On a Special Class of Hamiltonian Graphs.

On a Special Class of Hamiltonian Graphs

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One of the most basic questions asked about a graph (finite, undirected, without loops or multiple edges) is whether its structure is such that it can be traversed or traced in a certain manner. Undoubtedly, the two most important classes of graphs dealing with traversability are the eulerian graphs and the hamiltonian graphs. A graph G is *eulerian* if it has a closed path (called an eulerian path) containing every edge of G exactly once and every vertex of G at least once, while G is *hamiltonian* if it has a closed path containing every vertex of G exactly once, i.e.. if it has a hamiltonian cycle.

A graph G is said to be *randomly eulerian from a vertex v* if the following procedure always results in an eulerian path. Begin at the given vertex v and traverse any incident edge. On arriving at a vertex, choose any incident edge which has not yet been traversed. When no new edges are available the procedure terminates. These graphs have also been referred to as arbitrarily traversable from v and arbitrarily traceable from v and have been investigated by BÄBBLER [1], HARARY [3], and ORE [4].

This suggests the following concept. We define a graph G to be *randomly hamiltonian from the vertex v* if the following procedure always results in a hamiltonian cycle. Begin at the vertex v and proceed to any adjacent vertex. On arriving at a vertex, select any adjacent vertex not previously encountered. When no new vertices remain, then an edge exists between the final vertex chosen and v, and the procedure terminates. Thus in a graph G which is randomly hamiltonian from a vertex v, any path beginning at v can be extended to a hamiltonian cycle. Graphs which are randomly hamiltonian from every vertex were characterized in [2] and are called simply randomly hamiltonian graphs.

It is the object of this article to present a characterization of graphs which are randomly hamiltonian from a vertex, and thereby provide a classification of all such graphs.

It is convenient to introduce notation for several types of graphs which are encountered throughout the course of this article. The complete graph with p vertices is denoted by K_p, while C_p represents the cycle with $p \geqslant 3$ vertices. The complete bipartite graph $K(m, n)$ is the graph with $p=m+n$ vertices whose vertex set V can be partitioned as $V_1 \cup V_2$ such that $|V_1|=m$, $|V_2|=n$, and vertices u and v are adjacent if and only if $u \in V_i$ and $v \in V_j$, $i \neq j$. It was shown in [2] that a graph G with $p \geqslant 3$ vertices is randomly hamiltonian if and only if it is one of the graphs K_p, C_p, and $K(p/2, p/2)$.

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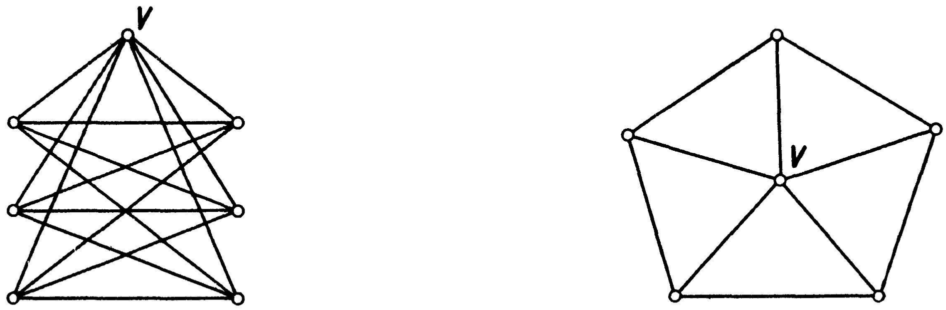
We express a graph G as $H+v$ provided v is a vertex of G adjacent to all other vertices of G, where then H is the graph obtained from G by the removal of v and all edges incident with v. For example, the graph C_n+v is often referred to as the wheel W_n. The graphs $K(3,3)+v$ and $W_5=C_5+v$ are illustrated in Figure 1. In each case, the graph is randomly hamiltonian from v.



$K(3,3)+v$ $W_5=C_5+v$

Figure 1
Two graphs which are randomly hamiltonian from the vertex v

Of course, if a graph G is randomly hamiltonian from a vertex, then G is hamiltonian and therefore has a hamiltonian cycle. Thus whenever we have a graph G with p vertices which is randomly hamiltonian from a vertex we assume the existance of a hamiltonian cycle C whose vertices are labeled cyclically $v_1, v_2, \dots v_p$. Each edge of G then either belongs to C, and is called a *cycle edge* of G, or joins two non-consecutive vertices of C and is called a *diagonal*.

If G is a graph which is randomly hamiltonian from some vertex (and which contains a hamiltonian cycle C labeled as earlier indicated), then any cycle of G containing exactly one diagonal of G is called an *outer cycle* of G. An *outer n-cycle* has length n, and an outer 3-cycle is also referred to as an *outer triangle*.

We now present the main result of the paper.

THEOREM. *A graph G is randomly hamiltonian from a vertex v if and only if G is randomly hamiltonian or $G=H+v$, where H is randomly hamiltonian.*

Proof. If G is a randomly hamiltonian graph containing a vertex v or if G is expressible as $H+v$, where H is randomly hamiltonian, then it is easily observed that G is randomly hamiltonian from v.

