A Notion of Krull Dimension for Differential Rings.

Autor(en): **Johnson, Joseph**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **44 (1969)**

PDF erstellt am: **23.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-33766

A Notion of Krull Dimension for Differential Rings

JOSEPH JOHNSON

It is known that if k is a field and if R is a commutative ring and a finitely generated k-algebra without zero divisors, then the dimension of R (defined to be the greatest length of a chain of prime ideals of R) is equal to the transcendance degree of R over k. In this paper an analogue of this theorem is shown for differential algebra. The principal theorem proven here has the fact just mentioned as a special case if in addition we assume that the characteristic of k is zero.

For an indication of how the theorem is actually stated, consider the special case of an ordinary differential field $\mathfrak{F}$ of characteristic zero and let R be a finitely generated differential algebra over $\mathfrak{F}$. If $\mathfrak{p}$ and $\mathfrak{q}$ are prime differential ideals of R with $\mathfrak{q}\supset\mathfrak{p}$, call the gap between $\mathfrak{q}$ and $\mathfrak{p}$ *large* if there exists an infinite strictly descending sequence $\mathfrak{q}=\mathfrak{q}_0 \supsetneqq \mathfrak{q} \supsetneqq_1 \cdots \supset \mathfrak{p}$ of prime differential ideals between $\mathfrak{q}$ and $\mathfrak{p}$. The main theorem of this paper (at the beginning of § 2) specializes in this instance to tell us that if $\mathfrak{p}$ is a prime differential ideal of R and if d is the differential transcendance degree of $R/\mathfrak{p}$ over $\mathfrak{F}$, then there exists a chain $\mathfrak{p}=\mathfrak{p}_0\subset\mathfrak{p}_1\subset\cdots\subset\mathfrak{p}_d$ of prime differential ideals of R with the gaps from $\mathfrak{p}_i$ to $\mathfrak{p}_{i-1}$ large for each i and that no chain of prime differential ideals of R all of which contain $\mathfrak{p}$ can have more than d large gaps.

This paper would have been substantially shorter were it not for the fact that very little is known about differential ideals in an arbitrary differential algebra finitely generated over a differential field. In §§ 3 and 4, some tools are developed which allow one to deal with these to a certain extent, and consequently these sections are quite technical in nature. Many readers will probably prefer to have an outline of the proof of the theorem mentioned above before digging deeply into these details. In § 5, the results from §§ 3 and 4 which are needed are succinctly summarized near the beginning so that one can defer reading those sections.

It is assumed that the reader is to some extent familiar with the material of [1] and [2]. In any instance where notation is unexplained there will be an explanation in one of these references.

§ 1. Combinatorial Type and Dimension

Let R be a ring and let $\mathfrak{A}$ be a family of ideals of R[1]. Adjoin to the set of integers $\mathbf{Z}$ a single element ∞ and extend the order on $\mathbf{Z}$ to $\mathbf{Z}\cup\{\infty\}$ by requiring that $p\leqslant\infty$ for every $p\in\mathbf{Z}$.

[1]) For the definitions and results in this section, $\mathfrak{A}$ can be any ordered set; the ring R plays no role.

LEMMA. *There is a unique way to define an element* $\mu_{\mathfrak{A}}(\mathfrak{a}, \mathfrak{a}')$ (or $\mu(\mathfrak{a}, \mathfrak{a}')$) *of* $\mathbf{Z} \cup \{\infty\}$ *for every pair* $(\mathfrak{a}, \mathfrak{a}') \in \mathfrak{A} \times \mathfrak{A}$ *with* $\mathfrak{a} \supset \mathfrak{a}'$ *such that the following two conditions hold:*

i) $\mu(\mathfrak{a}, \mathfrak{a}') \geqslant -1$

ii) *If d is any non-negative integer,* $\mu(\mathfrak{a}, \mathfrak{a}') \geqslant d$ *if and only if* $\mathfrak{a} \neq \mathfrak{a}'$ *and there exists an infinite sequence* $(\mathfrak{a}_i)_{i=0,1,\ldots}$ *of elements of* $\mathfrak{A}$ *such that* $\mathfrak{a} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}'$ *and* $\mu(\mathfrak{a}_{i-1}, \mathfrak{a}_i) \geqslant d-1$ *for* $i = 1, 2, \ldots$.

The reader can quickly prove this lemma by defining $\mu(\mathfrak{a}, \mathfrak{a}') = \infty$ if ii) holds for $d = 0, 1, 2, \ldots$ and $\mu(\mathfrak{a}, \mathfrak{a}') = d$ where $d \in \mathbf{Z}$ is the largest d such that ii) holds otherwise.

It follows from i) and ii) above that $\mu(\mathfrak{a}, \mathfrak{a}') \geqslant 0$ if $\mathfrak{a} \neq \mathfrak{a}'$ and that $\mu(\mathfrak{a}, \mathfrak{a}') = 0$ means that $\mathfrak{a} \neq \mathfrak{a}'$ and that every infinite descending chain of $\mathfrak{A}$ of the form $\mathfrak{a} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \cdots \supset \mathfrak{a}'$ satisfies $\mathfrak{a}_i = \mathfrak{a}_{i+1}$ for all sufficiently large i. On the other hand $\mu(\mathfrak{a}, \mathfrak{a}') \geqslant 1$ means that there exists an infinite strictly descending chain of $\mathfrak{A}$ of the form $\mathfrak{a} = \mathfrak{a}_0 \underset{\neq}{\supset} \mathfrak{a}_1 \underset{\neq}{\supset} \cdots \supset \mathfrak{a}'$.

