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# On a characterization of quasiconformal mappings<sup>1</sup>

by EDGAR REICH

## 1. Introduction

Let  $M(Q)$  denote the module of the quadrilateral  $Q$  (a JORDAN domain with four distinguished boundary points and two non-adjacent distinguished sides), A sense-preserving homeomorphism  $f$  of a region  $\Omega$  onto the image region  $f(\Omega)$  is said to be a quasiconformal mapping with maximal dilatation  $K(\Omega) = K_r(\Omega)$  if

$$\sup_{\bar{Q} \subset \Omega} \frac{M(f(Q))}{M(Q)} = K(\Omega) < \infty. \quad (1.1)$$

( $f(Q)$  denotes the quadrilateral with preimage  $Q$ .)

Instead of considering the effect of  $f$  on quadrilaterals it is natural to study the effect of  $f$  on ring domains (doubly connected regions). The modulus  $\mu(R)$  of a ring  $R$  is defined as  $(2\pi)^{-1} \log(r_2/r_1)$ , where  $\{r_1 < |w| < r_2\}$  is an annulus conformally equivalent to  $R$ . It is well known [1] that if (1.1) holds then

$$\frac{\mu(f(R))}{\mu(R)} \leq K(\Omega), \quad \bar{R} \subset \Omega. \quad (1.2)$$

In fact, the following has been proved by GEHRING and VÄISÄLÄ ([2], Theorem 3):

$$\sup_{\bar{R} \subset \Omega} \frac{\mu(f(R))}{\mu(R)} = K(\Omega). \quad (1.3)$$

Thus quasiconformal mappings  $f$  with maximal dilatation  $K(\Omega)$  may be characterized by (1.3) instead of (1.1).

In the proof of (1.3) in [2] essential use is made of the deep and rather difficult "analytic" characterization of quasiconformal maps (See, for instance, [4], Chapter 4). In view of the significance of (1.3) it appears desirable to obtain a more direct and more elementary proof of (1.3), starting with the definition (1.1). It is the object of the present note to provide such a proof.

## 2. A class of ring domains

Let  $Q$  be a quadrilateral with distinguished sides  $\alpha_1, \alpha_2$ . A ring  $R$  will be said to *link*  $Q$  if every closed curve  $\gamma$ , in  $R$ , separating the boundary com-

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ponents of  $R$ , contains an arc lying in  $Q$  which joins  $\alpha_1$  and  $\alpha_2$ . (See Fig.1 where the heavy lines are the boundary components of  $R$ , and the dashed line represents  $Q$ .)

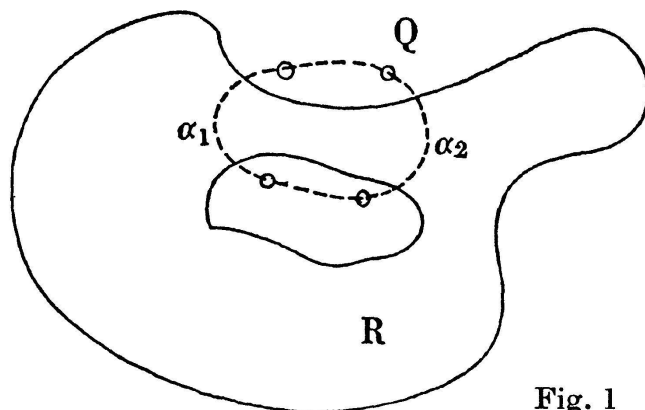


Fig. 1

We will first show that

$$\mu(R) \leq M(Q), \text{ if } R \text{ links } Q. \tag{2.1}$$

Let  $\varrho^*(z), z \in Q$ , be the extremal metric for  $Q$ ; that is,

$$M(Q) = \frac{A_{\varrho^*}(Q)}{L_{\varrho^*}^2(Q)}, \quad A_{\varrho^*}(Q) = \iint_Q \varrho^{*2} |dz|^2, \quad L_{\varrho^*}(Q) = \inf_{\gamma} \int_{\gamma} \varrho^* |dz|,$$

where  $\{\gamma\}$  is the set of locally rectifiable curves in  $Q$  joining  $\alpha_1, \alpha_2$ . On the other hand, (See, for instance, [3], Chapter 2),

$$\mu(R) = \inf_e \frac{A_e(R)}{L_e^2(R)} \tag{2.2}$$

where, this time,  $\{\gamma\}$  is the set of closed curves in  $R$  separating the boundary components. For  $z \in R$ , let

$$\varrho(z) = \begin{cases} \varrho^*(z), & z \in R \cap Q \\ 0, & z \in R - Q \end{cases}.$$

