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An Integral Equation and a General Existence Theorem for Harmonic Functions*)

by Leo Sario

The purpose of the present paper is to show that a Fredholm integral equation can be used to construct on an arbitrary Riemann surface W a harmonic function p that has preassigned singularities and that imitates the behavior of a given harmonic function σ in a neighborhood of the ideal boundary. Explicit upper and lower bounds will be given for the deviation of p from σ. Such bounds will be needed e.g. in the theory of complex analytic mappings into arbitrary Riemann surfaces [17, 18].

§ 1. Auxiliary functions

1. Let W be a compact bordered Riemann surface with border $\alpha \cup \beta_1 \cup \ldots \cup \beta_n$, where α and the β's are disjoint sets of analytic Jordan curves. Choose a point $z_0 \in W$ and a parametric disk D: $r < 1$, $r = |z - z_0|$. Consider the class G of harmonic functions g in $W - z_0$ such that

$$g \mid D = \frac{1}{2\pi} \log \frac{1}{r} + e(z)\,, \tag{1}$$

$$g \mid \alpha = 0\,, \tag{2}$$

$$\int_{\beta_j} dg^* = 0, \quad j = 1, \ldots, n, \tag{3}$$

where e is harmonic in D.

In the class G we single out the functions g_0, g_1 and g_{hk} defined by the conditions

$$\frac{\partial g_0}{\partial n} = 0 \text{ on } \beta = \bigcup_1^n \beta_j\,, \tag{4}$$

$$g_1 \mid \beta_j = c_j \text{ (const.)}, \; j = 1, \ldots, n, \tag{5}$$

$$g_{hk} = h\, g_0 + k\, g_1\,, \tag{6}$$

h, k being real constants with $h + k = 1$.

We shall first establish an extremal property of g_{hk} by evaluating the Dirichlet integral $D(g - g_{hk})$ over W. Denote by a, a_0, a_1 the values

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$e(z_0)$ for g, g_0, g_1 respectively and set $B(g) = \int_\beta g\, dg^*$, $B(g, \hat{g}) = \int_\beta g\, d\hat{g}^*$ for $g, \hat{g} \in G$.

Lemma 1. *The function g_{hk} minimizes $B(g) + (k - h)a$ in G. Explicitly,*

$$B(g) + (k - h)a = k^2 a_1 - h^2 a_0 + D(g - g_{hk}) . \tag{7}$$

Proof. We start with

$$D(g - g_{hk}) = B(g) + B(g_{hk}) - B(g, g_{hk}) - B(g_{hk}, g) .$$

To evaluate

$$B(g_{hk}) = h\, k\, [B(g_0, g_1) - B(g_1, g_0)]$$

we take a circle γ: $|z - z_0| = r_0 < 1$ and write $C(g, \hat{g}) = \int_\gamma g\, d\hat{g}^*$ for $g, \hat{g} \in G$. Then

$$B(g_{hk}) = h\, k\, [C(g_0, g_1) - C(g_1, g_0)] .$$

Here we have oriented β and γ so as to leave W and $|z - z_0| < r_0$ to the left, respectively. On denoting $(1/2\pi) \log (1/r)$ by ϱ we obtain

$$\begin{aligned} B(g_{hk}) &= hk\, [C(\varrho + e_0, \varrho + e_1) - C(\varrho + e_1, \varrho + e_0)] \\ &= hk\, [C(e_0, \varrho) - C(e_1, \varrho)] \\ &= hk\, [e_1(z_0) - e_0(z_0)] = hk(a_1 - a_0) . \end{aligned}$$

In the same manner we derive the equations

$$\begin{aligned} B(g, g_{hk}) &= k(a_1 - a) , \\ B(g_{hk}, g) &= h(a - a_0) , \end{aligned}$$

and conclude that (7) holds.

2. Now let W be a noncompact bordered Riemann surface with compact border α, and denote by Ω a compact bordered subregion with border $\alpha \cup \beta_{1\Omega} \cup \ldots \cup \beta_{n_\Omega \Omega}$. Here α and the β's are again disjoint sets of analytic Jordan curves, the border of any component of $W - \Omega$ being contained in one set only. For the various Ω, the $\beta_{j\Omega}$, $j = 1, \ldots, n_\Omega$, are required to form a consistent system of partitions ([1], p. 88). Specifically, for $\bar{\Omega} \subset \Omega'$, the partition of $\beta_{\Omega'}$ must be a refinement of the partition induced by that of β_Ω. The most important systems are the identity partition and the canonical partition.

