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On the Finsler and Doner–Tarski Arithmetical Hierarchies[1])

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In [5], [1; pg. 64] FINSLER set up a transfinite sequence of binary operations $\{h_c\}$ [2]) on the ordinals. The first four operations are $h_0(a, b)=a+1$, $h_1(a, b)=a+b$, $h_2(a, b)=a\cdot b$, and $h_3(a, b)=a^b$. The hierarchy satisfies the recursion formula $h_{c+1}(a, b+1)=h_c(h_{c+1}(a, b), a)$; this generalizes the formulae $a+(b+1)=(a+b)+1$, $a\cdot(b+1)=a\cdot b+a$, $a^{b+1}=a^b\cdot a$. DONER and TARSKI [3] have also set up a hierarchy $\{g_c\}$ [3]) of binary operations where each operation is related to the succeeding operation by this same recursion formula, and which, moreover, has a rather simple definition: $g_0(a, b)=a+b$, $g_c(a, b)=\bigcup_{d<b,\, e<c}[g_e(g_c(a, d), a)]$ for $c>0$. For every c and every $a\neq 0$, $g_c(a, x)$ is a continuous strictly increasing function of x. On the other hand, this can fail to be the case for $h_c(a, x)$ if c is a limit ordinal. Our basic result is that the Doner-Tarski hierarchy is *essentially* what one would get if one deleted from Finsler's hierarchy those operations whose subscript is a limit ordinal. We show also that this deletion does not diminish the representation power of the hierarchy in the sense that if $d=h_c(a, b)$ for some $a, b, c, <d$, then $d=g_{c'}(a', b')$ for some $a', b', c', <d$.

Let $\{f_c\}$ be the hierarchy of continuous increasing functions defined inductively by: $f_0(x)=\omega^x$, if $c\neq 0$ f_c enumerates in order those ordinals which are fixed points of f_s for all $s<c$ (existence proof given in [1]). The fixed points of the function $f_x(0)$ are called the *strongly critical epsilon numbers*. In [6] we showed that the least strongly critical epsilon number κ_0 is the least number greater then ω which is inaccessible by means of Finsler's hierarchy; by that we mean the least number $d>\omega$ such that $a, b, c, <d$ implies $h_c(a, b)<d$. Using the results announced in the above paragraph we will show that κ_0 plays the same role with respect to the Doner–Tarski hierarchy. FEFERMAN [4], SCHÜTTE [9], [10], and TAIT [11] have obtained results which show that κ_0 plays a significant role in ramified type theory.

DEFINITION. To each ordinal c we associate an ordinal c^* as follows: $c^*=c+1$ if $c=d+n$ where d is a limit ordinal and $0\leqslant n<\omega$, $c^*=c$ otherwise. It is easy to see that:

$$c^*+1=(c+1)^* \tag{1}$$

THEOREM 1. If $c\geqslant 4$ and $a\geqslant\omega$ then $g_{-1+c}(a, 1+b)=h_{c^*}(a, b)$.

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2) We write $h_c(a, b)$ for Finsler's $\phi_c(b, a)$ (Note the interchange of the variables). Finsler's restriction that $a, b, c<\Omega_1$ can be lifted [1; pg. 64].

3) We write $g_c(a, b)$ for Doner–Tarski's $a0_c b$.

Proof: By transfinite induction on $\Omega_i c+b$. Our induction hypothesis is that $c\geqslant 4$ and $\Omega_i c'+b'<\Omega_i c+b$ implies that $g_{-1+c'}(a, 1+b')=h_{(c')^*}(a, b')$ for all $a\geqslant\omega$.[4]) We must show under this assumption that $g_{-1+c}(a, 1+b)=h_{c^*}(a, b)$ for all $a\geqslant\omega$. Let $a\geqslant\omega$ be given:

Case 1. $c=4$; using [3; 3(iii)] and [5; pg. 80] we get:

$$g_{-1+c}(a, 1+b)=g_3(a, 1+b)=a^{(a^b)}=h_4(a, b)=h_c(a, b)=h_{c^*}(a, b).$$

Case 2. $c>4$;

Case 2.1 $b=0$; using [3; 17(iii)] and [5; th. 9] we get:

$$g_{-1+c}(a, 1+b)=g_{-1+c}(a, 1)=a=h_{c^*}(a, 0)=h_{c^*}(a, b).$$

Case 2.2 $b\neq 0$;

Case 2.2.1 b is a limit ordinal; using [3; 17(vi)], our induction hyp. and the fact that $h_{c^*}(a, x)$ is a continuous increasing function of x [1; pg. 65] we get:

$$g_{-1+c}(a, 1+b)=\sup_{d<b} g_{-1+c}(a, 1+d)=\sup_{d<b} h_{c^*}(a, d)=h_{c^*}(a, b).$$

Case 2.2.2 b is a successor ordinal $v+1$;

Case 2.2.2.1 c is a successor ordinal $d+1$; then using [3; 17(iv)] and the fact that $a\geqslant\omega$ we get:

$$\begin{aligned} g_{-1+c}(a, 1+b)&=g_{-1+(d+1)}(a, 1+(v+1))=g_{(-1+d)+1}(a, (1+v)+1)\\ &=g_{-1+d}(g_{(-1+d)+1}(a, 1+v), a)=g_{-1+d}(g_{(-1+d)+1}(a, 1+v), 1+a). \end{aligned}$$