Clearly,

$$A_{\varrho}(R) \leq A_{\varrho^*}(Q).$$

Since  $R$  links  $Q$ ,  $L_{\varrho}(R) \geq L_{\varrho^*}(Q)$ . Hence, by (2.2),

$$\mu(R) \leq \frac{A_{\varrho^*}(Q)}{L_{\varrho^*}^2(Q)} = M(Q),$$

as was to be shown.

In general, to obtain a bound on the modulus of a ring domain  $R$  from below, one attempts to make use of the fact ([3], Chapter 2) that

$$\frac{1}{\mu(R)} = \inf_e \frac{A_e(R)}{L_e^2(R)} \tag{2.3}$$

where, now,  $\{\gamma\}$  is the set of locally rectifiable curves in  $R$  joining the boun-

dary components. Fortunately, we shall require a bound on  $\mu(R)$  from below, for rings linking  $Q$ , only in the very special case considered next.

Suppose the set of positive numbers,  $a, p, h, s$ , is given, and

$$2p < \min(a, 1), h < a - 2p, p < s < a - p - h, h < 1.$$

We consider the ring  $R'$  (Fig. 2) obtained as follows.

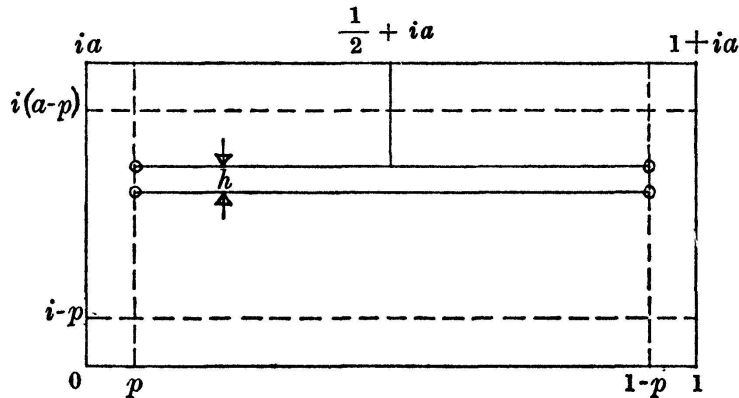


Fig. 2

The outer boundary component of  $R'$  consists of the union of (a) the boundary of the rectangle with vertices  $0, 1, 1 + ia, ia$  (b) the vertical segment joining the points  $\frac{1}{2} + ia, \frac{1}{2} + i(a - s)$  (c) the horizontal segment  $I_1 = \{z \mid \Im z = a - s, p \leq \Re z \leq 1 - p\}$ .

The inner boundary component of  $R'$  is the horizontal segment

$$I_2 = \{z \mid \Im z = a - s - h, p \leq \Re z \leq 1 - p\}.$$

Let  $T$  be the rectangle with horizontal sides  $I_1, I_2$ , with the vertical sides (of length  $h$ ) distinguished. The ring  $R'$  links  $T$ , and, therefore, by (2.1),  $\mu(R') \leq h/(1 - 2p)$ . We shall now show that

$$\mu(R') \geq \frac{h}{1 + ah^{\frac{1}{2}}}, \text{ if } h \leq p^4. \tag{2.4}$$

Let

$$\varrho(z) = \begin{cases} 1, & 0 < \Re z < 1, a - s - h < \Im z < a - s \\ h^{\frac{3}{4}}, & \text{elsewhere in } R'. \end{cases}$$

Let  $\{\gamma\}$  be the family of curves in  $R'$  joining  $I_2$  to the outer boundary component of  $R'$ . Clearly,

$$L_\varrho(R') = \min(h, h^{\frac{3}{4}}p), A_\varrho = h + h^{\frac{3}{2}}(a - h) \leq h + ah^{\frac{3}{2}}.$$

Hence, by (2.3),

$$\frac{1}{\mu(R')} \leq \frac{A_\varrho(R')}{L_\varrho^2(R')} \leq \frac{h + ah^{\frac{3}{2}}}{h^2} = \frac{1 + ah^{\frac{1}{2}}}{h}, \text{ if } h^{\frac{3}{4}}p \geq h.$$

This is equivalent to (2.4).

