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# The equivalence of two definitions of quasiconformal mappings<sup>1)</sup>

by LIPMAN BERS

We give here a new proof to the known fact (cf. MORI [7], BERS [3], YÛJÔBÔ [12], PFLUGER [11]) that the so-called analytic and geometric definitions of quasiconformality are equivalent. The proof uses a minimum of real variable techniques; no mention is made of absolute continuity in the sense of TONELLI. We rely instead on the theory of BELTRAMI's equations as exposed in AHLFORS-BERS [2] and on a theorem of BEURLING-AHLFORS [5].

Let  $z \rightarrow w(z) = u(x, y) + iv(x, y)$  be an orientation preserving homeomorphism of a plane domain  $D$  onto another. If the partial derivatives of  $w$ , in the sense of distribution theory, are locally square integrable functions, we denote by  $K_a(D, w)$  the smallest constant  $K \geq 1$  such that the inequality

$$\left| \frac{\partial w}{\partial x} \right|^2 + \left| \frac{\partial w}{\partial y} \right|^2 \leq \left( K + \frac{1}{K} \right) \frac{\partial(u, v)}{\partial(x, y)} \quad (1)$$

holds a.e. in  $D$ . If there is no such number, or if  $w$  does not have locally square integrable generalized derivatives, we set  $K_a(D, w) = \infty$ . If

$$K_a(D, w) < \infty,$$

$w$  is said to be  $K$ -quasiconformal according to the *analytic definition* [4, 6, 9].

A *topological rectangle*  $R$  is a conformal image of a closed rectangle

$$0 \leq \xi \leq m, \quad 0 \leq \eta \leq 1,$$

the images of the vertices being distinguished. We write  $m = \text{mod } R$ . For  $R \subset D$ ,  $w(R)$  is also a topological rectangle, in view of RIEMANN's mapping theorem. We set

$$K_g(D, w) = \sup \left( \frac{\text{mod } w(R)}{\text{mod } R} \right), \quad R \subset D. \quad (2)$$

It is immediate that  $K_g(D, w) = K_g(w(D), w^{-1})$ . Mappings with

$$K_g(D, w) < \infty$$

are called  $K$ -quasiconformal according to the *geometric definition* [1, 8, 10].

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**Theorem.**  $K_a(D, w) = K_g(D, w)$ .

The proof requires several lemmas. The crux of the argument is contained in Lemma 8 below.

If  $K_a(D, w) \leq K < \infty$ , then  $w(z)$  satisfies a BELTRAMI equation

$$\frac{\partial w}{\partial \bar{z}} = \mu(z) \frac{\partial w}{\partial z} \quad (3)$$

with a measurable coefficient  $\mu(z)$  satisfying

$$|\mu(z)| \leq \frac{K - 1}{K + 1}. \quad (4)$$

Indeed, inequality (1) may be written as

$$\left| \frac{\partial w}{\partial \bar{z}} \right| \leq \frac{K - 1}{K + 1} \left| \frac{\partial w}{\partial z} \right|.$$

Thus the theory exposed in [2] is applicable. In particular

$$K_a(D, \varphi \circ w \circ \psi) = K_a(\psi(D), w)$$

if  $\varphi$  and  $\psi$  are conformal.

Now let  $D$  and  $w(D)$  be JORDAN domains,  $z_0$  a point in  $D$  and  $\varphi$  and  $\psi$  conformal mappings of  $D$  and  $w(D)$  onto the unit disc with

$$\varphi(z_0) = \psi(w(z_0)) = 0.$$

Then  $W = \psi \circ w \circ \varphi^{-1}$  is a self-mapping of the unit disc with  $W(0) = 0$ . If  $K(D, w) \leq K < \infty$ ,  $W$  is a solution of a BELTRAMI equation with a coefficient satisfying (4). Using [2] we obtain