Define type$_{\mathfrak{A}} R$ (or type R) to be the least upper bound of all the $\mu(\mathfrak{a}, \mathfrak{a}') \in \mathbf{Z} \cup \{\infty\}$. Define $\dim_{\mathfrak{A}} R$ (or $\dim R$) to be the least upper bound of the $p \in \mathbf{Z}$ such that there exists a chain $\mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_p$ of elements of $\mathfrak{A}$ with $\mu(\mathfrak{a}_{i-1}, \mathfrak{a}_i) = \text{type}\, R$ for $i = 1, \ldots, p$. If $\mathfrak{A}$ is non-empty, type R and $\dim R$ are defined. If type $R < \infty$, then $\dim R \geqslant 1$, but if type $R = \infty$, we may have $\dim R = 0$. In the situations of interest here, type R will always be finite.

§ 2. Dimension in Differential Rings

In the sequel the following theorem will be shown.

THEOREM. *Let R be a differential integral domain with m derivation operators which is a finitely generated differential algebra over a differential field* $\mathfrak{F}$ *of characteristic zero. Let* $\mathfrak{A}$ *be the set of all prime differential ideals of R. Then if* diff tr $\deg_{\mathfrak{F}} R > 0$, type$_{\mathfrak{A}} R = m$ *and* $\dim_{\mathfrak{A}} R =$ diff tr $\deg_{\mathfrak{F}} R$. *If* diff tr $\deg_{\mathfrak{F}} R = 0$, *then* type$_{\mathfrak{A}} R < m$.

As a first step toward proving this theorem it will be shown that

i) type $R \leqslant m$,

ii) type $R < m$ if diff tr $\deg_{\mathfrak{F}} R = 0$ and

iii) if type $R = m$, $\dim R \leqslant$ diff tr $\deg_{\mathfrak{F}} R$.

Let $\eta_1, \ldots, \eta_n$ generate the differential algebra R over $\mathfrak{F}$. Let $\mathfrak{p}$ be a prime differential ideal of R and for the moment let $\bar{\eta}_1, \ldots, \bar{\eta}_n$ denote the images of $\eta_1, \ldots, \eta_n$ respectively in $R/\mathfrak{p}$. Let $\mathfrak{r}$ be an indeterminate. By the theorem in Chapter II of [1], there exists a polynomial $\chi_{\mathfrak{p}}(\mathfrak{r}) \in \mathbf{Q}[\mathfrak{r}]$ such that $\chi_{\mathfrak{p}}(r) = \text{tr}\deg_{\mathfrak{F}} \mathfrak{F}[\Theta(r)\bar{\eta}_1 \cup \ldots \cup \Theta(r)\bar{\eta}_n]$ for all sufficiently large $r \in \mathbf{Z}$.

Given polynomials $P_1(\mathfrak{r}), P_2(\mathfrak{r})$ with rational coefficients, write $P_1(\mathfrak{r}) \geqslant P_2(\mathfrak{r})$ if for all sufficiently large $r \in \mathbf{Z}$, $P_1(r) \geqslant P_2(r)$. If now $\mathfrak{p}$ and $\mathfrak{q}$ are prime differential ideals of R with $\mathfrak{p} \supset \mathfrak{q}$, then $\chi_{\mathfrak{p}}(\mathfrak{r}) \leqslant \chi_{\mathfrak{q}}(\mathfrak{r})$. Let $R_r = \mathfrak{F}[\Theta(r)\eta_1 \cup \cdots \cup \Theta(r)\eta_n]$. If $\chi_{\mathfrak{p}}(\mathfrak{r}) = \chi_{\mathfrak{q}}(\mathfrak{r})$, then $\text{tr}\deg_{\mathfrak{F}}(R_r/\mathfrak{p} \cap R_r) = \text{tr}\deg_{\mathfrak{F}}(R_r/\mathfrak{q} \cap R_r)$ for all sufficiently large

$r \in \mathbf{Z}$. It follows (c.f. [3], Lemma page 75) that $\mathfrak{p}=\mathfrak{q}$ if $\chi_{\mathfrak{p}}(\mathfrak{r})=\chi_{\mathfrak{q}}(\mathfrak{r})$. It will now be shown inductively that if $\mu(\mathfrak{p}, \mathfrak{q}) \geqslant p$, then $\deg(\chi_{\mathfrak{q}}(\mathfrak{r})-\chi_{\mathfrak{p}}(\mathfrak{r})) \geqslant p$.

The polynomials χ we are considering are *numerical*, that is $\chi(r)$ is an integer for all sufficiently large $r \in \mathbf{Z}$. If χ is a numerical polynomial, $\Delta\chi$ is also. (See [1], Chap. III § 2 for definition of $\Delta\chi$.) Let p be the degree of χ and write $\chi(\mathfrak{r})=(a/p!)\mathfrak{r}^p+a$ polynomial of degree less than p. A simple induction shows that $\Delta^p\chi=a$. It follows that if χ is numerical, a is an integer. Also if $\chi \geqslant 0$, it is clear that $a \geqslant 0$.

To show that $\deg(\chi_{\mathfrak{q}}(\mathfrak{r})-\chi_{\mathfrak{p}}(\mathfrak{r})) \geqslant p$ when $\mu(\mathfrak{p}, \mathfrak{q}) \geqslant p$, observe that in any case $\deg(\chi_{\mathfrak{q}}(\mathfrak{r})-\chi_{\mathfrak{p}}(\mathfrak{r})) \geqslant -1$. Also if $\mathfrak{p} \neq \mathfrak{q}$, then since $\chi_{\mathfrak{q}}(\mathfrak{r})-\chi_{\mathfrak{p}}(\mathfrak{r})$ is not the zero polynomial, $\deg(\chi_{\mathfrak{q}}(\mathfrak{r})-\chi_{\mathfrak{p}}(\mathfrak{r})) \geqslant 0$. This demonstrates the assertion for $p=-1$ and for $p=0$.