Conversely, let G be a graph with p vertices which is randomly hamiltonian from the vertex v. Thus G contains a hamiltonian cycle V whose vertices we label cyclically as $v=v_1, v_2, \dots, v_p$.

Suppose that G is not randomly hamiltonian so that G is none of the graphs K_p, C_p, $K(p/2, p/2)$. In particular, this implies that G contains diagonals so that G necessarily contains outer cycles. Hence the vertex v belongs to one or more outer cycles.

Let n be the length of the smallest outer cycle containing v. We first show that there exists an outer n-cycle containing v but in which v is not the endpoint of the associated diagonal. Suppose that the vertices of an outer n-cycle are $v_1, v_2, \ldots, v_n$. Consider the path which commences at v_1, proceeds to v_n along the diagonal $v_1 v_n$, and encounters in succession the vertices $v_{n+1}, v_{n+2}, \ldots, v_p$. Since G is randomly hamiltonian from $v=v_1$ and v belongs to no outer k-cycle, $k<n$, the diagonal $v_p v_{n-1}$ must be present in G. Hence v belongs to the outer n-cycle whose vertices are $v_p, v_1, \ldots, v_{n-1}$. In a similar way, one can show that if $v_{p-n+2} v_1$ is a diagonal of G, then $v_{p-n+3} v_2$ is a diagonal of G.

Thus we may assume the existence of an outer n-cycle whose vertices are $v_m, v_{m+1}, \ldots, v_p, v_1, \ldots v_{k-1}$, where $m=p-n+k$ and $3 \leqslant k \leqslant n$. We now show that the diagonals $v_{m-1} v_{k-2}$ and $v_{m+1} v_k$ are present in G in addition to $v_m v_{k-1}$. We begin a path at $v=v_1$ and proceed along C to $v_p, v_{p-1}, \ldots, v_m$. Following along the diagonal $v_m v_{k-1}$ to v_{k-1} and then taking $v_k, v_{k+1}, \ldots, v_{m-1}$, we see that $v_{m-1} v_{k-2}$ is a diagonal of G since G is randomly hamiltonian from v and v belongs to no outer t-cycle, $t<n$. Similarly, by applying the preceding arguments to the path $v_1, v_2, \ldots, v_{k-1}, v_m, v_{m-1}, v_{m-2}, \ldots, v_k$, we observe that $v_{m+1} v_k$ is a diagonal of G.

We now prove that $n<5$, for suppose, to the contrary, that $n \geqslant 5$. We have already seen that exists an outer n-cycle whose vertices are $v_m, v_{m+1}, \ldots, v_p, v_1, \ldots, v_{k-1}$, where $m=p-n+k$ and $3 \leqslant k \leqslant n$, and, in addition, the edges $v_{m-1} v_{k-2}$ and $v_{m+1} v_k$ belong to G. Furthermore, since $n \geqslant 5$, v_1 is not adjacent to both v_m and v_{k-1}. Let us say that v_1 is not adjacent to v_{k-1}, the other case being handled analogously. We now construct a path which beins at $v=v_1$ and takes in succession $v_p, v_{p-1}, \ldots, v_{m+1}$. We then proceed to v_k via the diagonal $v_{m+1} v_k$ and move along C in the order $v_{k+1}, v_{k+2}, \ldots, v_{m-1}$. On reaching v_{m-1}, we next take v_{k-2} (which is different from v), v_{k-1}, and then v_m. Since G is randomly hamiltonian from v, there exists either a vertex not yet encountered which is adjacent to v_m or the edge $v_m v$ which completes a hamiltonian cycle. In either case, there exists an edge $v_m u$, where u is one of the vertices $v_{m+2}, v_{m+3}, \ldots, v_p, v_1, \ldots, v_{k-3}$, which determines an outer cycle containing v having length less than n, and this is a contradiction.

We now show that $n \neq 4$. To prove this, we assume $n=4$ so that v belongs to an outer 4-cycle but not an outer triangle. From what we have shown above, we may assume, without loss of generality, that v_p, v_1, v_2, v_3 are the vertices of an outer 4-cycle. Since G is randomly hamiltonian from $v=v_1$, the path $v_1, v_2, v_3, v_p, v_{p-1}, v_{p-2}, \ldots, v_4$, which contains all vertices of G, implies that $v_1 v_4$ is an edge of G. The path $v_1, v_4, v_3, v_p, v_{p-1}, v_{p-2}, \ldots, v_5$ contains all the vertices of G with the exception of v_2; hence $v_2 v_5$ is an edge of G. Next the path $v_1, v_2, v_5, v_4, v_3, v_p, v_{p-1}, v_{p-2}, \ldots, v_6$ contains all the vertices of G and, as such, implies that $v_1 v_6$ is an edge of G. Continuing inductively, it is now easily verified that all edges of the type $v_1 v_{2m}$ belong to G as do all edges of the type $v_2 v_{2m+1}$. From this it now follows that every two vertices v_α

