Some Geometric Properties of Polynomial Surfaces.

Some Geometric Properties of Polynomial Surfaces[1]).

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In a recent paper [5] H. Huber proved that for every polynomial $P(x, y)$, the surface $z = P(x, y)$ is conformally equivalent to the plane. His method requires obtaining estimates on the length of the intersection of the surface with a sphere of radius r. We shall present an alternative proof in this paper by showing that an elementary argument gives a bound for the positive and negative parts of the *curvatura integra* of a polynomial surface. The result then follows immediately from a criterion of Blanc and Fiala [1] for parabolic surfaces.

As is clear from the method, the bound on total curvature extends to polynomial hypersurfaces in any number of dimensions. The details of this, together with other results related to the main theorem, are given in a series of remarks and additional theorems. Theorem 3 and the corollary to Lemma 2 concern the number of real finite intersections of real algebraic curves.

Theorem 1. *Let $P(x, y)$ be a polynomial of degree n. Let K be the* Gauss *curvature of the surface S defined by $z = P(x, y)$, and let dA be the area element of S. Then*

$$\iint_S |K|\, dA \leq 2\pi(n-1)^2 . \tag{1}$$

Proof. Let S^* denote the part of S for which $K \neq 0$. Then the Gauss spherical map is a local homeomorphism at each point of S^*, and the image of S^* is an unbranched covering of a part of the unit sphere. If the area of this covering is I, we have $I = \iint_{S^*} |K|\, dA = \iint_S |K|\, dA$. Since only points of the upper hemisphere (or lower, depending on the choice of normal direction) are covered, the theorem will be proved as soon as we show that no point is covered more than $(n-1)^2$ times. But that is equivalent to saying that there are at most $(n-1)^2$ points on S^* satisfying $\dfrac{\partial P}{\partial x} = a$, $\dfrac{\partial P}{\partial y} = b$, for any given a and b. If we set

$$u = \frac{\partial P}{\partial x}, \quad v = \frac{\partial P}{\partial y}, \tag{2}$$

then the condition $K \neq 0$ is equivalent to $\dfrac{\partial(u, v)}{\partial(x, y)} \neq 0$ which means that points on S^* correspond to isolated intersections of the curves $\dfrac{\partial P}{\partial x} - a = 0$ and $\dfrac{\partial P}{\partial y} - b = 0$. Since these are both algebraic curves of degree at most

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$n - 1$, they can have at most $(n-1)^2$ isolated intersections by BEZOUT's Theorem, which is the required result.

Corollary. For every polynomial $P(x, y)$, *the surface* $z = P(x, y)$ *is of parabolic type.*

Proof. Let $K^+ = \max\{K, 0\}$ and $K^- = \max\{-K, 0\}$. Then

$$\iint_S |K|\, dA = \iint_S K^+ dA + \iint_S K^- dA\,,$$

and we have in particular

$$\iint_S K^- dA \leq 2\pi(n-1)^2\,. \tag{3}$$

But the theorem of BLANC and FIALA [1] (see also A. HUBER [4] for a different proof and generalizations) states that a complete simply-connected surface with an analytic RIEMANNian metric satisfying $\iint_S K^- dA < \infty$ must always be of parabolic type. Since a polynomial surface is clearly complete, simply connected, and analytic, the result follows.

Remarks. 1. If $P(x_1, \ldots, x_k)$ is a polynomial of degree n in k variables we may again consider the GAUSS curvature K of the hypersurface $S: x_{k+1} = P(x_1, \ldots, x_k)$, which is defined as the ratio of the volume elements under the normal map of S into the unit sphere $U: x_1^2 + \ldots + x_{k+1}^2 = 1$. Exactly the same reasoning gives the result $\int_S |K|\, dV \leq \frac{1}{2} c_k (n-1)^k$, where c_k is the volume of U.

2. If $P(x, y)$ is linear, then obviously both sides of (1) are zero. If $P(x, y)$ is quadratic, then the gradient map (2) is a linear transformation which will be singular if and only if $K \equiv 0$. Thus, for $n = 2$, the left-hand side of (1) can be only 0 or 2π. For $n > 2$, we shall show (in the corollary to Theorem 3) that equality can never be attained in (1). However, the bound is still the correct one, since for each n we can find polynomials of degree n for which the left-hand side of (1) comes arbitrarily close to the right. For example, if $Q(t) = (t-1)(t-2)\ldots(t-n)$, then we may set

$$P(x, y) = M[Q(x) + Q(y)]\,.$$

The equations (2) take the form $u = M\,Q'(x)$, $v = M\,Q'(y)$, and by choosing M sufficiently large we may cover any prescribed compact part of the u, v-plane $(n-1)^2$ times.