Now let $p>0$ and let $\mathfrak{p}=\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{q}$ be an infinite chain of prime differential ideals of P with $\mu(\mathfrak{p}_{i-1}, \mathfrak{p}_i) \geqslant p-1$ for $i=1,2,\dots$. If for some i, $\deg(\chi_{\mathfrak{p}_i}(\mathfrak{r})-(\chi_{\mathfrak{p}_{i-1}}(\mathfrak{r})) \geqslant p$, then surely since $\chi_{\mathfrak{q}}(\mathfrak{r})-\chi_{\mathfrak{p}}(\mathfrak{r}) \geqslant \chi_{\mathfrak{p}_i}(\mathfrak{r})-\chi_{\mathfrak{p}_{i-1}}(\mathfrak{r})$, $\deg(\chi_{\mathfrak{q}}(\mathfrak{r})-\chi_{\mathfrak{p}}(\mathfrak{r})) \geqslant p$. Otherwise we have $\deg(\chi_{\mathfrak{p}_i}(\mathfrak{r})-\chi_{\mathfrak{p}_{i-1}}(\mathfrak{r}))=p-1$ for every i. Write $\chi_{\mathfrak{p}_i}(\mathfrak{r})-\chi_{\mathfrak{p}}(\mathfrak{r})=a_i\mathfrak{r}^{p-1}/(p-1)!+$a polynomial of degree $<p-1$. It is clear that $a_{i-1}<a_i$ for all i. Because $\chi_{\mathfrak{q}}-\chi_{\mathfrak{p}} \geqslant \chi_{\mathfrak{p}_i}-\chi_{\mathfrak{p}}$ for all i, the degree of $\chi_{\mathfrak{q}}-\chi_{\mathfrak{p}}$ is forced to be $\geqslant p$ proving the assertion that was made.

For any prime differential ideal $\mathfrak{p}$ of R, $\deg \chi_{\mathfrak{p}}(\mathfrak{r}) \leqslant m$ and is $<m$ if diff tr $\deg_{\mathfrak{F}} R=0$. It follows that type $R \leqslant m$ and is $<m$ if diff tr $\deg_{\mathfrak{F}} R=0$. Suppose now that type $R=m$ and that $\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_d$ is a chain of prime differential ideals of R with $\mu(\mathfrak{p}_{i-1}, \mathfrak{p}_i)=m$ for $i=1,\dots,d$. Write $\chi_{\mathfrak{p}_d}(\mathfrak{r})-\chi_{\mathfrak{p}_0}(\mathfrak{r})=\mathrm{a}\mathfrak{r}^m/m!+$ a polynomial of degree $<m$ with $\mathrm{a} \in \mathbf{Z}$. Then since $\chi_{\mathfrak{p}_i}(\mathfrak{r})-\chi_{\mathfrak{p}_{i-1}}(\mathfrak{r})=a_i\mathfrak{r}^m/m!+$ a polynomial of degree $<m$ with a_i an integer and >0, $a \geqslant d$. On the other hand $a=\Delta^m(\chi_{\mathfrak{p}_d}(\mathfrak{r})-\chi_{\mathfrak{p}_0}(\mathfrak{r})) \leqslant \Delta^m(\chi_{\mathfrak{p}_d}(\mathfrak{r})) \leqslant \Delta^m(\chi_{(0)}(\mathfrak{r}))=$ diff tr $\deg_{\mathfrak{F}} R$, so $d \leqslant$ diff tr $\deg_{\mathfrak{F}} R$. Hence $\dim R \leqslant$ diff tr $\deg_{\mathfrak{F}} R$. This completes the proof of assertions i), ii) and iii).

Before going on to prove the rest of the theorem, let it be remarked that if χ is a differential dimension polynomial for R over $\mathfrak{F}$, it is not in general true that type $R=\deg \chi$. To see this take $m=1$, let $\mathfrak{F}$ be a differential field of constants and let x be an algebraic indeterminate over $\mathfrak{F}$ with $x'=1$. Then $\mathfrak{F}[x]$ has no non-trivial differential ideals so that type $\mathfrak{F}[x]=-1$. On the other hand $\deg \chi$ must be zero.

The proof of the rest of the theorem starts with what is essentially the consideration of a special case. (In the statement of the lemma, [ ] means *differential ideal generated by*.) (Consult [2], § 0 for explanations of the notation used here.)

Lemma.[2]) *Let $\mathfrak{F}$ be a differential field of characteristic zero and let $y_1,\dots,y_n$ be differential indeterminates. Let $\mathfrak{A}$ be the set of all differential ideals of $\mathfrak{F}\{y_1,\dots,y_n\}$ of the form $[\theta_1 y_{i(1)},\dots,\theta_p y_{i(p)}]$ where $p \geqslant 0$, $\theta_1,\dots,\theta_p \in \Theta$ and $1 \leqslant i(1),\dots,i(p) \leqslant n$. Then* $\mathrm{type}_{\mathfrak{A}} \mathfrak{F}\{y_1,\dots,y_n\}=m$ *and* $\dim_{\mathfrak{A}} \mathfrak{F}\{y_1,\dots,y_n\}=n$.

---

[2]) The special choice of $\mathfrak{A}$ makes the proof of the lemma much less awkward than it might otherwise be. It turns out however to be also necessary to have the lemma in this form in order to successfully carry out the proof of the theorem.

