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On the conformal types of algebraic surfaces of euclidean space[1]

by ADRIANO M. GARSIA

Introduction. It has been an open question for some time whether or not the classical (i.e. C^2 surfaces) of Euclidean space (3-dimensional) exhaust all possible conformal types. In the non compact case the question is still open. In the compact case it can be shown (see [1] and [2]) that among the C^∞ surfaces of Euclidean space there are surfaces conformally equivalent to any given compact RIEMANN surface.

In this paper we are to improve the results in [1] and [2]. It will be shown that for any given compact RIEMANN surface conformally equivalent models can be found among the algebraic surfaces of ordinary space. Here by "algebraic surface" we mean a surface satisfying an equation of the type

$$F(x, y, z) = 0$$

where $F(x, y, z)$ is a real polynomial in its arguments.

The proof of this result will be based upon the methods and results contained in [1] and [2]. Thus it will be a purely existential one. In our first approach to the problem of imbedding RIEMANN surfaces we were able to give some explicit constructions (see [3] and [4]), and these illustrate in a striking way some conformal properties of compact RIEMANN surfaces and bring out some interesting geometrical interpretations of the SCHOTTKY parameters. However, general constructive approaches may be very difficult to obtain.

As an illustration of the difficulties that arise in carrying out explicit computations we point out that it is not known whether or not the affine images of the tori of revolution contain all conformal types of surfaces of genus one.

Perhaps it should be noted that the ease with which the results in [2] and specially those of this paper are obtained illustrate once more the power of the TEICHMÜLLER results on quasiconformal mappings and the usefulness of the concept of TEICHMÜLLER space for the study of families of compact RIEMANN surfaces.

The contents of this paper are divided into two sections. In the first section we shall give a proof of a theorem on uniform algebraic approximation of C^k manifolds, which is of some interest in itself. In the second section this result is combined with the results of [1) and [2], to obtain the proof of the above mentioned results on algebraic imbeddings.

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1. A theorem on uniform algebraic approximation of C^k compact manifolds of a compact family

1.1 Let Σ_ξ be a family of compact $C^k (k \geq 2)$ submanifolds imbedded[2]) in an Euclidean space E_n. Suppose that all the Σ_ξ are differentiably homeomorphic to a fixed real algebraic manifold Σ of $E_m (m \gtreqless n)$. We shall assume that the parameter ξ varies in a compact space and that Σ_ξ depends continuously on ξ. We shall express these conditions in a more precise way. Let X be a variable point in E_m and let

$$F(X) = 0$$

be the equation of Σ. Suppose that $F(X)$ is a vector valued function whose components are real polynomials in the components of X. We shall assume that the differentiable homeomorphism between Σ and Σ_ξ for each ξ is given by the vector function

$$Y = Y(X, \xi) \qquad \text{(defined for } X \in \Sigma) .$$

In other words for each ξ $Y(X, \xi)$ gives a one-to-one bicontinuous and bidifferentiable map of Σ onto Σ_ξ. In addition we shall assume that ξ varies in some compact portion C of a parameter space and that the mapping $Y(X, \xi)$ as well as its derivatives are continuous functions of (X, ξ) in $\Sigma \times C$.

Under these assumptions the following theorem holds.

Theorem. *We can approximate the submanifolds of the family* Σ_ξ *by those of another family* $\Sigma_{n, \xi}$ (where n is an integer) *defined by maps*

$$Y_n = Y_n(X, \xi)$$

satisfying for n *sufficiently large the same condition as* $Y(X, \xi)$ *but in addition:*

1. *Each component of* Y_n *is a rational function with real coefficients of the components of* X.

2. *For each* $\epsilon > 0$, *by choosing* n *large enough we can achieve that*

$$\text{2a)} \quad | Y_n(X, \xi) - Y(X, \xi) | < \epsilon$$
$$\text{2b)} \quad | dY_n - dY | \leq \epsilon \, | dX |$$

for all $(X, \xi) \in \Sigma \times C$.

Before proceeding with the proof we shall make a few observations. First of all we note that 2b) implies that $\Sigma_{n, \xi}$ approximates uniformly Σ_ξ also

[2]) It should be understood that all manifolds and surfaces referred to in this paper are to be considered deprived of self-intersections.

in the metric. In fact we can set

$$dY_n = dY + o(|dX|)$$

then

$$dY_n^2 = dY^2 + 2\,o(dX)\cdot dY + o(dX^2)\,.$$

However, the term $o(dX)\cdot dY$ will also be uniformly small in view of the compactness of Σ.