Now by [3; corr. 5(i)] we note that $g_{-1+d}(a, 1+v)\geqslant a\geqslant\omega$, so using successive applications of our induction hypothesis followed by (1), [5; pg. 6] and (1) again, we get:

$$\begin{aligned} g_{-1+d}(g_{(-1+d)+1}(a, 1+v), 1+a)&=h_{d^*}(g_{-1+(d+1)}(a, 1+v), a)\\ &=h_{d^*}(h_{(d+1)^*}(a, v), a)=h_{d^*}(h_{d^*+1}(a, v), a)\\ &=h_{d^*+1}(a, v+1)=h_{(d+1)^*}(a, b)=h_{c^*}(a, b). \end{aligned}$$

Case 2.2.2.2. c is a limit ordinal; using [3; 17(v)] we get:

$$\begin{aligned} g_{-1+c}(a, 1+b)&=g_c(a, 1+(v+1))=g_c(a, (1+v)+1)\\ &=\sup_{d<c} g_d(g_c(a, 1+v), a)=\sup_{d<c} g_{-1+d}(g_c(a, 1+v), a)\\ &=\sup_{d<c} g_{-1+d}(g_{-1+c}(a, 1+v), a). \end{aligned}$$

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[4]) Our induction is up to Ω_i^2, where Ω_i is an initial ordinal and $a, b, c<\Omega_i$.

Now by [3; corr. 5(i)] and induction hyp. $\omega \leqslant g_{-1+c}(a, 1+v) = h_{c^*}(a, v)$ so

$$\sup_{d<c} g_{-1+d}(g_{-1+c}(a, 1+v), a) = \sup_{d<c} g_{-1+d}(h_{c^*}(a, v), a) = \sup_{d<c} g_{-1+d}(h_{c^*}(a, v), 1+a).$$

Now by a further application of the induction hypothesis, Finsler's definition, and [5, pg. 6] we get:

$$\sup_{d<c} g_{-1+d}(h_{c^*}(a, v), 1+a) = \sup_{d<c} h_{d^*}(h_{c^*}(a, v), a)$$
$$= \sup_{d<c} h_d(h_{c^*}(a, v), a) = h_c(h_{c^*}(a, v), a) = h_c(h_{c+1}(a, v), a)$$
$$= h_{c+1}(a, v+1) = h_{c^*}(a, b) \quad \text{(Q.E.D.)}.$$

COROLLARY 1. *If* $a, b, c, < \kappa_0$ *then* $g_c(a, b) < \kappa_0$.

Proof: Follows from the main theorem together with the fact that Finsler's hierarchy has the same property [6, th. 3].

COROLLARY 2. *If* $x \geqslant 1$, *then*

$$f_c(-1+x) = \begin{cases} g_{2c+2}(\omega, \omega x) & \text{if } 1 \leqslant c < \omega \\ g_c(\omega, 1+x) & \text{if } c \text{ is a limit ordinal} \\ g_{a+2n}(\omega, \omega x) & \text{if } c = a+n \text{ where } a \text{ is a limit ordinal and } 1 \leqslant n < \omega \end{cases}$$

Proof: Follows from the main theorem together with [6, th. 1].

THEOREM 2. *If* $d > \omega$ *and* $d = h_c(a, b)$ *where* $a, b, c, < d$, *then* $d = g_u(v, w)$ *where* $u, v, w < d$.

Proof: By [6, th. 3] $d \neq f_d(0)$; but then by [8; (5.1)] $d = a+b$ where $a, b < d$ or $d = f_c(a)$ where $c, a < d$:

CASE 1. $d = a+b$ where $a, b < d$; then $d = g_0(a, b)$.

CASE 2. $d = f_c(a)$ where $c, a < d$; write $a = -1+b$ where $b \geqslant 1$:

CASE 2.1 $c \neq 0$; then d is an epsilon number, so $a < d$ implies $b < d$, $1+b < d$, and $\omega b < d$.

CASE 2.1.1 c is a limit ordinal; then by corollary 2:

$$c, 1+b, \omega < d = f_c(-1+b) = g_c(\omega, 1+b).$$

CASE 2.1.2 $c = e+n$ where e is a limit ordinal and $1 \leqslant n < \omega$; then by corollary 2:

$$e+2n, \omega, \omega b < d = f_c(-1+b) = g_{e+2n}(\omega, \omega b).$$

CASE 2.1.3 c is finite; then by corollary 2:

$$2c+2, \omega, \omega b < d = f_c(-1+b) = g_{2c+2}(\omega, \omega b).$$

CASE 2.2 $c=0$; now $a \geqslant 2$ because $\omega < d = f_0(a) = \omega^a$, thus by [3, 3(ii)] $d = f_2(\omega, a) > 2, a, \omega$. (Q.E.D.)

COROLLARY 3. κ_0 *(the least strongly critical epsilon number) is the least ordinal inaccessible by means of the Doner–Tarski hierarchy.*

Proof: κ_0 is inaccessible by corollary 1. That it is the least such number follows from the main theorem together with the result of [6] that κ_0 is the least ordinal greater than ω which is inaccessible by means of Finsler's hierarchy.

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