Because $\mathfrak{A}$ is contained in the family of all prime differential ideals of $\mathfrak{F}\{y_1,\dots,y_n\}$, it is readily apparent from what has already been shown that $\text{type}_{\mathfrak{A}}\mathfrak{F}\{y_1,\dots,y_n\}\leqslant m$, and if $\text{type}_{\mathfrak{A}}\mathfrak{F}\{y_1,\dots,y_n\}=m$, then $\dim_{\mathfrak{A}}\mathfrak{F}\{y_1,\dots,y_n\}\leqslant n$. Upon considering therefore the chain $[y_1,\dots,y_n]\supset[y_1,\dots,y_{n-1}]\supset\cdots\supset[y_1]\supset(0)$ of $\mathfrak{A}$, it becomes evident that it is enough to show that $\mu_{\mathfrak{A}}([y_1,\dots,y_p],[y_1,\dots,y_{p-1}])\geqslant m$ for $1\leqslant p\leqslant n$. If $\mathfrak{A}'=\{\mathfrak{a}\in\mathfrak{A}:[y_1,\dots,y_p]\supset\mathfrak{a}\supset[y_1,\dots,y_{p-1}]\}$, it will suffice to show that $\mu_{\mathfrak{A}'}([y_1,\dots,y_p],[y_1,\dots,y_{p-1}])\geqslant m$. We have $\mathfrak{F}\{y_1,\dots,y_n\}/[y_1,\dots,y_{p-1}]\approx\mathfrak{F}\{y_p,\dots,y_n\}$ canonically and under this isomorphism, the set $\mathfrak{A}'$ is mapped order isomorphically onto the set $\mathfrak{B}$ consisting of all those differential ideals of $\mathfrak{F}\{y_p,\dots,y_n\}$ which are generated by the finite subsets of Θy_p. Let $\mathfrak{B}'$ consist of all those differential ideals of $\mathfrak{F}\{y_p\}$ which are generated by finite subsets of Θy_p. Because $\mathfrak{b}\in\mathfrak{B}'$ implies $(\mathfrak{b}\,\mathfrak{F}\{y_p,\dots,y_n\})\cap\mathfrak{F}\{y_p\}=\mathfrak{b}$, it follows quickly that the map from $\mathfrak{B}$ to $\mathfrak{B}'$ obtained by intersecting the elements of $\mathfrak{B}$ with $\mathfrak{F}\{y_p\}$ is an order preserving isomorphism. This reduces our problem to the case $n=1$, i.e. it will suffice to show that if y is a differential indeterminate over $\mathfrak{F}$ and if $\mathfrak{A}_1$ is the class of all differential ideals of $\mathfrak{F}\{y\}$ generated by the finite subsets of Θy, $\mu_{\mathfrak{A}_1}([y],(0))\geqslant m$.

Considering the chain $[y]\supset\cdots\supset[\delta_m^r y]\supset[\delta_m^{r+1}y]\supset\cdots\supset(0)$ of elements of $\mathfrak{A}_1$, it is obvious that it will suffice to show that $\mu_{\mathfrak{A}_1}([\delta_m^r y],[\delta_m^{r+1}y])\geqslant m—1$. Similarly to what was done before, let $\mathfrak{A}_1'$ be the family of all $\mathfrak{a}\in\mathfrak{A}_1$ such that $[\delta_m^r y]\supset\mathfrak{a}\supset[\delta_m^{r+1}y]$. Let ζ be the element $\delta_m^r y+[\delta_m^{r+1}y]$ of the differential ring $\mathfrak{F}\{\delta_m^r y\}/[\delta_m^{r+1}y]$. As before one reasons that $\mathfrak{A}_1'$ is order isomorphic to the family $\mathfrak{B}_1'$ of all differential ideals of $\mathfrak{F}\{\zeta\}=\mathfrak{F}\{\delta_m^r y\}/[\delta_m^{r+1}y]$ generated by finite subsets of $\Theta^{(m-1)}\zeta$, where $\Theta^{(m-1)}\subset\Theta$ is the subset of all those derivative operators which do not involve δ_m.

Let $\Delta^{(m-1)}=\Delta-\{\delta_m\}$. As the family of the $\theta\zeta$ for $\theta\in\Theta^{(m-1)}-$ is algebraically independent over $\mathfrak{F}$, ζ is a $\Delta^{(m-1)}$-differential indeterminate. The structure of Δ-differential ring on $\mathfrak{F}\{\zeta\}$ is determined by the additional condition $\delta_m\zeta=0$. It is clear that every $\Delta^{(m-1)}-$differential ideal of $\mathfrak{F}\{\zeta\}$ generated by a finite subset of $\Theta^{(m-1)}\zeta$ is Δ-differential and generated by the same finite subset of $\Theta^{(m-1)}\zeta$ and the converse is true too. Consequently $\mathfrak{A}_1'$ is order isomorphic to the set of all $\Delta^{(m-1)}$-differential ideals of $\mathfrak{F}\{\zeta\}$ generated by finite subsets of $\Theta^{(m-1)}\zeta$ so the proof of the lemma follows by induction.

To go from the lemma that has just been demonstrated to the proof of the theorem an analogue will be developed for differential algebra of the following very easily proven fact of commutative algebra.

Lemma. *Let k be a field, R an algebra of finite type over R without zero divisors. If $x_1,\dots,x_p$ is a transcendance basis for R over k, there exists a non-zero $D\in k[x_1,\dots,x_p]$ such that $R[1/D]$ is an integral extension of $k[x_1,\dots,x_p,1/D]$.*

Using this lemma one can prove the theorem for the case $m=0$. What it is necessary to do is to show the existence of a chain $\mathfrak{P}_p\underset{\neq}{\supset}\mathfrak{P}_{p-1}\underset{\neq}{\subset}\cdots\underset{\neq}{\supset}\mathfrak{P}_0$ of prime ideals of R of length p. We have (assuming characteristic $k=0$) that k is infinite, so there exist $\xi_1, \ldots, \xi_p \in k$ such that $D(\xi_1, \ldots, \xi_p) \neq 0$. Let $\mathfrak{p}_i = (x_1 - \xi_1, \ldots, x_i - \xi_i)$. Then $\mathfrak{p}_p \underset{\neq}{\supset} \mathfrak{p}_{p-1} \underset{\neq}{\supset} \cdots \underset{\neq}{\supset} \mathfrak{p}_0$ and $D \notin \mathfrak{p}_p$. By integrality this chain lifts to a chain of prime ideals $\mathfrak{Q}_p \underset{\neq}{\supset} \mathfrak{Q}_{p-1} \underset{\neq}{\supset} \cdots \underset{\neq}{\supset} \mathfrak{Q}_0$ of $R[1/D]$ (i.e., we have $\mathfrak{Q}_i \cap k[x_1, \ldots, x_p] = \mathfrak{p}_i$). Letting $\mathfrak{P}_i = \mathfrak{Q}_i \cap R$ we obtain a chain of the desired type.