Let G be the class of harmonic functions g on $W - z_0$ defined by the conditions (1), (2) and $\int_{\beta_{j\Omega}} dg^* = 0$ for all Ω and j. The functions $g_{0\Omega}$, $g_{1\Omega}$, and $g_{hk\Omega}$ in $\Omega - z_0$ are defined by obvious modifications of conditions (4)–(6). We shall show that these functions converge uniformly in compact subsets to unique harmonic functions g_0, g_1, g_{hk} on $W - z_0$ as $\Omega \to W$.

3. Let $\bar{\Omega} \subset \Omega'$ and apply (7) to $g = g_{0\Omega'}$ and $g_{hk} = g_{0\Omega}$. Since $B_\Omega(g_{i\Omega'}) \leqq B_{\Omega'}(g_{i\Omega'}) = 0$ for $i = 0, 1$, we obtain

$$D_\Omega(g_{0\Omega'} - g_{0\Omega}) \leqq a_{0\Omega} - a_{0\Omega'} .$$

Similarly

$$D_\Omega(g_{1\Omega'} - g_{1\Omega}) \leqq a_{1\Omega'} - a_{1\Omega} ,$$

and

$$D_\Omega(g_{0\Omega} - g_{1\Omega}) \leqq a_{0\Omega} - a_{1\Omega} .$$

We conclude that the directed limits

$$a_i = \lim_{\Omega \to W} a_{i\Omega} , \tag{8}$$

$i = 0, 1$, exist. Consequently

$$\lim_{\Omega \to W} D_\Omega(g_{i\Omega'} - g_{i\Omega}) = 0 . \tag{9}$$

We reflect W, Ω, Ω' across α to form the doubles $\hat{W}$, $\hat{\Omega}$, $\hat{\Omega}'$ ([1], p. 119) and infer that the corresponding integral $D_{\hat{\Omega}}$ of the harmonic extension $\hat{g}_{i\Omega'} - \hat{g}_{i\Omega}$ to $\hat{\Omega}$ of $g_{i\Omega'} - g_{i\Omega}$ also tends to zero. By virtue of the triangle inequality, $D_{\hat{\Omega}_0}(\hat{g}_{i\Omega} - \hat{g}_{i\Omega'}) \to 0$ for any fixed Ω_0 as Ω, Ω' independently of each other exhaust W. Since $\hat{g}_{i\Omega} = 0$ on α, one infers ([1], p. 147) that $\hat{g}_{i\Omega}$ converges uniformly on compact subsets of $\hat{W} - z_0 - z_0^*$ (z_0^* the reflection of z_0), hence on those of $W - z_0$. We obtain the limiting functions g_0, g_1, and

$$g_{hk} = \lim_{\Omega \to W} g_{hk\Omega} = h\, g_0 + k\, g_1 . \tag{10}$$

One can show that (7) remains valid on the noncompact W but we shall not need this information.

§ 2. Linear operators

4. Let again W be a compact bordered RIEMANN surface with border $\alpha \cup \beta_1 \cup \ldots \cup \beta_n$. Let f be a harmonic function on α and consider the class H of harmonic functions u on W with

$$u \mid \alpha = f , \tag{11}$$

$$\int_{\beta_j} du^* = 0 , \tag{12}$$

$j = 1, \ldots, n$. In H take the functions u_0, u_1, u_{hk} defined by

$$\frac{\partial u_0}{\partial n} = 0 \quad \text{on} \quad \beta = \bigcup_1^n \beta_j , \tag{13}$$

$$u_1 \mid \beta_j = c_j \text{ (const.)}, \tag{14}$$

$$u_{hk} = h\,u_0 + k\,u_1\,. \tag{15}$$

Let $A(u) = \int\limits_\alpha u\,du^*$, $B(u) = \int\limits_\beta u\,du^*$.