3. Inequality (1) implies in particular the existence of the *curvatura integra* $\iint_S K\, dA$. For a complete surface this quantity is bounded above by $2\pi\chi$, where χ is the EULER characteristic of S (COHN-VOSSEN [2]. See also HUBER [4].). For a simply-connected surface we have $\chi = 1$. Thus, for every polynominal surface,

$$\iint_S K\, dA \leq 2\pi\,. \tag{4}$$

Inequality (3) gives us also a lower bound, but it turns out that a much stronger result is true (Theorem 2). In order to obtain it we shall have to study polynomial mappings more closely, and we shall restrict ourselves to two dimensions, making use of winding numbers.

We recall that if Γ is a piecewise smooth closed curve in the u, v-plane, and if (u_0, v_0) is any point not on Γ, then in a neighborhood of each point of Γ the function $\theta = \tan^{-1}\dfrac{v - v_0}{u - u_0}$ is defined up to an additive constant, so that the differential $d\theta$ is uniquely defined along Γ. We may define the *winding number* of Γ about (u_0, v_0) by

$$n(\Gamma; u_0, v_0) = \frac{1}{2\pi}\int_\Gamma d\theta\,.$$

The following lemma connecting winding numbers with the degree of a mapping is known under much more general conditions, but we include a simple proof which covers the only case that we shall need.

Lemma 1. *Let D denote the disk $x^2 + y^2 \leq R^2$ and let C be its circumference, where C is given the positive orientation with respect to D. Let*

$$u(x, y),\ v(x, y)$$

be continuously differentiable functions in D, let $J(x, y) = \dfrac{\partial(u, v)}{\partial(x, y)}$, and let Γ be the image of the curve C. Assume that (u_0, v_0) is a point not on Γ, whose inverse image consists of a finite number of points at each of which $J(x, y) \neq 0$. If $J(x, y) > 0$ at p of these points and $J(x, y) < 0$ at r of them, then we have

$$p - r = n(\Gamma; u_0, v_0)\,. \tag{5}$$

Proof. Choose $\varepsilon > 0$ so that the disk $\Delta: (u - u_0)^2 + (v - v_0)^2 \leq \varepsilon^2$ does not intersect Γ, and such that the inverse image of Δ consists of $r + p$ disjoint neighborhoods N_i in D, each of which is mapped one-to-one onto Δ. Denote by C_i the boundaries of N_i, and by γ the boundary of Δ. Then the function $\tilde{\theta} = \tan^{-1}\dfrac{v(x, y) - v_0}{u(x, y) - u_0}$ is defined up to an additive constant in a neighborhood of each point of $D' = D - \bigcup N_i$. The differential $d\tilde{\theta}$ is therefore a locally exact, hence closed differential in D', so that $\int_{\partial D'} d\tilde{\theta} = 0$. Hence

$$2\pi n(\Gamma; u_0, v_0) = \int_\Gamma d\theta = \int_C d\tilde{\theta} = \sum_{i=1}^{p+r}\int_{C_i} d\tilde{\theta} = (p - r)\int_\gamma d\theta = 2\pi(p - r)$$

since p of the curves C_i map onto γ with the positive orientation and r of them map onto γ with the negative orientation.

We wish next to apply this lemma to the case of polynomial mappings. First we make the following general comments.

Let $u(x,y), v(x,y)$ be continuously differentiable and let $J(x,y) = \dfrac{\partial(u,v)}{\partial(x,y)}$. A point (u_0, v_0) is called a *regular value* if at each point of its inverse image $J(x,y) \neq 0$. If $u(x,y)$ and $v(x,y)$ are polynomials, then either $J(x,y) \equiv 0$ in which case the whole plane will map into a curve, or else the set $J(x,y) = 0$ is an algebraic curve C whose image will lie on an algebraic curve Γ in the u, v-plane. Thus, given any neighborhood in the u, v-plane, all points of the neighborhood not on Γ will be regular. For a regular value it is perfectly clear what is meant by the number of times it is covered under the mapping. Note also that (u_0, v_0) is a regular value if and only if all intersections of $u(x,y) = u_0, v(x,y) = v_0$ are simple, and the number of these intersections is precisely the number of times that the point (u_0, v_0) is covered.