The proof of the theorem for $m>0$ will run along lines similar to those just given, but is quite a bit more complicated.

§ 3. Some Preliminary Results on Differential Vector Spaces

For the time being $\mathfrak{G}$ will be a differential field of any characteristic. It will however be assumed that the subfield C of $\mathfrak{G}$ consisting of all constants (elements c of $\mathfrak{G}$ satisfying $\delta(c)=0$ for all $\delta \in \Delta$) contains infinitely many elements.

Let $\delta_1, \ldots, \delta_m$ be the elements of Δ. If $\sigma \in GL(m, C)$ is any non-singular $m \times m$ matrix over C, Δ^σ will be the set $\{\delta_1^\sigma, \ldots, \delta_m^\sigma\}$ where δ_i^σ is the formal linear combination $\delta_i^\sigma = \sum_{j=1}^{m} \sigma_{ij} \delta_j$. If M is any differential vector space over $\mathfrak{G}$ or a differential algebra over $\mathfrak{G}$, the set Δ^σ will be made to act on M in the obvious way, i.e. we will define $\delta_i^\sigma(x) = \sum_{j=1}^{m} \sigma_{ij}\delta_j(x)$ for $x \in M$. We have clearly that $\delta_i^\sigma(\delta_j^\sigma(x)) = \delta_j^\sigma(\delta_i^\sigma(x))$ since the coefficients of σ lie in C. In this way $\mathfrak{G}$ and all differential structures over $\mathfrak{G}$ will be viewed as having Δ or Δ^σ as their fundamental set of derivation operators. It is clear (since σ is invertible) that if M is a differential vector space over $\mathfrak{G}$ or a differential algebra over $\mathfrak{G}$, M is finitely generated with respect to Δ if and only if M is finitely generated with respect to Δ^σ.

In the sequel when the phrase "for an appropriate choice of Δ" precedes a statement P, this will mean that there exists a $\sigma \in GL(m, C)$ such that P holds true when Δ is replaced by Δ^σ in the manner indicated above. Similarly when the phrase "for almost all choices of Δ" precedes a statement P, then there exists a non-empty Zariski open subset U of $GL(m, C)$ such that P holds true when Δ is replaced by any Δ^σ with $\sigma \in U$.

If $0 \leqslant p \leqslant m$, $\Delta^{(p)}$ will be the set $\{\delta_1, \ldots, \delta_p\}$. Then any differential structure with Δ as its fundamental set of derivation operators inherits a differential structure with $\Delta^{(p)}$ as its fundamental set of derivation operators by simply ignoring the given action of $\delta_{p+1}, \ldots, \delta_m$. Denote by $\Theta^{(p)}$ the subset of Θ consisting of all elements of the form $\delta_1^{i(1)} \cdots \delta_p^{i(p)}$. Similarly if $\mathfrak{F}$ is a differential ring with fundamental set of derivation operators Δ and if R is a differential ring extension of $\mathfrak{F}$ with respect to Δ or with respect to $\Delta^{(p)}$, let $\mathfrak{F}\{\eta_1, \ldots, \eta_n\}^{(p)}$ be the $\Delta^{(p)}$-differential subring of R generated by

$\mathfrak{F}$ and $\eta_1, \ldots, \eta_n$ whenever $\eta_1, \ldots, \eta_n$ are given elements of R. Similar uses of a superscript $^{(p)}$ whose meaning seems clear will be made without further explanation. The following theorem has its origin in Theorem 6 of [4].

THEOREM. *(Reduction of the number of derivation operators.)*
Let M be a finitely generated differential vector space over the differential field $\mathfrak{G}$. Then if diff $\dim_{\mathfrak{G}} M=0$ *and if the field of constants of $\mathfrak{G}$ is infinite, for almost all choices of Δ, M is finitely generated as a $\Delta^{(m-1)}$-differential vector space.*

As diff $\dim_{\mathfrak{G}} M=0$, all the elements of M are differentially linearly dependent over $\mathfrak{G}$. Let $\mathfrak{D}$ be the ring of linear differential operators in Δ with coefficients in $\mathfrak{G}$, and let $\eta_1, \ldots, \eta_n$ be a set of generators of M. It will suffice to show that if $1 \leqslant i \leqslant n$, then for almost all choices of Δ, the differential vector space $\mathfrak{D}\eta_i$ is finitely generated as a $\Delta^{(m-1)}$-differential vector space over $\mathfrak{G}$. Indeed if we do this for each $i=1, \ldots, n$, we obtain non-empty Zariski open subsets $U_1, \ldots, U_n$ of $GL(m, C)$ such that if we replace Δ by Δ^σ with $\sigma \in U_i$, $\mathfrak{D}\eta_i$ is a finitely generated $\Delta^{(m-1)}$-differential vector space over $\mathfrak{G}$. Then $U_1 \cap \cdots \cap U_n$ is non-empty and if $\sigma \in U_1 \cap \cdots \cap U_n$, $M=\mathfrak{D}\eta_1+\cdots+\mathfrak{D}\eta_n$ is a finitely generated $\Delta^{(m-1)}$-differential vector space over $\mathfrak{G}$ when Δ has been replaced by Δ^σ. It follows that it will be enough to prove the theorem when M is generated by a single element η.

