} = 0 \,.$$

Putting $l = j$, $k = i$, we get

$$P_{ji}(P_{ij} - P_{ji}) - 2(P_{ji} - P_{ij}) P_{ji} = 0 \,,$$

i.e. $$3(P_{ji} - P_{ij}) P_{ji} = 0 \qquad (i, j \text{ not summed}).$$

This gives

$$P_{ji} = P_{ij} \,, \tag{5.3}$$

and consequently, (5.1) becomes

$$R_{kji}{}^h = -A_k^h P_{ji} + A_j^h P_{ki} \,. \tag{5.4}$$

Since, conversely, (5.1) and (5.2) are easy consequences of (5.4), (5.3) and (5.2′), we have

Theorem 5.1. *An A_N is a symmetric P_N, iff*

$$R_{kji}{}^h = - A_k^h P_{ji} + A_j^h P_{ki} ,$$

$$P_{ji} = P_{ij} , \quad \nabla_k P_{ji} = 0 .$$

The properties stated in Theorem 5.1 characterize a well-known type of projectively flat space; in particular, if P_{ji} is of rank n, it is RIEMANNian and of constant curvature (cf. EISENHART [1], p. 97 and p. 166). We can easily prove that in contrast, a P_N with recurrent curvature, i.e. a P_N^*, is never a RIEMANNian space.

Obviously, a P_N^* with a non-symmetric RICCI tensor cannot be RIEMANNian. If a P_N^* with a symmetric RICCI tensor is RIEMANNian, let g_{ji} be a fundamental tensor whose CHRISTOFFEL symbol $\left\{ {h \atop j\,i} \right\}$ is equal to Γ_{ij}^h. Then the covariant components

$$R_{kjih} = R_{kji}{}^a g_{ha}$$

of the curvature tensor satisfy the well-known equation

$$R_{kjih} = R_{ihkj} ,$$

which on account of (3.1) reduces to

$$g_{kh} \xi_j \xi_i = g_{ij} \xi_h \xi_k .$$

But this contradicts the fact that g_{ji} is of rank $N > 1$. Therefore, no P_N^* is RIEMANNian.

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