Next we should point out that while the functions $Y(X, \xi)$ need only be defined for $X \in \Sigma$, the functions $Y_n(X, \xi)$ are rational functions of the unrestricted X and therefore are defined in a neighborhood of Σ and induce an algebraic transformation of E_m into E_n.

Perhaps we should mention that results of a nature similar to some of those implied by this theorem already exist in the literature (see for instance [5], [6]); however, under our assumptions we can give a much simpler and more direct proof. To achieve this we need to introduce a few tools.

Let $A(x)$ denote a positive bell-shaped function of the real variable x C^∞ in $(-\infty, +\infty)$, satisfying the conditions

i) $A(x) = A(-x) \geq 0$
ii) $A(x) = 1$ for $0 \leq x \leq 1$
iii) $A(x) = 0$ for $3 \leq x$
iv) $\dot{A}(x) \leq 1$ for all x.

Let $A_n(x) = A(nx)$ and let $P_n(x)$ be a sequence of real polynomials such that for each n $P_n(x)$ approximates $A_n(x)$ and $\dot{P}_n(x)$ approximates $\dot{A}_n(x)$ by an error of at most e^{-n} in the whole interval $(-M, M)$. Here M is defined by

$$M = \operatorname*{Max}_{X, X' \in \Sigma} |X - X'|^2\,. \qquad (1.11)$$

We can assume without restriction that $P_n(X) \geq 0$[3]) and that $P_n(x) = P_n(-x)$.

We then define a function $K_n(X, X')$ in $E_m \times E_m$ by setting

$$K_n(X, X') = \frac{P_n(|X - X'|^2)}{\int_\Sigma P_n(|X - T|^2)\, d\Sigma_T} \qquad (1.12)$$

where by $d\Sigma_T$ we mean the area element on Σ. Clearly $K_n(X, X')$ is well defined and regular in X, X' at least when X is sufficiently close to Σ.

We shall set

$$Y_n(X, \xi) = \int_\Sigma K_n(X, X')\; Y(X', \xi)\, d\Sigma_X\,, \qquad (1.13)$$

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[3]) By replacing $P_n(x)$ by $P_n^2(x)$ if necessary.

and define $\Sigma_{n,\xi}$ to be the locus of $Y_n(X, \xi)$ as X describes Σ. Our contention is that by choosing n large enough we can satisfy all the requirements of the theorem. The rest of the proof consists of a verification that this assertion is correct.

Property 2a) is almost obvious. In fact, by the continuity of $Y(X, \xi)$ and the compactness of $\Sigma \times C$ we can achieve that

$$| Y(X_1, \xi) - Y(X_2, \xi) | < \epsilon/2 \tag{1.14}$$

holds for all $\xi \in C$ and X_1, X_2 in Σ such that

$$\overline{X_1 X_2} < \delta_\epsilon \tag{1.15}$$

for some suitable δ_ϵ.

However

$$Y_n(X, \xi) - Y(X, \xi) = \int_\Sigma K_n(X, X') \left[Y(X', \xi) - Y(X, \xi)\right] d\Sigma_{X'} ,$$

thus

$$| Y_n(X, \xi) - Y(X, \xi) | \leq \epsilon/2 + 2L \int_{\Sigma \cap (\overline{XX'} \geq \delta_\epsilon)} K_n(X, X') \, d\Sigma_{X'}$$

where we have set

$$L = \max_{(X, \xi) \in \Sigma \times C} | Y(X, \xi) | .$$

Then, in view of the properties of $K_n(X, X')$ the inequality 2a) will hold as soon as n is sufficiently large.

1.2 To prove the remaining part of the theorem we shall first introduce a covering of Σ by a finite number of open coordinate cells

$$0_i (i = 1, 2, \ldots, N)$$

with the property that as soon as $\overline{X'X''} < \delta_0$, both X' and X'' belong to the same coordinate cell. In each of these cells, say O_i, we need to have a parametrization of Σ in the form

$$X = X(U) \qquad U = (u_1, u_2, \ldots . u_\sigma) \qquad (\sigma = \text{dimension of } \Sigma)$$

where $X(U)$ gives a continuously differentiable homeomorphism of the open ball $\sum_{i=1}^{\sigma} u_i^2 < 1$ onto O_i.