Lemma 2. *The function* u_{hk} *minimizes the functional* $B(u) + (k-h)\,A(u)$ *in* H *and the value of the minimum is* $k^2 A(u_1) - h^2 A(u_0)$:

$$B(u) + (k-h)\,A(u) = k^2 A(u_1) - h^2 A(u_0) + D(u - u_{hk})\,. \tag{16}$$

The proof is completely analogous to that of Lemma 1, with $g_0, g_1, g_{hk}, a_0, a_1, a$ replaced by $u_0, u_1, u_{hk}, A(u_0), A(u_1), A(u)$.

5. If W is noncompact with compact border α, one again considers a consistent system of partitions of the boundaries β_Ω. The family H consists, by definition, of harmonic functions u on W with $u = f$ on α and $\int\limits_{\beta_{j\Omega}} du^* = 0$ for all Ω, j. In the same manner as in No. 3 one proves that $A(u_{0\Omega})$ decreases, $A(u_{1\Omega})$ increases with increasing Ω and that the directed limits

$$A(u_i) = \lim_{\Omega \to W} A(u_{i\Omega})\,,$$

$i = 0, 1$, exist. The integrals $D_\Omega(u_{i\Omega'} - u_{i\Omega})$ tend to zero and we have the limiting functions u_0, u_1 and

$$u_{hk} = \lim_{\Omega \to W} u_{hk\Omega} = h\,u_0 + k\,u_1\,. \tag{17}$$

Lemma 2 can be shown to remain valid for the limiting functions, but this property will not be needed.

6. The operators L_{hk} are defined by

$$u_{hk} = L_{hk} f\,, \tag{18}$$

with L_0, L_1 standing for $L_{1,0}, L_{0,1}$ respectively. They satisfy the conditions

$$L_{hk} f \mid \alpha = f\,, \tag{19}$$

$$L_{hk}(c_1 f_1 + c_2 f_2) = c_1 L_{hk} f_1 + c_2 L_{hk} f_2\,, \tag{20}$$

$$\int\limits_\alpha d\,(L_{hk} f)^* = 0\,, \tag{21}$$

$$\min f \leqq L_{hk} f \leqq \max f \tag{22}$$

for compact W. By virtue of uniform convergence the same is true for noncompact W.

The operator L_{hk} has an integral representation in terms of g_{hk}. If W is bordered compact, we apply GREEN's formula to u_i, g_i along $\beta - \alpha - \gamma$, and let γ shrink to z_0. The resulting equations $u_i(z_0) = \int\limits_\alpha f\,dg_i^*$ give

$$u_{hk}(z_0) = \int_\alpha f\,dg_{hk}^*\,. \tag{23}$$

Because of uniform convergence of both $u_{hk\Omega}$ and $g_{hk\Omega}$ on a noncompact W we have:

Lemma 3. *On an arbitrary bordered* RIEMANN *surface with compact border* α,

$$L_{hk} f(z_0) = \int_\alpha f \, dg^*_{hk} , \tag{24}$$

where α *is oriented so as to leave* W *to its right.*

7. The following is a general property of harmonic functions:

Lemma 4. *Let* E *be a compact set on an arbitrary open* RIEMANN *surface* W. *Consider the class of harmonic functions* u *on* W *with*

$$\min_E u \leqq 0, \quad \sup_W u \geqq 0 . \tag{25}$$

Then there exists a constant q, $0 < q < 1$, *independent of* u, *such that*

$$u \mid E \leqq q \sup_W u . \tag{26}$$

In the more restricted class of functions u with $sgn\, u \mid E \neq$ const. the lemma implies

$$q \inf_W u \leqq u \mid E \leqq q \sup_W u \tag{27}$$

as can be seen by applying (26) to $-u$. Less sharply, $max_E \mid u \mid \leqq q \sup_E \mid u \mid$.

Proof. If $\sup_W u = 0$ or ∞, there is nothing to prove. In other cases we normalize by a multiplicative constant so as to make $\sup_W u = 1$. For $v = 1 - u$ we have $\inf_W v = 0$, $\max_E v \geqq 1$, hence $v > 0$ on W. We are to establish the existence of a $q' \in (0, 1)$ such that $\min_E v \geqq q'$.

Without loss of generality we may assume that E is connected, for if this is not the case we first replace E by a larger connected compact set.