Since η is differentially algebraically dependent over $\mathfrak{G}$, there exists a non-zero $D \in \mathfrak{D}$ such that $D\eta=0$. Let r be the order of D. Consider any $\sigma \in GL(m, C)$ and let $\tau=\sigma^{-1}$. Writing $\varepsilon_i=\delta_i^\sigma$ we have $\delta_i=\sum_{j=1}^{m} \tau_{ij}\varepsilon_j$. If $\delta_1^{j(1)} \cdots \delta_m^{j(m)}$ with $j(1)+\cdots+j(m)=r$ is an arbitrary element of order r of

$$\Theta,\ \delta_1^{j(1)} \ldots \delta_m^{j(m)} = \left(\sum_{j=1}^{m} \tau_{1j}\varepsilon_j\right)^{j(1)} \cdots \left(\sum_{j=1}^{m} \tau_{mj}\varepsilon_j\right)^{j(m)} = \tau_{1m}^{j(1)} \ldots \tau_{mm}^{j(m)} \varepsilon_m^r + E,$$

where $E \in \mathfrak{D}_r$ and can be written as a polynomial in $\varepsilon_1, \ldots, \varepsilon_m$ with coefficients in C of degree $\leqslant r$ and such that the coefficient of ε_m^r is zero. Let I be the set of all m-tuples $j=(j(1), \ldots, j(m))$ with $j(1), \ldots, j(m)$ non-negative integers and $j(1)+\cdots+j(m)=r$. Then we can write $D=\sum_{j \in I} a_j \delta_1^{j(1)} \ldots \delta_m^{j(m)}+D'$ where $D' \in \mathfrak{D}_{r-1}$ and the a's belong to $\mathfrak{G}$. Then

$$(*) \quad D=\sum_{j \in I} a_j \tau_{1m}^{j(1)} \ldots \tau_{mm}^{j(m)} \varepsilon_m^r + E' + D'$$

where E' is a linear combination with coefficients in $\mathfrak{G}$ of elements of $\mathfrak{D}$ which have the properties listed above for E.

It is possible to choose a non-zero polynomial function P on $GL(m, C)$ such that if $P(\sigma) \neq 0$ and if letting $\alpha=\sum_{j \in I} a_j \tau_{1m}^{j(1)} \ldots \tau_{mm}^{j(m)}$, $\alpha \neq 0$. Let $\mathfrak{D}^{(m-1)}$ be the subring of $\mathfrak{D}$ generated by $\mathfrak{G}$ and $\varepsilon_1, \ldots, \varepsilon_{m-1}$. Because $D\eta=0$, it follows that if $\alpha \neq 0$, $\varepsilon_m^r \eta \in \mathfrak{D}^{(m-1)}\eta+$ $+\mathfrak{D}^{(m-1)}\varepsilon_m\eta+\cdots+\mathfrak{D}^{(m-1)}\varepsilon_m^{r-1}\eta$ because of the relation $(*)$. This implies that

$\mathfrak{D}^{(m-1)}\eta+\cdots+\mathfrak{D}^{(m-1)}\varepsilon_m^{r-1}\eta$ is mapped into itself by ε_m. Since it is certainly mapped into itself by $\varepsilon_1, \ldots, \varepsilon_{m-1}$, it must be all of M, and this proves the theorem.

COROLLARY. *Let $\mathfrak{F}$ be a differential field of characteristic zero, $\mathfrak{G}$ a finitely generated differential field extension of $\mathfrak{F}$. If* diff tr $\deg_{\mathfrak{F}} \mathfrak{G}=0$, *then for almost all choices of Δ, $\mathfrak{G}$ is a finitely generated $\Delta^{(m-1)}$-differential field extension of $\mathfrak{F}$.*

Let $\Omega=\Omega^1_{\mathfrak{G}/\mathfrak{F}}$ be the module of Kahler differentials of $\mathfrak{G}$ over $\mathfrak{F}$ and recall ([2], § 1) that Ω has a structure of differential vector space over $\mathfrak{G}$ such that for $\delta\in\Delta$ and $\eta\in\mathfrak{G}$ we have $\delta(d(\eta))=d(\delta(\eta))$. Choose $\eta_1, \ldots, \eta_n\in\mathfrak{G}$ such that $d\eta_1, \ldots, d\eta_n$ generate Ω as a $\Delta^{(m-1)}$-differential vector space, and such that $\eta_1, \ldots \eta_n$ generate G as a differential field over F. Let $\mathfrak{H}_0$ be the $\Delta^{(m-1)}$-differential field extension of $\mathfrak{F}$ generated by $\eta_1, \ldots, \eta_n$, let $\mathfrak{H}$ be the $\Delta^{(m-1)}$-differential field extension of $\mathfrak{F}$ generated by $\eta_1, \ldots, \eta_n$, $\delta_m(\eta_1), \ldots, \delta_m(\eta_n)$. It will be shown that $\mathfrak{H}=\mathfrak{G}$.

Letting η be one of the η_i, it will suffice to show that all the $\delta_m^s\eta$ for $s\geqslant 0$ belong to $\mathfrak{H}$. Since $d(\delta_m\eta)\in d(\mathfrak{H}_0)$, there exists an $F\in\mathfrak{H}_0[X]$, X an indeterminate, such that $F(\delta_m\eta)=0$ and $S=dF/dX(\delta_m\eta)\neq 0$. We have $0=\delta_m(F)=S\,\delta_m^2\eta+$ an element of $\mathfrak{H}$. This shows that $\delta_m^2\eta\in\mathfrak{H}$. Since $\delta_m^2\eta_1, \ldots, \delta_m^2\eta_n\in\mathfrak{H}$, it follows that if $s>2$ and $\delta_m^{s-1}\eta\in\mathfrak{H}$, then $\delta_m\eta\in\mathfrak{H}$ so that by induction all the $\delta_m\eta\in\mathfrak{H}$. This completes the proof of the corollary.

§ 4. Parametric Subrings

Let R and $\mathfrak{F}$ be differential integral domains and suppose that R is an extension of $\mathfrak{F}$. The notion of a *parametric subring of R over $\mathfrak{F}$* (or parametric subring) will be defined by induction on m.

If $m=0$, a subring R_0 of R will be called *parametric* if the quotient field of R is finite algebraic over the quotient field of R_0 and if there exist finitely many elements $\eta_1, \ldots, \eta_n$ of R_0 algebraically independent over $\mathfrak{F}$ and such that $R_0=\mathfrak{F}[\eta_1, \ldots, \eta_n]$. Observe that in this case R_0 is a parametric subring of R if and only if R_0 is a parametric subring of the quotient field of R.