**Lemma 1.** *If  $K_a(D, w) < \infty$ ,  $D$  and  $w(D)$  are JORDAN domains and  $z_0 \in D$ , then  $w$  has a uniform modulus of continuity depending only on  $K_a(D, w)$ ,  $D$ ,  $w(D)$ ,  $z_0$  and  $w(z_0)$ .*

Let  $R$  be a topological rectangle,  $\mu(z)$ ,  $z \in R$ , a measurable function satisfying (4) for some  $K \geq 1$ ,  $w$  and  $w_1$  two homeomorphisms of  $D$  satisfying (3). Then  $w \circ w_1^{-1}$  is a conformal mapping so that  $\text{mod } w(R) = \text{mod } w_1(R)$ . Hence we may define:  $\text{mod } (R, \mu) = \text{mod } w(R)$ .

**Lemma 2.** *Let  $R$  be a topological rectangle and  $\{\mu_j(z)\}$  a sequence of measurable functions in  $R$  such that  $|\mu_j(z)| \leq k < 1$  and  $\mu_j(z) \rightarrow \mu(z)$  a.e. Then  $\text{mod } (R, \mu_j) \rightarrow \text{mod } (R, \mu)$ .*

*Proof.* We may assume that  $R$  is the unit disc made into a topological rectangle by choosing four "vertices"  $\zeta_1, \zeta_2, \zeta_3, \zeta_4$  on the boundary. Let  $W^{\mu_j}$  be the homeomorphism of  $|z| \leq 1$  onto itself with  $W^{\mu_j}(0) = 0$ ,  $W^{\mu_j}(1) = 1$

and  $\partial W^{\mu_j}/\partial \bar{z} = \mu_j(z) \partial W^{\mu_j}/\partial z$  and let  $W^\mu$  be defined similarly. By [2], p. 399,  $W^{\mu_j} \rightarrow W^\mu$  uniformly in the closed unit disc. Since  $\text{mod}(R, \mu_j) = \text{mod } W^{\mu_j}(R)$  is a continuous function of the cross-ratio of the points  $W^{\mu_j}(\zeta_i)$ ,  $i = 1, 2, 3, 4$ , and similarly for  $\text{mod}(R, \mu)$ , the conclusion follows.

**Lemma 3.** *Let  $R$  be a topological rectangle,  $\mu(z)$ ,  $z \in R$ , a measurable function satisfying (4). Then  $\text{mod}(R, \mu) \leq K \text{mod } R$ .*

*Proof.* If  $\mu(z)$  is smooth, every homeomorphic solution of (3) is smooth and has a positive jacobian (cf. [2], p. 391). In this case the desired inequality follows by GRÖTZSCH's classical argument [7]. The general case is reduced to this special one by Lemma 2, since it is easy to find a sequence of smooth  $\mu_j$  satisfying (4) and converging a.e. to  $\mu$ .

**Lemma 4.**  $K_g(D, w) \leq K_a(D, w)$ .

This is an immediate corollary of Lemma 3.

**Lemma 5.** *If  $K_a(D, w) < \infty$ , then  $K_g(D, w) = K_a(D, w)$ .*

*Proof.* Set  $K_a(D, w) = K$  and assume that  $1 < K < \infty$ . (Otherwise there is nothing to prove.) In view of Lemma 4 it suffices to show that for every  $\delta$ ,  $0 < \delta < K - 1$ , there exists a sequence of squares  $Q_j \subset D$  with

$$\lim \text{mod } w(Q_j) \geq K - \delta. \quad (5)$$

Set  $\mu(z) = (\partial w/\partial \bar{z})/(\partial w/\partial z)$  for  $\partial w/\partial z \neq 0$ ,  $\mu(z) = 0$  for  $\partial w/\partial z = 0$ ; then  $w$  satisfies (4) and  $\text{ess. sup } |\mu(z)| = (K - 1)/(K + 1)$ . Let  $\Delta$  denote the annulus

$$\frac{K - \delta - 1}{K + \delta + 1} \leq |\mu| \leq \frac{K - 1}{K + 1}$$

in the  $\mu$ -plane; then  $\mu^{-1}(\Delta) \subset D$  has positive measure. Let  $\epsilon_j \downarrow 0$  be a given sequence. We can find a sequence of measurable sets  $\Delta_j$  such that

$$\Delta_{j+1} \subset \Delta_j \subset \Delta, \quad \text{diam } \Delta_j \leq \epsilon_j, \quad \text{mes } \mu^{-1}(\Delta_j) > 0.$$