Lemma 2. *Let $P(x,y), Q(x,y)$ be arbitrary polynomials. For any regular value (u_0, v_0) let the number of intersections of the curves $P(x,y) = u_0$, $Q(x,y) = v_0$ where $\dfrac{\partial(P,Q)}{\partial(x,y)} > 0$ be p, and the number where $\dfrac{\partial(P,Q)}{\partial(x,y)} < 0$ be r. Then*

$$|p - r| \leq \min\{\deg P, \deg Q\}. \tag{6}$$

Proof. Set $u = P(x,y)$, $v = Q(x,y)$. Let (u_0, v_0) be regular, so that all intersections of $P(x,y) = u_0$, $Q(x,y) = v_0$ are simple. By BEZOUT's Theorem there are only a finite number, and hence, for R sufficiently large they will all be contained in a disk $x^2 + y^2 < R^2$. By Lemma 1 it is sufficient to show that $|n(\Gamma; u_0, v_0)| \leq \min\{\deg P, \deg Q\}$, where Γ is the image of $x^2 + y^2 = R^2$. Let us assume for definiteness that the degree of $P(x,y)$ is less than or equal to the degree of $Q(x,y)$, and let the degree of $P(x,y)$ be k. For all sufficiently large R the curve $P(x,y) = u_0$ can intersect $x^2 + y^2 = R^2$ in at most a finite number of points, and again by BEZOUT's Theorem the total number of intersections is at most $2k$. Hence if there are k_1 points on $x^2 + y^2 = R^2$ where $u = u_0, v > v_0$ and k_2 points where $u = u_0$, $v < v_0$, we see that $\min\{k_1, k_2\} \leq k$. But the number of times that Γ intersects each ray $u = u_0$, $v > v_0$ and $u = u_0$, $v < v_0$ must be at least $|n(\Gamma; u_0, v_0)|$. Hence we have $|n(\Gamma; u_0, v_0)| \leq \min\{k_1, k_2\} \leq k = \min\{\deg P, \deg Q\}$, which proves (6).

Corollary. For any polynomials $P(x,y)$, $Q(x,y)$, if $\dfrac{\partial(P,Q)}{\partial(x,y)} \geq 0$ or $\dfrac{\partial(P,Q)}{\partial(x,y)} \leq 0$ everywhere, then for all (u_0, v_0) the number of simple intersections of

$$P(x,y) = u_0, \quad Q(x,y) = v_0$$

is at most the minimum degree of $P(x,y)$ and $Q(x,y)$.

Proof. If $P(x,y) = u_0$, $Q(x,y) = v_0$ have n simple intersections, then for all points (u_1, v_1) in some neighborhood of (u_0, v_0) the curves

$$P(x, y) = u_1, \; Q(x, y) = v_1$$

must have at least n intersections. In particular, if (u_1, v_1) is a regular point, then since the Jacobian always has the same sign we find that
$n \leq \{$the number of intersections of $P = u_1,\ Q = v_1\} \leq \min\{\deg P, \deg Q\}$.

Theorem 2. *With the same notation as in Theorem 1, we have*

$$\left| \iint_S K\, dA \right| \leq 2\pi(n-1). \tag{7}$$

Equality is attained for all harmonic polynomials.

Proof. Applying Lemma 2 to the map $u = \dfrac{\partial P}{\partial x}$, $v = \dfrac{\partial P}{\partial y}$, we see that for every regular value (u_0, v_0) we have $|p - r| \leq n - 1$. As in the proof of Theorem 1, if we consider the normal map of S into the unit sphere U, and if we denote by U^* the part of U corresponding to regular points (u_0, v_0), then denoting the area element of U by $d\sigma$ we have

$$\left| \iint_S K\, dA \right| = \left| \iint_{S^*} K^+ dA - \iint_{S^*} K^- dA \right| = \left| \iint_{U^*} (p - r)\, d\sigma \right| \leq 2\pi(n-1).$$

The second statement in the theorem follows immediately from the fact that if $P(x, y)$ is a harmonic polynomial of degree n, then $\dfrac{\partial P}{\partial x} - i \dfrac{\partial P}{\partial y}$ is an analytic polynomial of degree $n - 1$, so that except for a finite number of points (u_0, v_0), the equation $\dfrac{\partial P}{\partial x} - i \dfrac{\partial P}{\partial y} = u_0 - i v_0$ will have $n - 1$ distinct (complex) roots.

Theorem 3. *Let $P(x, y)$, $Q(x, y)$ be arbitrary polynomials of degree m, n respectively. Then unless $P(x, y)$ and $Q(x, y)$ are both linear, there always exists an open set in the u, v-plane such that for any point (u_0, v_0) in this set, the number of (real, finite) intersections of $P(x, y) = u_0$, $Q(x, y) = v_0$ is strictly less than mn.*

Proof. Let $J(x, y) = \dfrac{\partial(P, Q)}{\partial(x, y)}$.