We can also assume, without restriction, that all our maps $X(U)$ satisfy the inequalities

$$m \leq \frac{\overline{X(U_1) X(U_2)}}{\overline{U_1 U_2}} \leq M . \tag{1.21}$$

We shall assume that m and M have been chosen in such a way that for all

$(X, \xi) \in \Sigma \times C$ we have also

$$m\,\overline{XX'} \leq \overline{Y(X,\xi)\ \ Y(X',\xi)} \leq M\,\overline{XX'}\,. \tag{1.22}$$

This is clearly possible since $\Sigma \times C$ is compact, the maps $Y(X, \xi)$ are supposed continuously differentiable[4]) and by assumption neither Σ nor any of the Σ_ξ have selfintersections.

Suppose now that X is restricted to one of the coordinate cells. We can write then $X = X(U)$ and set

$$Y_n(U, \xi) = Y_n(X(U), \xi)\,.$$

Let $\dfrac{d}{d\varrho} = \sum\limits_{i=1}^{\sigma} a^i \dfrac{\partial}{\partial u^i}$ be a local tangent vector to Σ (defined in O_i) whose components a^i are constants. We shall denote by ϱ the column vector whose components are $a^1, a^2, \ldots, a^\sigma$. Then, using matrix notation, we can set

$$\frac{d}{d\varrho}\,Y_{n,\xi} = \int\limits_{\Sigma} Y(X',\xi)\,\nabla_X K_n(X(U),X')\,\frac{dX}{dU}\cdot\varrho\,d\Sigma_{X'}\,. \tag{1.23}$$

By $\nabla_X K_n(X, X')$ we denote a row vector whose i^{th} component is

$$\frac{\partial}{\partial x^i}\,K_n(X, X')$$

and $\dfrac{dX}{dU}$ is to denote the Jacobian matrix of the map $X = X(U)$. Observe now that

$$\nabla_X K_n(X, X') = -\,\nabla_{X'} K_n(X, X') + K_n(X, X')\int\limits_{\Sigma}\nabla_T K_n(X, T)\,d\Sigma_T\,. \tag{1.24}$$

In addition for any $\delta > 0$ we have

$$\frac{d}{d\varrho}\,Y_n = \int\limits_{\Sigma\cap\{\overline{XX'}<\delta\}} Y(X(V),\xi)\,\nabla_X K_n(X(U),X(V))\,\frac{dX(U)}{dU}\cdot\varrho\left|\frac{d\Sigma}{dV}\right|dV + o_n(1)\,|\,\varrho\,|\text{[5]}), \tag{1.25}$$

where by $\left|\dfrac{d\Sigma}{dV}\right|$ we mean the element of area in Σ in the V-coordinate system.

In view of the continuity of $\dfrac{dX}{dU}$ we can set

$$\frac{dX(U)}{dU} = \frac{dX(V)}{dV} + o_{\overline{UV}}(1) \tag{1.26}$$

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[4]) Both ways.

[5]) By $o_n(1)$ we mean a quantity which can be made uniformly small for sufficiently large n independently of $\varrho, \xi \in C$ and $U \in O_i$.

and $o_{\overline{UV}}(1)$ can be made uniformly small in each 0_i. Substituting 1.24 in 1.25 and using 1.26, after some manipulations we obtain

$$\frac{dY_n}{d\varrho} = -\int_{\Sigma\cap\{XX'\le\delta\}} Y(X(V),\xi)\, \nabla_{X'} K_n(X(U),X(V))\, \frac{dX(V)}{dV} \cdot \varrho \left|\frac{d\Sigma}{dV}\right| dV +$$

$$+\int_{\Sigma} Y(X',\xi)\, K_n(X,X')\, d\Sigma_{X'} \int_{\Sigma\cap\{X(U)X(W)\le\delta\}} \nabla_T K_n(X(U),T)\,|_{T=X(W)}\, \frac{dX(W)}{dW} \cdot$$

$$\cdot\, \varrho \left|\frac{d\Sigma}{dW}\right| dW + o_{n,\delta}(1)^{6)}\, |\,\varrho\,|\,.$$