Indeed, if  $\Delta$  is subdivided into finitely many measurable sets of diameter not exceeding  $\epsilon_1$ , at least one of them, say  $\Delta_1$  must be such that  $\text{mes } \mu^{-1}(\Delta) > 0$ . If  $\Delta_1$  is subdivided into finitely many measurable sets of diameter not exceeding  $\epsilon_2$ , at least one of them, say  $\Delta_2$ , is such that  $\text{mes } \mu^{-1}(\Delta_2) > 0$ , etc. Let  $\mu_0 = |\mu_0| e^{i\alpha}$  be the intersection of the closures of the  $\Delta_j$ . For each  $j$  let  $z_j \in D$  be a point at which the set  $\mu^{-1}(\Delta_j)$  has metric density one; such points exist by LEBESGUE's theorem. Each  $z_j$  is the center of a square  $Q_j$  with one side parallel to the ray  $z = r e^{i\alpha/2}$ ,  $0 < r < \infty$ , and such that

$$Q_j \subset D, \quad \text{mes } Q_j = m_j^2 < \epsilon_j, \quad \text{mes } [Q_j \cap \mu^{-1}(\Delta_j)] \geq (1 - \epsilon_j) m_j^2.$$

Hence

$$\text{mes} \{z \mid z \in Q_j, \mid \mu(z) - \mu_0 \mid > \epsilon_j\} \leq \epsilon_j m_j^2.$$

Let  $Q$  be the square obtained from  $Q_j$  by the mapping  $z \rightarrow (z - z_j)/m_j$ . For  $z \in Q$  set  $\mu_j(z) = \mu(z_j + m_j z)$ . Then  $\mu_j(z) \rightarrow \mu_0$  in measure. Selecting if need be a subsequence we may assume that  $\mu_j(z) \rightarrow \mu_0$  a. e. in  $Q$ . By Lemma 2

$$\text{mod} (Q, \mu_j) \rightarrow \text{mod} (Q, \mu_0) = \frac{1 + \mid \mu_0 \mid}{1 - \mid \mu_0 \mid} \geq K - \delta.$$

Noting that  $\text{mod} w(Q_j) = \text{mod} (Q, \mu_j)$  we obtain (5).

**Lemma 6.** *If  $w_j \rightarrow w$  uniformly in  $D$  and  $K_a(D, w_j) \leq K < \infty$ , then  $K_a(D, w) \leq K$ .*

*Proof.* Set  $w = u + iv$ ,  $w_j = u_j + iv_j$ . Let  $D_0$  be a relatively compact subdomain of  $D$ . It suffices to show that  $K_a(D_0, w) \leq K$ . By [2] Theorem 5, and the hypothesis

$$\int \int_{D_0} \frac{\partial(u_j, v_j)}{\partial(x, y)} dx dy = \text{mes} w_j(D_0) = O(1).$$

Since (1) holds for each  $w_j$ ,

$$\int \int_{D_0} \left( \left| \frac{\partial w_j}{\partial x} \right|^2 + \left| \frac{\partial w_j}{\partial y} \right|^2 \right) dx dy = O(1).$$

This shows that the partial derivatives of  $w_j$  are square-integrable functions in  $D_0$  and that we may assume, selecting if need be a subsequence, that

$$\frac{\partial w_j}{\partial x} \rightarrow \frac{\partial w}{\partial x}, \quad \frac{\partial w_j}{\partial y} \rightarrow \frac{\partial w}{\partial y} \quad \text{weakly in } L_2(D_0). \tag{6}$$