### 3. Proof of (1.3)

Let  $\epsilon$ ,  $0 < \epsilon < \frac{1}{2}$ , be given. In the notation of Section 1, there exists a quadrilateral  $Q, \bar{Q} \subset \Omega$ , with

$$\frac{M(f(Q))}{M(Q)} \geq (1 - \epsilon) K(\Omega).$$

Let the transformation  $\Phi$  map  $f(Q)$  conformally onto a rectangle  $V$  with vertices  $0, 1, 1 + ia, ia$ , with the vertical sides distinguished. Thus

$$a = M(V) = M(f(Q)).$$

Let  $V_p$  be the rectangle, interior to  $V$  and oriented as  $V$ , such that the distance between corresponding sides of  $V_p$  and  $V$  is  $p$ ,  $0 < p < \frac{1}{2} \min(a, 1)$

Let  $Q_p = f^{-1} \Phi^{-1}(V_p)$ . There exists [1] a number  $\delta(\epsilon) > 0$  such that

$$\frac{M(V_p)}{M(Q_p)} = \frac{a - 2p}{(1 - 2p) M(Q_p)} \geq (1 - 2\epsilon) K(\Omega), \text{ if } p \leq \delta(\epsilon);$$

that is,  $Q_p$  has nearly the same module as  $Q$  if  $p$  is small.

We now divide  $V_p$  into  $n$  equal horizontal strips,  $\sigma_k$ ,  $k = 1, 2, \dots, n$ .  $\sigma_k$  is an  $h_n \times 1 - 2p$  rectangle,  $h_n = (a - 2p) n^{-1}$ , whose vertical sides are distinguished. The quadrilaterals

$$q_k = f^{-1} \Phi^{-1}(\sigma_k), \quad k = 1, 2, \dots$$

which lie in  $\Omega$ , subdivide  $Q_p$ . According to the subadditivity property for modules [1],

$$\sum_{k=1}^n M(q_k) \leq M(Q_p) \leq \frac{a - 2p/1 - 2p}{(1 - 2\epsilon) K(\Omega)}.$$

Hence for some  $k$ , say  $k = k_n$ ,

$$M(q_{k_n}) \leq \frac{a - 2p/1 - 2p}{(1 - 2\epsilon) n K(\Omega)} = \frac{h_n/1 - 2p}{(1 - 2\epsilon) K(\Omega)}.$$

If we define  $N(p)$  as  $N(p) = (a - 2p) p^{-4}$ , then  $n \geq N(p)$  implies  $h_n \leq p^4$ .

Let  $R'_{pn}$  be the ring linking  $\sigma_{k_n}$  formed as in Fig.2, with the present  $h_n$  serving as the  $h$  of Fig.2. Then, by (2.4),

$$\mu(R'_{pn}) \geq \frac{h_n}{1 + ah_n^{\frac{1}{2}}}, \text{ if } n \geq N(p).$$

Consider the ring  $R_{pn} = f^{-1} \Phi^{-1}(R'_{pn})$ , in  $\Omega$ . Since  $R_{pn}$  links  $q_{k_n}$ , we have, by (2.1),

$$\mu(R_{pn}) \leq M(q_{kn}) \leq \frac{h_n/1 - 2p}{(1 - 2\varepsilon)K(\Omega)}, \text{ if } p \leq \delta(\varepsilon).$$

Now  $\mu(f(R_{pn})) = \mu(R'_{pn})$ . Therefore,

$$\frac{\mu(f(R_{pn}))}{\mu(R_{pn})} \geq \frac{(1 - 2\varepsilon)(1 - 2p)K(\Omega)}{1 + ah_n^{\frac{1}{2}}}, \text{ if } p \leq \delta(\varepsilon), n \geq N(p). \quad (3.1)$$

Once  $p$  is chosen,  $h_n$  can be made arbitrarily small by taking  $n$  sufficiently large. Thus (3.1) shows that there exist rings  $R, \bar{R} \subset \Omega$ , such that

$$\frac{\mu(f(R))}{\mu(R)} \geq (1 - 3\varepsilon)K(\Omega).$$

Together with (1.2) the above establishes (1.3).

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