Integrating by parts and including the new small terms in $o_{n,\delta}(1)$ we get

$$\frac{dY_n}{d\varrho} = \int_{\Sigma\cap\{\overline{X(U)X(V)}\le\delta\}} K_n(X(U),X(V))\, \frac{dY}{d\varrho}(X(V),\xi) \left|\frac{d\Sigma}{dV}\right| dV +$$

$$+\int_{\Sigma\cap\{\overline{X(U)X(V)}\le\delta\}} Y(X(V),\xi)\, K_n(X(U),X(V)) \left[\frac{d}{d\varrho}\log\left|\frac{d\Sigma}{dV}\right|\right] d\Sigma_{X(V)} +$$

$$-\int_{\Sigma} Y(X',\xi)\, K_n(X,X')\, d\Sigma_{X'} \int_{\Sigma\cap\{\overline{X(U)X(W)}\le\delta\}} K_n(X(U),X(W)) \left[\frac{d}{d\varrho}\log\left|\frac{d\Sigma}{dW}\right|\right] \cdot$$

$$\cdot\, d\Sigma_{X(W)} + o_{n,\delta}(1)\,|\,\varrho\,|\,. \tag{1.27}$$

We then easily verify that

1) $\displaystyle\int_{\Sigma\cap\{XX'\le\delta\}} K_n(X,X')\, \frac{d}{d\varrho}\, Y(X',\xi)\, d\Sigma_{X'} = \frac{d}{d\varrho}\, Y(X,\xi) + o_{n,\delta}(1)\,|\,\varrho\,|\,.$

2) $\displaystyle\int_{\Sigma\cap\{XX'\le\delta\}} Y(X',\xi)\, K_n(X,X')\, \frac{d}{d\varrho}\left[\log\left|\frac{d\Sigma}{dV}\right|\right] d\Sigma_{X'} =$

$$= Y(X,\xi)\left[\frac{d\log}{d\varrho}\left|\frac{d\Sigma}{dV}\right|\right]_{V=U} + o_{\delta,n}(1)\,|\,\varrho\,|\,.$$

3) $\displaystyle\int_{\Sigma\cap\{\overline{XX'}\le\delta\}} K_n(X(U),X(W))\left[\frac{d}{d\varrho}\left|\frac{d\Sigma}{dW}\right|\right] d\Sigma_{X(W)} = \frac{d}{d\varrho}\log\left|\frac{d\Sigma}{dW}\right|_{W=U} + o_{n,\delta}(1)|\varrho|\,.$

Combining these results with 1.27 we get

$$\frac{d}{d\varrho}\, Y_n(X,\xi) = \frac{d}{d\varrho}\, Y(X,\xi) + o_{n,\delta}(1)\,|\,\varrho\,|\,. \tag{1.28}$$

Now, 1.21 implies that

$$|\,dU\,|\,\mathrm{m} \le |\,dX\,| \le M\,|\,dU\,|$$

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[6]) Here $o_{n,\delta}(1)$ represents a quantity which can be made uniformly small for all sufficiently large n provided δ is chosen sufficiently small.

and setting $\varrho = dU, dY_n = \frac{dY_n}{dU} dU$, and $dY = \frac{dY}{dU} dU$ in 1.28 we finally obtain

$$dY_n = dY + o_{n,\delta}(1) \mid dX \mid . \tag{1.29}$$

This establishes 2b).

1.3 We are left to check whether or not for n large enough the sets $\Sigma_{n,\xi}$ are really submanifolds.

In view of 1.29 we need only check whether we can prevent selfintersections. To this end we observe that for $\overline{XX'} \geq \delta_0$[7]) by 1.22 we shall have

$$\overline{Y(X,\xi)\; Y(X',\xi)} \geq m\,\delta_0$$

for all $\xi \in C$. Thus, as soon as n is so large that

$$\overline{Y_n(X,\xi)\; Y(X,\xi)} < \frac{m\delta_0}{3}$$

we shall also have

$$\overline{Y_n(X,\xi)\; Y_n(X',\xi)} > \tfrac{1}{3}\, m\delta_0$$

for all $\overline{XX'} \geq \delta_0$ and $\xi \in C$.