Next, let  $\omega$  be a smooth function with compact support in  $D_0$ . Then

$$\begin{aligned} \int \int_{D_0} \omega \frac{\partial(u_j, v_j)}{\partial(x, y)} dx dy &= \int \int_{D_0} v_j \frac{\partial(\omega, u_j)}{\partial(x, y)} dx dy, \\ \int \int_{D_0} \omega \frac{\partial(u, v)}{\partial(x, y)} dx dy &= \int \int_{D_0} v \frac{\partial(\omega, u)}{\partial(x, y)} dx dy. \end{aligned} \tag{7}$$

If  $w$  and  $w_j$  are smooth, this follows by integration by parts. In the general case one approximates  $w$  (or  $w_j$ ) together with its first derivatives, in the mean, by smooth functions. If  $\omega$  is also non-negative, then

$$\int \int_{D_0} \omega \left( \left| \frac{\partial w_j}{\partial x} \right|^2 + \left| \frac{\partial w_j}{\partial y} \right|^2 \right) dx dy \leq \left( K + \frac{1}{K} \right) \int \int_{D_0} \omega \frac{\partial(u_j, v_j)}{\partial(x, y)} dx dy. \tag{8}$$

But by (6)

$$\iint_{D_0} \omega \left( \left| \frac{\partial w}{\partial x} \right|^2 + \left| \frac{\partial w}{\partial y} \right|^2 \right) dx dy \leq \liminf \iint_{D_0} \omega \left( \left| \frac{\partial w_j}{\partial x} \right|^2 + \left| \frac{\partial w_j}{\partial y} \right|^2 \right) dx dy$$

and by (6) and (7)

$$\lim \iint_{D_0} \omega \frac{\partial(u_j, v_j)}{\partial(x, y)} dx dy = \iint_{D_0} \omega \frac{\partial(u, v)}{\partial(x, y)} dx dy,$$

so that by (8)

$$\iint_{D_0} \omega \left( \left| \frac{\partial w}{\partial x} \right|^2 + \left| \frac{\partial w}{\partial y} \right|^2 \right) dx dy \leq \left( K + \frac{1}{K} \right) \iint_{D_0} \omega \frac{\partial(u, v)}{\partial(x, y)} dx dy.$$

A simple limiting argument shows that this holds also if  $\omega$  is the characteristic function of a rectangle in  $D_0$ . This implies that (1) holds a.e.

We note now a corollary of the BEURLING-AHLFORS theorem [5].

**Lemma 7.** *For every  $K \geq 1$  there exists a number  $K^*$  with the following property. Let  $K_g(D, w) \leq K < \infty$  and let  $D_0$  be a relatively compact JORDAN subdomain of  $D$ . Then there exists a homeomorphism  $\Omega$  of the closure of  $D_0$  onto that of  $w(D_0)$  with  $K_a(D, \Omega) \leq K^*$  and  $w(z) = \Omega(z)$  on the boundary  $\dot{D}_0$  of  $D_0$ .*

*Proof.* Choose a point  $\hat{z}$  on  $\dot{D}_0$  and set  $\hat{Z} = w(\hat{z})$ . Let  $z \rightarrow \varphi(z)$  and  $Z \rightarrow \psi(Z)$  be conformal homeomorphisms, of  $D_0$  and  $w(D_0)$ , respectively, onto the half-plane  $U = \{ \zeta \mid \text{Im } \zeta > 0 \}$  with  $\varphi(\hat{z}) = \psi(\hat{Z}) = \infty$ . Set

$$\gamma(\xi) = \psi \circ w \circ \varphi^{-1}(\xi), \quad -\infty < \xi < +\infty.$$

