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By PETER LAPPAN, Michigan State University

1. Let D and C denote the unit disk and unit circle, respectively, let f denote a complex-valued function defined in D, and let W denote the extended complex plane. By a path γ in D we mean the image of the interval $0 \leqq x < 1$ under a continuous function g. A path γ is called an *asymptotic path* if (1) $|g(t)| \to 1$ as $t \to 1$ and (2) there exists a number $w \in W$ such that $f(g(t)) \to w$ as $t \to 1$. The number w is called the *asymptotic value* for the asymptotic path γ. If γ is a path, the set $C \cap \bar{\gamma}$ is called the end of γ, and we say that γ ends in $C \cap \bar{\gamma}$. It is clear that the end of an asymptotic path must be either a point or an arc of C. A path γ is called a *point asymptotic path* if γ is an asymptotic path which ends in a single point.

The following sets will be considered:

(i) $C_\gamma(f, \zeta)$, where ζ is a point of C and γ is a path which ends in ζ, denotes the set $\{w \in W$: there exists a sequence of points $\{z_n\}$ in γ such that $|z_n| \to 1$ and $f(z_n) \to w\}$;

(ii) $\prod(f, \zeta)$, where ζ is a point of C, denotes the intersection $\bigcap_\gamma C_\gamma(f, \zeta)$, where the intersection is taken over all paths γ which end at ζ;

(iii) $\prod_\infty(f) = \{\zeta \in C\colon \infty \in \prod(f, \zeta)\}$;

(iv) $\Gamma(f) = \{w \in W$: there exists an asymptotic path γ for which the corresponding asymptotic value is $w\}$;

(v) $A(f) = \{\zeta \in C$: there exists an asymptotic path γ for which the end contains the point $\zeta\}$;

(vi) $A_P(f) = \{\zeta \in C$: there exists a point asymptotic path for which the end is $\zeta\}$.

The sets $\Gamma(f)$, $A(f)$, and $A_P(f)$ have been studied by many persons, with two of the more complete treatments being given by Collingwood and Cartwright [3] and MacLane [5]. The main focus of this paper will be results concerning $\prod_\infty(f)$. We first prove that if f is a holomorphic function then $\prod_\infty(f) - \overline{A(f)}$ is an open subset of C. Next, it is shown that if f is a continuous function in the extended sense, then $\prod_\infty(f)$ is a measurable subset of C. Finally, it is proved that if f is a normal holomorphic function then $\prod_\infty(f)$ is nowhere dense in C. We conclude with some unsolved questions relating to $\prod_\infty(f)$.

2. We begin by proving a lemma.

LEMMA. *Let f be a function holomorphic in D, let β be a subarc of C with endpoints at ζ_1 and ζ_2, and let $\zeta_1 \notin \prod_\infty(f)$ and $\zeta_2 \notin \prod_\infty(f)$. Then there exists an asymptotic path for which the end is a subset of β.*

Proof. Since $\zeta_1 \notin \prod_\infty(f)$ and $\zeta_2 \notin \prod_\infty(f)$, there exist paths γ_1 and γ_2 leading from

0 to ζ_1 and ζ_2, respectively, such that $\gamma_1 \cap \gamma_2 = \{0\}$ and f is bounded on $\gamma_1 \cup \gamma_2$. Let H be the region bounded by $\gamma_1 \cup \gamma_2 \cup \beta$. If f is bounded on H then, by Fatou's Theorem [6, p. 5], f has point asymptotic paths to almost every point of β. If f is unbounded in H, then there exists a point $z_0 \in H$ such that $|f(z_0)|$ is greater than the bound of $|f(z)|$ on $\gamma_1 \cup \gamma_2$. Let L be the ray described by $\{w\colon w = tf(z_0),\ t \geqq 1\}$. The component of $f^{-1}(L)$ which contains z_0 is an asymptotic path in H with its end contained in $\bar{\beta}$ (since $f^{-1}(L) \cap (\gamma_1 \cup \gamma_2) = \emptyset$). Thus the lemma is proved.

THEOREM 1. *Let f be a function holomorphic in D. Then* $\prod_\infty(f) - \overline{A(f)}$ *is an open subset of C.*

Proof. Let $\zeta \in \prod_\infty(f) - \overline{A(f)}$. Then there exists a neighborhood N of ζ such that $N \cap C \cap A(f) = \emptyset$. Suppose that ζ_1 and ζ_2 are two points of $N \cap C$ such that $\zeta_1 \notin \prod_\infty(f)$ and $\zeta_2 \notin \prod_\infty(f)$. If β is the subarc of $N \cap C$ with ζ_1 and ζ_2 as endpoints, then $A(f) \cap \bar{\beta} \neq \emptyset$ according to the Lemma. But this would violate the condition that $N \cap C \cap A(f) = \emptyset$. Thus $N \cap C$ may contain at most one point which is not in $\prod_\infty(f)$. But this means that ζ is an interior point of $\prod_\infty(f)$, and the theorem is proved.