If $\overline{XX'} < \delta_0$, both X and X' will belong to the same coordinate cell O_i; we can thus write $X = X(U), X' = X(V)$ and

$$Y_n(X',\xi) = Y_n(X,\xi) + \int_0^1 dY_n(X(t),\xi) \tag{1.31}$$

where the integral is to be taken along the curve

$$X(t) = X(U + t(V - U)) \qquad 0 \leq t \leq 1 .$$

Substituting 1.29 in 1.31 we have

$$Y_n(X',\xi) - Y_n(X,\xi) = Y(X',\xi) - Y(X,\xi) + \int_0^1 o_{n,\delta}(1) \mid dX \mid .$$

But $\mid dX \mid \leq M \mid dU \mid$, so we can conclude that

$$\overline{Y_n(X',\xi)\; Y_n(X,\xi)} \geq \overline{Y(X',\xi)\; Y(X,\xi)} - M \mid o_{n,\delta}(1) \mid \overline{UV},$$

Finally in view of 1.21 and 1.22

$$\overline{Y_n(X',\xi)\; Y_n(X,\xi)} \geq \left(m - \mid o_{n,\delta}(1) \mid \frac{M}{m}\right)\overline{XX'} \geq \frac{m}{2}\, \overline{XX'} .$$

as soon as $o_{n,\delta}(1)$ is made sufficiently small. This completes the proof of the theorem.

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[7]) For the definition of δ_0 see the beginning of section 1.2.

2. Existence of algebraic imbeddings

Suppose we are given a RIEMANN surface Σ_0 of genus $g \geq 1$ and we are to show that there exists an algebraic surface of E_3 which is conformally equivalent to Σ_0. We shall have to distinguish the cases $g = 1$ and $g > 1$.

2.1 The case $g = 1$. We take as canonical models for these surfaces the parallelograms

$$P_\omega = \{z : z = \lambda + \mu\omega, 1 \geq \lambda \geq 0,\ 1 \geq \mu \geq 0\} \qquad (\omega \text{ a complex number})$$

with points on opposite sides identified in the usual fashion. To fix our ideas we shall take a model P_{ω_0} conformally equivalent to Σ_0 in such a way that

$$| R_e\, \omega_0 | \leq \tfrac{1}{2}, \quad | \omega_0 | \geq 1 .$$

From the results of [1] it can be deduced that for $\delta > O$ sufficiently small, we can construct a one-paramter family of C^∞ surfaces Σ_ω of E_3 one for each ω such that $| \omega - \omega_0 | \leq \delta$, and each surface Σ_ω is a small deformation of the parallelogram model P_ω.

More precisely, we can construct a family of C^∞ vector functions $X_\omega(z)$ mapping P_ω into E_3 which are periodic in z with periods 1 and ω; in addition the metric of the surface Σ_ω described by $X_\omega(z)$ can be made so close to that of P_ω that if $P_{\omega'(\omega)}$ denotes the canonical parallelogram associated with the map $X = X_\omega(z)$ of P_ω onto Σ_ω we have

$$| \omega'(\omega) - \omega | < \delta/2 \text{ for } | \omega - \omega_0 | \leq \delta .\text{[8]} \tag{2.11}$$

Furthermore the function $X_\omega(z)$, its derivatives and $\omega'(\omega)$ are continuous n $\{ | \omega - \omega_0 | \leq \delta \} \times P_\omega$ and $| \omega - \omega_0 | \leq \delta$ respectively.

For our purposes it will be necessary to alter the maps $X_\omega(z)$. This will be done as follows. We define $\zeta = \pi X$ as the map of the familiar torus Σ whose equations are

$$X(u, v) = \begin{Bmatrix} (2 + \cos 2\pi u) \cos 2\pi v \\ (2 + \cos 2\pi u) \sin 2\pi v \\ \sin 2\pi u \end{Bmatrix} \tag{2.12}$$

into the parallelogram model P_i defined by setting

$$\pi X(u, v) = u + iv .$$

We then let v_ω be the affine transformation which deforms P_i into P_ω and sends the sides 1 and i of P_i onto the sides 1 and ω of P_ω. We finally define

$$Y(X, \omega) = X_\omega(v_\omega \cdot \pi X)$$

[8]) Actually in the construction given in [1] the maps $X_\omega(z)$ are not isometries only on regions of P_ω of arbitrarily small area. Furthermore in these regions the distortion in the metric is kept uniformly (with respect to ω) bounded.

as a map of Σ onto Σ_ω. It is easy to see that the maps $Y(X, \omega)$ defined above have the properties required by the theorem on algebraic approximations. Here, ξ is to be replaced by ω and C by the set $|\omega - \omega_0| \leq \delta$.