If $m>0$, a subring R_0 of R will be called *parametric* if there exist elements $\eta_1, \ldots, \eta_n$ of R_0 differentially algebraically independent over $\mathfrak{F}$ and such that R_0 is a parametric subring of R over $\mathfrak{F}\{\eta_1, \ldots, \eta_n\}$ when these are considered to be differential rings with respect to the set of derivation operators $\Delta^{(m-1)}$.

From the definition, the following proposition follows immediately by induction on m.

PROPOSITION. *Let R and $\mathfrak{F}$ be differential integral domains, R being an extension of $\mathfrak{F}$. Then*

i) *A subring R_0 of R containing $\mathfrak{F}$ is a parametric subring of R over $\mathfrak{F}$ if and only if R_0 is a parametric subring of the quotient field of R over $\mathfrak{F}$.*

ii) *If R_0 is a parametric subring of R over $\mathfrak{F}$, then R_0 is a polynomial ring extension of $\mathfrak{F}$ and the quotient field of R is finite algebraic over the quotient field of R_0.*

THEOREM. *Let R and $\mathfrak{F}$ be differential integral domains of characteristic zero, R being an extension of $\mathfrak{F}$. Suppose that the quotient field of R is a finitely generated differential field extension of the quotient field of $\mathfrak{F}$. Let $\eta_1, \ldots, \eta_d$ be a differential transcendance basis for R over $\mathfrak{F}$. Then for an appropriate choice of Δ, R contains a subring parametric over $\mathfrak{F}\{\eta_1, \ldots, \eta_d\}$.*

First suppose that $m=0$. Then the quotient field of R is finite algebraic over the quotient field of $\mathfrak{F}[\eta_1, \ldots, \eta_d]$, and hence $\mathfrak{F}[\eta_1, \ldots, \eta_d]$ is a parametric subring of R over $\mathfrak{F}$.

Suppose now that $m>0$. Then for almost all choices of Δ, the quotient field of R considered as a $\Delta^{(m-1)}$-differential field is a finitely generated extension of the quotient field of $\mathfrak{F}\{\eta_1, \ldots, \eta_d\}$. Choose such a Δ. By induction we may assume that after an appropriate change of $\Delta^{(m-1)}$ (and hence ultimately one more change of Δ) there exists a subring R_0 of R parametric over $\mathfrak{F}\{\eta_1, \ldots, \eta_d\}$ when these last are considered as $\Delta^{(m-1)}$-differential rings. This establishes the theorem.

LEMMA. *Let R and $\mathfrak{F}$ be differential integral domains with $\mathfrak{F}$ a differential subring of R. Suppose that R_0 is a subring of R parametric over $\mathfrak{F}$. Let K be the quotient field of R, K_0 the quotient field of R_0. Let $a_1, \ldots, a_h$ be a basis for K over K_0. Then there exists $D \in R_0-(0)$ such that $R_0[1/D]\, a_1 + \cdots + R_0[1/D]\, a_h$ is a finitely generated differential algebra over $\mathfrak{F}$.*

Let $M = R_0 a_1 + \cdots + R_0 a_h$. It is necessary to find $D \in R_0-(0)$ such that $R_0[1/D]\, M$ is a finitely generated differential algebra over $\mathfrak{F}$. If $m=0$, it suffices to choose D such that all the $D a_i a_j$ belong to M. Suppose now that $m>0$. Choose $\eta_1, \ldots, \eta_d \in R_0$ which are differentially algebraically independent over $\mathfrak{F}$ and such that R_0 is parametric for R over $\mathfrak{F}\{\eta_1, \ldots, \eta_d\}$ when R and $\mathfrak{F}\{\eta_1, \ldots, \eta_d\}$ are considered as $\Delta^{(m-1)}$-differential rings. By induction there exists $D \in R_0-(0)$ such that $R_0[1/D]\, M$ is a finitely generated $\Delta^{(m-1)}$-differential algebra over $\mathfrak{F}\{\eta_1, \ldots, \eta_d\}$.

It is now only a question of finding $E \in R_0-(0)$ such that $R_0[1/DE]\, M$ is mapped into itself by δ_m. Let $\zeta_1, \ldots, \zeta_q$ be generators of $R_0[1/D]\, M$ as a $\Delta^{(m-1)}$-differential algebra over $\mathfrak{F}\{\eta_1, \ldots, \eta_d\}$. Choose $E \in R_0-(0)$ such that $E\delta_m(\zeta_1), \ldots, E\delta_m(\zeta_q) \in R_0[1/D]\, M$. Now $R_0[1/DE]\, M$ is closed under $\Delta^{(m-1)}$. It will suffice to show that if $i=1, \ldots, q$ and $\theta \in \Theta^{(m-1)}$, $\delta_m \theta \zeta_i \in R_0[1/DE]\, M$. But this is clear since $\delta_m \zeta_i \in R_0[1/DE]\, M$. This completes the proof of the lemma.

COROLLARY. *Suppose that R and $\mathfrak{F}$ are differential integral domains with R a finitely generated differential ring extension of $\mathfrak{F}$ and suppose that R_0 is a ring parametric for R over $\mathfrak{F}$. Then there exists $D \in R_0-(0)$ such that $R[1/D]$ is a finite module over $R_0[1/D]$.*

In the lemma take $a_1, \ldots a_h$ to be in R. Choose a D as given by the lemma and let $\xi_1, \ldots, \xi_n$ generate R as a differential algebra over $\mathfrak{F}$. Multiply D by a non-zero element of R_0 so as to get $\xi_1, \ldots, \xi_n \in R_0[1/D]\, a_1 + \cdots + R_0[1/D]\, a_h$. Then as $R[1/D] = R_0[1/D]\, a_1 + \cdots + R_0[1/D]\, a_h$, $R[1/D]$ is a finite module over $R_0[1/D]$. This proves the corollary.