For a real  $\xi$  and an  $h > 0$  make  $D_0$  into a topological rectangle  $R_{\xi, h}$  by choosing as "vertices" the points  $\varphi^{-1}(\xi - h)$ ,  $\varphi^{-1}(\xi)$ ,  $\varphi^{-1}(\xi + h)$  and  $\hat{z}$ . Then  $\text{mod } R_{\xi, h} = 1$ . The "vertices" of  $w(R_{\xi, h})$  are the points

$$\psi^{-1}(\gamma(\xi - h)), \quad \psi^{-1}(\gamma(\xi)), \quad \psi^{-1}(\gamma(\xi + h))$$

and  $\hat{Z}$ , and  $\text{mod } w(R_{\xi, h})$  is a continuous function of the ratio

$$(\gamma(\xi + h) - \gamma(\xi)) / (\gamma(\xi) - \gamma(\xi - h)).$$

Since  $K^{-1} \leq \text{mod } w(R_{\xi, h}) \leq K$ , there exists a  $\varrho > 0$  depending only on  $K$  such that

$$0 < \frac{1}{\varrho} \leq \frac{\gamma(\xi + h) - \gamma(\xi)}{\gamma(\xi) - \gamma(\xi - h)} \leq \varrho. \quad (9)$$

According to [5], condition (9) implies that

$$\zeta = \xi + i\eta \rightarrow F(\zeta) = \frac{1}{2} \int_0^1 [(1 + i)\gamma(\xi + \tau\eta) + (1 - i)\gamma(\xi - \tau\eta)] d\tau$$

is a homeomorphism of the closed upper half-plane onto itself with  $F(\xi) = g(\xi)$  and  $K_a(U, F) = K^* < \infty$ , where  $K^*$  depends only on  $K$ . Set  $\Omega = \psi^{-1} \circ F \circ \varphi$ ; this mapping has the required properties.

**Lemma 8.** *If  $K_g(D, w) < \infty$ , then  $K_a(D, w) < \infty$ .*

*Proof.* Set  $K_g(D, w) = K$ . We show that for every square  $Q \subset D$ ,

$$K_a(Q, w) \leq K^*,$$

the number in Lemma 7. For every integer  $j > 0$  subdivide  $Q$  into  $4^j$  congruent squares. Lemma 7 implies that there exists a homeomorphism  $w_j$  of  $Q$  such that  $K_a(q, w_j) \leq K^*$  for each of the  $4^j$  small squares  $q$  and  $w_j = w$  on the boundary of each small square. Hence  $w_j$  is a homeomorphism of  $Q$  onto  $w(Q)$  and  $K_a(Q, w_j) \leq K^*$ . By Lemma 1 the  $w_j$  are equicontinuous and, by construction,  $w_j \rightarrow w$  on a dense set. Hence  $w_j \rightarrow w$  uniformly and, by Lemma 6,  $K_a(Q, w) \leq K^*$ .

Combining Lemmas 5 and 8 we obtain the theorem.

Now set

$$K_1(D, w) = \inf \text{mod } w(R) \quad \text{for all } R \subset D \text{ and } \text{mod } R = 1$$

where  $R$  is a topological rectangle.

The argument used in proving Lemma 5 shows that  $K_1(D, w) = K_a(D, w)$  whenever  $K_a(D, w) < \infty$ . The argument used in proving Lemmas 7 and 8 shows that  $K_a(D, w)$  is finite whenever  $K_1(D, w)$  is. Thus

$$K_1(D, w) = K_a(D, w).$$

The geometric definition can be given a local form by setting (cf. PFLUGER [9]).

$$K^*(D, w) = \sup_{z_0 \in D} \lim_{r \rightarrow 0} K_g(S_r(z_0), w)$$

$S_r(z_0)$  being the disc  $|z - z_0| < r$ . We have that

$$K^*(D, w) = K_g(D, w);$$

the proof is immediate via the equivalence theorem.

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