Consequently we can find a family of algebraic surfaces Σ_ω defined by the maps $Y = Y^n(X, \omega)$ whose metrics are arbitrarily close to the metrics of the corresponding surfaces Σ_ω. If we denote by $P_{\omega^n(\omega)}$ the canonical parallelogram model associated with the map $Y^n(X, \omega)$ of Σ onto Σ^n_ω, in view of the deformation lemma established in [1] we shall have that

i) $\omega_n(\omega)$ is continuous in ω for $|\omega - \omega_0| \leq \delta$.

ii) By choosing n large enough we can assure that

$$|\omega^n(\omega) - \omega'(\omega)| < \delta/2 . \tag{2.13}$$

Combining 2.11 with 2.13 we have that

$$|\omega^n(\omega) - \omega| < \delta \text{ for } |\omega - \omega_0| \leq \delta .$$

Thus, by continuity we can be sure that there exists at least one ω_0' such that

$$\omega^n(\omega_0') = \omega_0 .$$

This establishes our result in the case $g = 1$.

2.2 The case $g > 1$. We start by choosing a fixed imbedded C^∞ surface Σ_1 of genus g, and let $X = X_1(u, v)$ be a parametrization of Σ_1, which is given by the Poincaré uniformization of Σ_1. To be specific, let U denote the open unit disk in the $w = u + iv$ plane. We have a Fuchsian group G such that U/G is a Poincaré model conformally equivalent to Σ, We then choose a fundamental region π for the group G and let $X = X(w)$ be the conformal map of U onto Σ_1 which uniformizes the universal covering of Σ_1. We obtain then all of Σ_1 as an image of π under $X(w)$.

We represent every element of Teichmüller space T_g for genus g, as a couple (Σ, τ) of a Riemann surface of genus g and a topological map τ of Σ_1 onto Σ. We recall that the couples $(\Sigma_a, \tau_a), (\Sigma_b, \tau_b)$ such that Σ_a can be mapped conformally onto Σ_b in the homotopy class of $\tau_b^{-1}\tau_a$ are to be identified.

We then choose a basis

$$\Phi_1(w), \Phi_2(w). \dots, \Phi_N(w) \qquad (N = 3g - 3)$$

for the quadratic differentials of Σ_1, and as customary we introduce in T_g the Teichmüller coordinates corresponding to Σ_1 and this choice of

$$\Phi_1, \Phi_2, \dots, \Phi_N .$$

This is done as follows.

We represent each point ξ of the open ball B in $2N$-dimensional EUCLIDEAN space in the form

$$\xi = (\xi_1, \xi_2, \ldots, \xi_N)$$

where the ξ_i's are complex numbers satisfying the conditions

$$|\xi|^2 = \sum_{i=1}^{N} |\xi_i|^2 < 1 .$$

For each $\xi \in B$ we set

$$\Phi_\xi(w) = \sum_{i=1}^{N} \left| \frac{\xi i}{\xi} \right| \Phi_i(w)$$

and

$$ds_\xi^2 = |\Phi_\xi| \; |dw + |\xi| \; (\overline{\Phi}_\xi / |\Phi_\xi|) \overline{dw}|^2 .$$

We then use ds_ξ^2 as a new conformal metric on Σ_1 to define a RIEMANN surface $\Sigma_\xi = \Sigma_1(ds_\xi^2)$ for each $\xi \in B$.

It is a consequence of TEICHMÜLLER theorem that for each couple (Σ, τ) there exists a unique ξ such that (Σ, τ) is equivalent to (Σ_ξ, I): where I is the identity map in Σ_1, or better, the projection $\Sigma_1(ds_\xi^2) \to \Sigma_1$. We take the models $\Sigma_\xi = \Sigma_1(ds_\xi^2)$ as canonical models for the elements of T_g.