Cover E by a finite number of disks D_i, $i = 1, \ldots, N$, centered at z_i, such that slightly smaller disks $D'_i = \mid z - z_i \mid < 1 - d$, $d > 0$, already cover E. By HARNACK's inequality, $v(z)/v(z_i)$ for $z \in D'_i$ is in the interval (c, c^{-1}), where $c = d/(2 - d)$. For any $z, z' \in D'_i$ we have $v(z)/v(z') \in (c^2, c^{-2})$. There is a point $\zeta \in E$ where $v(\zeta) \geqq 1$ and this point can be connected with any $z \in E$ by a sequence of points

$$\zeta = z_0, z_1, \ldots, z_m = z,\ m \leqq N, \text{ the pair } z_{j-1}, z_j (j = 1, \ldots, m)$$

being in some disk D'_i. Consequently c^{2N} qualifies as q'.

8. On an arbitrary open RIEMANN surface W let $\overline{W}_0$ be a bordered compact subregion with border α_0. Let $\overline{W}_1$ with border $\alpha_1 \subset W_0$ be the complement of a relatively compact bordered subregion of W. For a real-valued function $f \in C$ on α_0 let Lf be the solution of the DIRICHLET problem in $\overline{W}_0$. The operator

L_{hk} acting on $Lf \mid \alpha_1$ gives on W_1 the harmonic function Kf with

$$K = L_{hk} L . \tag{28}$$

The *n th* iterate of K is denoted by K^n.

Let q be the constant of Lemma 4 applied to the compact set α_1 on the RIEMANN surface W_0.

Lemma 5. *If* sgn $K^i f \mid \alpha_1 \neq$ *const.*, $i = 1, \ldots, n$, then

$$q^n \min f \leqq K^n f \mid \alpha_0 \leqq q^n \max f . \tag{29}$$

For $n = 1$ this follows from (27) and (22). For each iteration we obtain another factor q.

§ 3. An integral equation

9. Orient α_0 and α_1 so as to leave $W_0 \cap W_1$ to the left and right respectively. In $\overline{W}_0 \cap \overline{W}_1$ let ω be the harmonic function with conjugate $\omega^* = s$ such that $\omega \mid \alpha_1 = 0$, $\omega \mid \alpha_0 =$ const., $\int_{\alpha_1} ds = 1$. Choose branches of s with the property that the curves

$$\alpha_0 : z = t_0(s), \quad \alpha_1 : z = t_1(s) \tag{30}$$

are traced as s goes from 0 to 1. Let $\partial/\partial n$ denote the normal derivative on α_0 and α_1 interior to $W_0 \cap W_1$.

Designate by $g(z, \zeta)$ the GREEN's function on W_0 with the singularity. at ζ. Given harmonic functions f_0, f_1 on α_0, α_1 respectively, the L-operators have the following integral representations:

$$Lf_0 \mid t_1(s) = \int_0^1 f_0(t_0(y)) \frac{\partial g(t_0(y), t_1(s))}{\partial n} dy , \tag{31}$$

$$L_{hk} f_1 \mid t_0(x) = \int_0^1 f_1(t_1(s)) \frac{\partial g_{hk}(t_1(s), t_0(x))}{\partial n} ds . \tag{32}$$

We introduce the kernel

$$K(x, y) = \int_0^1 \frac{\partial g(t_0(y), t_1(s))}{\partial n} \frac{\partial g_{hk}(t_1(s), t_0(x))}{\partial n} ds \tag{33}$$

and have

$$K f_0 \mid t_0(x) = \int_0^1 K(x, y) f_0(t_0(y)) dy . \tag{34}$$

10. Given a harmonic function σ on $\overline{W}_1$ and an operator L_{hk}, we wish to construct on W a harmonic function p such that $p - \sigma = L_{hk}(p - \sigma)$ on W_1.

Here W could be an open RIEMANN surface R_0 punctured at a finite number of points z_j, and W_1 could consist of neighborhoods D_j of the z_j and a neighborhood D_β of the ideal boundary β of R_0. The harmonic function σ in D_j may have an isolated singularity at z_j, and on D_β it may be an arbitrarily behaving function as one approaches β. Thus we are dealing with the problem of constructing on R_0 a harmonic function with given singularities and a prescribed behavior near the ideal boundary.