and v_β, where α and β are of opposite parity, are adjacent. To see this, let v_{2r} and v_{2s+1} be two non-consecutive vertices of C, where v_{2r} is different from v_2 and v_{2s+1} is not v_1. There are two cases to consider according to whether the path $v_{2r}, v_{2r+1}, \ldots, v_{2s+1}$ does or does not contain the vertex v. We treat here only the latter case, the former case being handled in a similar manner. We construct a path which begins at v_1, proceeds along a diagonal to v_{2s}, then along C to the vertices $v_{2s-1}, v_{2s-2}, \ldots, v_{2r+1}$, from where we move to v_2 by way of the diagonal $v_2 v_{2r+1}$. Next we proceed to v_{2r-1} via the diagonal $v_2 v_{2r-1}$ and then take $v_{2r-2}, v_{2r-3}, \ldots, v_3, v_p, v_{p-1}, \ldots, v_{2s+1}$, which produces a path failing only to contain v_{2r}. Since G is randomly hamiltonian from v, the edge $v_{2r}v_{2s+1}$ must be present in G. Finally, we show that if α and β are of the same parity, then v_α and v_β are not adjacent. We consider here only the case where α and β are odd, the other case following similarly. Assume, to the contrary, that the vertices v_{2r+1} and v_{2s+1} are adjacent, where $2r+1<2s+1$, say. The path $v_1, v_{2s+2}, v_{2s+3}, \ldots, v_p, v_3, v_4, \ldots, v_{2r+1}, v_{2s+1}, v_{2s}, \ldots, v_{2r+2}$ fails only to contain the vertex v_2; thus $v_2 v_{2r+2}$ is an edge of G. From this we see that the path $v_1, v_p, v_{p-1}, \ldots, v_{2r+2}, v_2, v_{2r+1}, v_{2r}, \ldots, v_3$, which contains all vertices of G, implies that $v_1 v_3$ is a diagonal of G. However, this contradicts the fact that v_1 belongs to no outer triangle. Hence, v_α and v_β are adjacent if and only if α and β are of opposite parity. This implies that p is even since $v_1 v_p$ is an edge of G. Furthermore, be letting $V_1=\{v_{2n} \mid n=1, 2, \ldots, p/2\}$ and $V_2=\{v_{2n-1} \mid n=1, 2, \ldots, p/2\}$, we see that G is the graph $K(p/2, p/2)$, which, as noted earlier, is randomly hamiltonian. However, this is a contradiction since it is contrary to our assumption that G is not randomly hamiltonian.

We now arrive at the conclusion that the only possible value is $n=3$; thus v belongs to an outer triangle. From methods similar to those we have already employed, it is immediately established that G contains the edges $v_1 v_3$, $v_2 v_p$, and $v_1 v_{p-1}$. Thus v_1 is adjacent to each of the vertices v_2, v_3, and v_p. However, v_1 is necessarily adjacent to all other vertices of G, for if $v_1 v_k$ is an edge of G, $3 \leqslant k < p-1$, then so too is $v_1 v_{k+1}$ an edge of G since the path $v_1, v_k, v_{k+1}, \ldots, v_2, v_p, v_{p-1}, \ldots, v_{k+1}$ contains all vertices of G and therefore v_1 is adjacent to v_{k+1}. The result then follows by induction.

Hence we may express G as $H+v$. The only remaining detail now is to verify that H is randomly hamiltonian. In order to prove this, it is necessary to show that any path $u_1, u_2, \ldots, u_k$ of H can be extended to a hamiltonian cycle of H. Since v is adjacent to u_1: $v, u_1, u_2, \ldots, u_k$ is a path of G and can be extended to a hamiltonian cycle $v, u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_{p-1}, v$ of G. However, v is also adjacent to u_2; thus $v, u_2, u_3, \ldots, u_{p-1}$ is also a path of G and can be extended to a hamiltonian cycle of G. This implies that $u_{p-1}u_1$ is an edge of G so that $u_1, u_2, \ldots, u_{p-1}, u_1$ is a hamiltonian cycle of H. Hence H is randomly hamiltonian, completing the proof.

The preceding theorem now indicates that the only graphs with p vertices which are randomly hamiltonian from some vertex v are C_p, K_p, $K(p/2, p/2)$, $C_{p-1}+v$, and $K((p-1)/2, (p-1)/2)+v$. As one final observation, we state the following.

COROLLARY. *The number of vertices in a graph G with p vertices from which G is randomly hamiltonian is either* 0, 1, *or p.*

REFERENCES

[1] F. BÄBLER, *Über eine spezielle Klasse Euler'scher Graphen*, Comment. Math. Helv. *27* (1953), 81–100.
[2] G. CHARTRAND and H. V. KRONK, *Randomly Traceable Graphs*, J. Soc. Indust. Appl. Math. (to appear).
[3] F. HARARY, *On Arbitrarily Traceable Graphs and Directed Graphs*, Scripta Math. *23* (1957), 37–41.
[4] O. ORE, *A Problem Regarding the Tracing of Graphs*, Elem. Math. *6* (1951), 49–53.

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