Now, let two numbers ϱ and δ such that $0 < \delta < \varrho < 1$ be given and set

$$B_\varrho = \{\xi : |\xi| \leq \varrho\} .$$

In [2], for any such choice of ϱ and δ we have presented a method of constructing a family $X(w, \xi)$ of C^∞ deformations of Σ_1 which has the following properties:

i) The functions $X(w, \xi)$ for each $\xi \in B_\varrho$, as w varies in π, describe a regularly imbedded C^∞ surface Ω_ξ.

ii) The vector $X(w, \xi)$ and its w-derivatives are continuous functions of (w, ξ) in $U \times B_\varrho$.

iii) If we denote by $\Sigma_{\xi'(\xi)}$ the canonical model corresponding to the couple (Ω_ξ, τ), where τ is the map defined by letting $X(w)$ be the image of $X(w, \xi)$, the function $\xi'(\xi)$ is continuous in B_ϱ.

iv) Finally for each $\xi \in B_\varrho$ we have

$$|\xi'(\xi) - \xi| < \delta . \tag{2.21}$$

We also have that if ds_a^2 and ds_b^2 are two metrics on Σ_1, and we set

$$k = \max_{\Sigma_1} \sqrt{\frac{\max ds_b^2 / ds_a^2}{\min ds_b^2 / ds_a^2}}$$

then the extremal quasiconformal map of $\Sigma_1(ds_a^2)$ onto $\Sigma_1(ds_b^2)$ has a dilatation which satisfies the inequality

$$k_0 \leq k .$$

By known results (cfr. [7]) on TEICHMÜLLER coordinates, we can assure that if Σ_{ξ_a} and Σ_{ξ_b} are the canonical models corresponding to $\Sigma_1(ds_a^2)$ and $\Sigma_1(ds_b^2)$, then we shall have

$$|\xi_a - \xi_b| < \delta \tag{2.22}$$

provided ξ_a is kept in a compact portion of B, say B_ϱ, and

$$k \leq 1 + \epsilon(\delta) \tag{2.23}$$

for some sufficiently small $\epsilon(\delta)$.

2.3 With these premises, we pick a fixed algebraic surface Σ of genus g and let $w(Z) = w$ be the equation of a twice continuously differentiable map[9]) of Σ onto Σ_1. We then set

$$Y = Y(Z, \xi) = X(\nu(Z), \xi) , \tag{2.31}$$

and apply the theorem of section 1 to the family of surfaces Ω_ξ and the family of maps in 2.31. Here C is to be the set B_ϱ.

We will then have a family of algebraic surfaces Ω_ξ^n described by the functions

$$Y = Y^n(Z, \xi) .$$

In view of the possibility of arbitrarily accurate uniform metric approximation of Ω_ξ by Ω_ξ^n it will be possible to select n in such a way that

$$\max_{\Sigma} \sqrt{\frac{\max (d Y^n)^2/(d Y)^2}{\min (d Y^n)^2/(d Y)^2}} \leq 1 + \epsilon(\delta) . \tag{2.32}$$

In addition, since $Y^n(Z, \xi)$ and its Z-derivatives will be continuous in $\pi \times B_\varrho$, we shall have that if $\Sigma_{\xi''(\xi)}$ is the canonical model corresponding to Ω_ξ^n:

1. The function $\xi''(\xi)$ will be continuous in B_ϱ.
2. By 2.32 and 2.22 it will follow that

$$|\xi''(\xi) - \xi'(\xi)| < \delta . \tag{2.33}$$

Combining this inequality with 2.21 we deduce then

$$|\xi''(\xi) - \xi| < 2\delta \text{ for all } \xi \in B_\varrho . \tag{2.34}$$

[9]) Both ways.

Let now Σ_{ξ_0} be a canonical model conformally equivalent to the given surface Σ_0 to be imbedded, and suppose that ϱ and δ have been chosen in such a way that

$$|\xi_0| < \varrho - 3\delta$$

We shall have then the following situation

a) The function $\xi'' = \xi''(\xi)$ is continuous in the sphere $|\xi - \xi_0| \leq 3\delta$
b) For all $|\xi - \xi_0| < 3\delta$ we have $|\xi''(\xi) - \xi| < 2\delta$.

These two conditions assure the existence of a point η_0 in the sphere

$$|\xi - \xi_0| \leq 3\delta$$

such that

$$\xi''(\eta_0) = \xi_0 .$$

This establishes our result in the case $g > 1$.

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