THEOREM 2. *Let f be a function holomorphic in D. Then the complement of* $\prod_\infty(f) \cup \overline{A(f)}$ *in C is a finite (or empty) set.*

Proof. Let E be the complement of $\prod_\infty(f) \cup \overline{A(f)}$ in C. Then $C = \prod_\infty(f) \cup \overline{A(f)} \cup E$. If $\zeta \in E$, there exists a neighborhood N of ζ such that $N \cap C \cap A(f) = \emptyset$. Suppose there exists a point $\zeta' \in E \cap N \cap C$, $\zeta' \neq \zeta$. Then by the Lemma we have $A(f) \cap N \cap C \neq \emptyset$, in violation of the choice of N. Thus ζ must be an isolated point of E. Therefore, each point of E must be an isolated point, and E is a finite set.

THEOREM 3. *If f is a holomorphic function in D, then* $\prod_\infty(f) \cup A(f)$ *is a dense subset of C.*

Theorem 3 is an immediate consequence of Theorem 2.

We note that Theorem 3 need not be true when f is a meromorphic function in D, as is illustrated by the Schwarz triangle functions, for which both $\prod_\infty(f)$ and $A(f)$ are empty.

3. We now let f be a continuous complex-valued function in the extended sense.

THEOREM 4. *If f is a continuous function in D, then* $\prod_\infty(f)$ *is a measurable set.*

Proof. We will show that $C - \prod_\infty(f)$ is a measurable set.

If $\zeta \in C - \prod_\infty(f)$, there exists an integer n such that ζ is an accessible boundary point of $A(n) = \{z \in D\colon |f(z)| < n\}$. For each n, $A(n)$ has a finite or a countable number of components $\{A(n, i)\colon i = 1, 2, 3, \ldots\}$.

Let $B(n) = \{z \in D\colon |f(z)| > n\}$ and let $\{B(n, j)\colon j = 1, 2, 3, \ldots\}$ be the components of $B(n)$. Let $E(n, i, j)$ be the set of points of $C \cap \overline{B(n, j)}$ which are accessible from within

$A(n, i)$. Let $K(n, j)$ be the set of all points in $C-\overline{B(n, j)}$. By a result of Kaczynski [4, Lemma 1, p. 590], $E(n, i, j)$ contains at most two points, so that $E(n)=\bigcup_{i,j}E(n, i, j)$ is a countable subset of C. But $K(n, j)$ is an open subset of C, and $K(n)=\bigcap_j K(n, j)$ is a G_δ set. Then $E(n)\cup K(n)$ is a measurable set. But

$$C-\prod_\infty(f)=\bigcup_n[E(n)\cup K(n)]$$

and thus $C-\prod_\infty(f)$ is a measurable set, and therefore $\prod_\infty(f)$ is also a measurable subset of C.

We have already noted that $\prod_\infty(f)$ may be empty and thus have measure zero, where f is a meromorphic function. Likewise, f may be holomorphic and $\prod_\infty(f)$ may equal C, as in the case of annular functions in the sense of Bagemihl and Erdös [1].

4. We now consider the case where f is a normal holomorphic function.

THEOREM 5. *If f is a normal holomorphic function in D, then $\prod_\infty(f)$ is nowhere dense in C.*

Proof. Suppose there exists an arc α of C such that $\alpha\subset\prod_\infty(f)$. Let β be a subarc of α with endpoints ζ_1 and ζ_2 in the interior of α. Let S_1 and S_2 be the radii to ζ_1 and ζ_2, respectively, and let H be the sector of D bounded by $S_1\cup S_2\cup\beta$. For each n, let $D_n=\{z\in\bar{H}\cap D\colon |f(z)|<|f(0)|+n\}$, and let F_n be the component of D_n which contains 0. Since $\beta\subset\prod_\infty(f)$, we must have $\bar{F}_n\cap C=\emptyset$ for each n. However, $H\subset\bigcup_n F_n$. For each n, the boundary of F_n contains a component which meets both S_1 and S_2. Thus for each n there exists a Jordan arc J_n leading from a point on S_1 to a point on S_2 such that $|f(z)|>n-1$ for $z\in J_n$, and $J_n\subset F_n$. The sequence $\{J_n\}$ forms a Koebe sequence of arcs relative to β such that $f(z)\to\infty$ along $\{J_n\}$. By a result of Bagemihl and Seidel [2, Theorem 1, p. 10], f must be identically ∞ and hence not holomorphic. Thus the theorem is proved.

We remark that Theorem 5 remains true when f is a normal meromorphic function. To prove this, we need only to modify the proof above by choosing S_1 and S_2 which do not contain poles of f, and by showing that the sequence $\{J_n\}$ does not allow limit points in D. For if $\{J_n\}$ had a limit point in D, then $\{J_n\}$ would have uncountably many such limit points in D, and f would be identically ∞.

We further remark that Theorem 5 is valid if f is assumed to be of bounded characteristic, but not necessarily normal. However, an example of MacLane [5, Example 3, p. 57] shows that Theorem 5 may fail if the assumption that f is normal is removed, even though $A_P(f)$ may be dense in C.

5. The following questions concerning $\prod_\infty(f)$ are still unanswered.

QUESTION 1. *If f is a normal holomorphic function in D, can $\prod_\infty(f)$ have positive measure in C?*

QUESTION 2. *If f is a holomorphic or meromorphic function in D which is the sum of two normal functions, must $\prod_\infty(f)$ be nowhere dense in C?*

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Michigan State University

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