Case 1. Suppose $J(x, y) \geq 0$ or $J(x, y) \leq 0$ everywhere. Then for all regular values (u_0, v_0), (and in particular for all points not lying on a certain curve) we have by Lemma 2 that the number of intersections of

$$P(x, y) = u_0, \; Q(x, y) = v_0$$

is at most the minimum of m and n. But for any positive integers m, n, we always have $\min\{m, n\} < mn$ unless $m = n = 1$.

Case 2. Suppose the homogeneous polynomials of highest degree in $P(x, y)$ and $Q(x, y)$ have a common factor. Then for all (u_0, v_0) the number mn of intersections of $P(x, y) = u_0$, $Q(x, y) = v_0$ from BEZOUT's Theorem will include at least one intersection (real or complex) on the line at infinity, and hence the number of finite intersections can be at most $mn - 1$.

Case 3. Suppose that $J(x, y)$ takes on both positive and negative values, and that $P(x, y)$ and $Q(x, y)$ do not have a common factor in their highest degree terms. In this case we shall show that for some point (u_0, v_0) the equations $P(x, y) = u_0$, $Q(x, y) = v_0$ will have a common (non-real) complex solution. The same will then be true in a neighborhood of (u_0, v_0) and in fact there must be a pair of complex solutions, so that the number of real intersections can be at most $mn - 2$.

To prove this statement, we consider a branch C of the curve $J(x, y) = 0$ which separates a region where $J(x, y) > 0$ from one where $J(x, y) < 0$. If the image of C were a single point (u_0, v_0) it would mean that the curves $P(x, y) - u_0 = 0$, $Q(x, y) - v_0 = 0$ have more than a finite number of intersections, and hence that the polynomials $P(x, y) - u_0$, $Q(x, y) - v_0$ have a common factor. That would in turn imply that the terms of highest degree in $P(x, y)$ and $Q(x, y)$ also have a common factor, which was our Case 2. Thus the image of C must be a curve Γ. The correspondence between C and Γ is locally one-to-one at every point where either grad u or grad v is not perpendicular to C, But there must always be a point on C where one of these is not perpendicular, since otherwise both u and v would be constant on C. Let us assume for definiteness that grad u is not perpendicular to C at some point, and choose a regular point of C for which this is true. By a rotation in the x, y-plane we may assume that the tangent to C is horizontal at this point and that $u_x > 0$. The curves $u = c$ will fill out a neighborhood D, and we may choose this neighborhood sufficiently small so that $u_x > 0$ throughout. Then each point of D lies above, below, or on C, along a unique arc of a level curve $u = c$. Let us suppose that $J(x, y) > 0$ above C and $J(x, y) < 0$ below C. This means precisely that v must be decreasing as we approach C along a level curve $u = c$, either moving down from above or up from below. Thus the image of this whole neighborhood lies above or on an arc of Γ. If we take a point (u_0, v_0) on this arc, then it will correspond to a unique point (x_0, y_0) of C lying in the neighborhood D, while no point (u_0, v_1) with $v_1 < v_0$ will correspond to any point of D. However, if we allow the full complex neighborhood N of (x_0, y_0) then there must always be an intersection of $P(x, y) = u_1$, $Q(x, y) = v_1$ lying in N for (u_1, v_1) sufficiently near (u_0, v_0) by the theorem on continuous dependence of roots of polynomials on the coefficients. Hence for those points lying below Γ there will be at least one non-real complex root of $P(x, y) = u_1$, $Q(x, y) = v_1$. This completes the proof.

Corollary. Inequality (1) *is always strict if* $n > 2$. This follows immediately from the proof of Theorem 1, using Theorem 3.

Final remarks. It is quite likely that a stronger version of Theorem 3 is true, in the sense that there may always exist an open set of points (u_0, v_0) such that the number of intersections of $P(x, y) = u_0$, $Q(x, y) = v_0$ is much lower than mn, conceivably even min $\{m, n\}$ which would be the best possible result.

It would be interesting to obtain results in the opposite direction, stating that under certain conditions every point is covered at least a certain number of times. For example, it seems likely that if $J(x, y) > 0$ everywhere, then the map is a one-to-one map of the whole x, y-plane onto the whole u, v-plane. This is known to be true under the stronger assumption that $J(x, y) \equiv c \neq 0$ [3]. The analogous result is still not known in higher dimensions.

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