We may assume that $\sigma \mid \alpha_1 = 0$, for otherwise we can replace σ by $\sigma - L_{hk}\sigma$. It is necessary that the flux $\int_{\alpha_1} d\sigma^*$ vanish, for so do $\int_{\alpha_1} dp^*$ and $\int_{\alpha_1} d(L_{hk}(p-\sigma))^*$.

Our problem will be solved if we can find $p \mid \alpha_0$, for then $p \mid W_0 = Lp$ and $p \mid W_1 = \sigma + L_{hk} p$. On α_0 we have, by $L_{hk} L = K$,

$$p = \sigma + Kp \tag{35}$$

or, more precisely,

$$p(t_0(x)) = \sigma(t_0(x)) + \int_0^1 K(x, y)\, p(t_0(y))\, dy\,. \tag{36}$$

Thus we are dealing with a FREDHOLM integral equation. It is known that its solution is $p = \Sigma_0^\infty K^n \sigma$ provided the series converges uniformly. In fact, the K-operator can then be applied term by term and gives $Kp = \Sigma_1^\infty K^n \sigma = p - \sigma$.

11. For the convergence proof we first observe that, by virtue of GREEN's formula,

$$\int_{\alpha_0} u\, ds = \int_{\alpha_1} u\, ds \tag{37}$$

for any harmonic function on $\overline{W}_0 \cap \overline{W}_1$ with $\int_{\alpha_1} du^* = 0$. The functions $\sigma, Lf_0, L_{hk}f_1$, and Kf_0 qualify as u.

It is easy to see that

$$\int_{\alpha_1} K^i \sigma\, ds = 0 \tag{38}$$

for all $i \geqq 0$. In fact, for $i = 0$ this is so by assumption. Suppose then (38) holds for $i = m - 1$. Then $\int_{\alpha_0} K^{m-1} \sigma\, ds = 0$. Here the integrand can be replaced by L acting on it, whence $\int_{\alpha_1} LK^{m-1} \sigma\, ds = 0$. The operator L_{hk} can now be applied to the integrand, and (38) follows for $i = m$.

We conclude that $K^i\sigma$, $i = 1, 2, \ldots$, is not of constant sign on α_1. Lemma 5 gives to $p \mid \alpha_0 = \Sigma_0^\infty K^n \sigma \mid \alpha_0$ the upper bound $\max_{\alpha_0} \sigma/(1-q)$ and the lower bound $\min_{\alpha_0} \sigma/(1-q)$. By the maximum-minimum principle the same bounds hold for $p \mid W_0$, hence for $p \mid \alpha_1$ and $p - \sigma \mid \alpha_1$ and a fortiori for $p - \sigma$ on all of W_1.

We have established the following result:

Theorem. *The solution* $p_{hk} \mid \alpha_0 = \Sigma_0^\infty K^n \sigma$ *of the* FREDHOLM *integral equation* (36) *gives a harmonic function* p_{hk} *on* W *such that* $p - \sigma = L_{hk}\sigma$ *on* W_1 *and*

$$\frac{\min_{\alpha_0} \sigma}{1-q} \leqq p_{hk} \mid W_0 \leqq \frac{\max_{\alpha_0} \sigma}{1-q} \tag{39}$$

$$\frac{\min_{\alpha_0} \sigma}{1-q} \leqq p_{hk} - \sigma \mid W_1 \leqq \frac{\max_{\alpha_0} \sigma}{1-q}. \tag{40}$$

12. For L_0 and L_1 the existence of principal functions p_0, p_1 has been known [10, 1] and has applications to conformal mapping, extremum problems, capacity functions, classification theory, kernel functions, and value distribution theory (see Bibliography). The relation to integral equations is new and so are the bounds (39), (40). These bounds only depend on $\min_{\alpha_0} \sigma$ and $\max_{\alpha_0} \sigma$, not on σ otherwise. This makes it possible to give bounds simultaneously for uniformly bounded families of functions σ. In essence, if $\sigma \mid \alpha_0$ is $O(1)$, then so are $p \mid W_0$ and $p - \sigma$, all uniformly. Such bounds are needed e.g. in general value distribution theory [17, 18].

The construction of principal functions p_{hk} can also be carried out in locally Euclidian spaces. This leads to extremum, capacity, and classification problems in higher dimensions [19].

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