§ 5. Conclusion of the Proof of the Main Theorem

We return to the situation of the theorem of § 2. Hence again $\mathfrak{F}$ is a differential field of characteristic zero, and R is a differential overring of $\mathfrak{F}$ which is finitely generated as a differential algebra over $\mathfrak{F}$ and without zero divisors. Let $d = \text{diff tr deg}_{\mathfrak{F}} R$. Because the assertions i), ii) and iii) which immediately follow the statement of the theorem have already been demonstrated, it will suffice to show that if $\mathfrak{C}$ is the class of all prime differential ideals of R and if $d>0$, then $(m, d)\leqslant(\text{type}_{\mathfrak{C}} R, \dim_{\mathfrak{C}} R)$ where $\mathbf{Z}\times\mathbf{Z}$ has been ordered lexicographically.

Let $\eta_1, \ldots, \eta_d$ be a differential transcendance basis for R over $\mathfrak{F}$. By the results of §§ 3, 4 (corollary of § 4 and proposition of § 4), there exists a subring R_0 of R and a non-zero element D of R_0 such that

i) R_0 is a polynomial ring extension of $\mathfrak{F}\{\eta_1, \ldots, \eta_d\}$.

ii) $R[1/D]$ is a finite $R_0[1/D]$-module.

LEMMA. *Let A and B be rings with B an overring of A. Let $\mathfrak{A}$ and $\mathfrak{B}$ be families of ideals of A and B respectively such that if* $\mathfrak{b}\in\mathfrak{B}$, $\mathfrak{b}\cap A\in\mathfrak{A}$. *Suppose*

a) If $\mathfrak{a}, \mathfrak{a}'\in\mathfrak{A}$ with $\mathfrak{a}\subset\mathfrak{a}'$, then there exist $\mathfrak{b}, \mathfrak{b}'\in\mathfrak{B}$ with $\mathfrak{b}\cap A=\mathfrak{a}$, $\mathfrak{b}'\cap A=\mathfrak{a}'$ and $\mathfrak{b}\subset\mathfrak{b}'$.

b) If $\mathfrak{b}_0, \mathfrak{b}_2\in\mathfrak{B}$ with $\mathfrak{b}_0\subset\mathfrak{b}_2$ and if $\mathfrak{a}_1\in\mathfrak{A}$ with $(\mathfrak{b}_0\cap A)\subset\mathfrak{a}_1\subset(\mathfrak{b}_2\cap A)$, then there exists $\mathfrak{b}_1\in\mathfrak{B}$ such that $\mathfrak{b}_0\subset\mathfrak{b}_1\subset\mathfrak{b}_2$ and $\mathfrak{b}_1\cap A=\mathfrak{a}_1$.

Under these assumptions, $(\text{type}_{\mathfrak{A}} A, \dim_{\mathfrak{A}} A)\leqslant(\text{type}_{\mathfrak{B}} B, \dim_{\mathfrak{B}} B)$.

Let $\mathfrak{a}, \mathfrak{a}'\in\mathfrak{A}$ with $\mathfrak{a}'\supset\mathfrak{a}$, $\mathfrak{b}, \mathfrak{b}'\in\mathfrak{B}$ with $\mathfrak{b}'\supset\mathfrak{b}$ and suppose that $\mathfrak{b}\cap A=\mathfrak{a}$, $\mathfrak{b}'\cap A=\mathfrak{a}'$. If $p\geqslant 0$ and $\mu_{\mathfrak{A}}(\mathfrak{a}', \mathfrak{a})\geqslant p$, consider an infinite chain in $\mathfrak{A}$ of the form $\mathfrak{a}'=\mathfrak{a}_0\supset\mathfrak{a}_1\supset\cdots\supset\mathfrak{a}$ with $\mu(\mathfrak{a}_i, \mathfrak{a}_{i-1})\geqslant p-1$ for $i=1, 2, \ldots$. By condition b) there exists an infinite chain of $\mathfrak{B}$ of the form $\mathfrak{b}'=\mathfrak{b}_0\supset\mathfrak{b}_1\supset\cdots\supset\mathfrak{b}$ such that for $i=0, 1, \ldots$, $\mathfrak{b}_i\cap A=\mathfrak{a}_i$. By induction we may assert that $\mu_{\mathfrak{B}}(\mathfrak{b}_i, \mathfrak{b}_{i-1})\geqslant p-1$ and hence, since $\mathfrak{b}'\neq\mathfrak{b}$, that $\mu(\mathfrak{b}', \mathfrak{b})\geqslant p$. It follows from this and a) that $\text{type}_{\mathfrak{A}} A\leqslant\text{type}_{\mathfrak{B}} B$.

Suppose now that $\text{type}_{\mathfrak{A}} A=\text{type}_{\mathfrak{B}} B$. Any chain $\mathfrak{a}_0\supset\mathfrak{a}_1\supset\cdots\supset\mathfrak{a}_d$ of $\mathfrak{A}$ with $\mu_{\mathfrak{A}}(\mathfrak{a}_{i-1}, \mathfrak{a}_i)=\text{type}_{\mathfrak{A}} A$ for $i=1, \ldots, d$ can by a) and *b*) be lifted to a chain of $\mathfrak{B}$, say $\mathfrak{b}_0\supset\mathfrak{b}_1\supset\cdots\supset\supset\mathfrak{b}_d$. As $\mu_{\mathfrak{B}}(\mathfrak{b}_{i-1}, \mathfrak{b}_i)=\text{type}_{\mathfrak{B}} B$ and as $\dim_{\mathfrak{A}} A$ is the least upper bound of such d, it follows that $\dim_{\mathfrak{A}} A\leqslant\dim_{\mathfrak{B}} B$, and this completes the proof of the lemma.

LEMMA. *Let B_0 and B be rings with B an overring of B_0 and integral over B_0. Then if $\mathfrak{p}$ is a prime ideal of B_0 and if $\mathfrak{q}$ is a prime ideal of B such that $\mathfrak{q}\cap B_0=\mathfrak{p}$, $\mathfrak{q}$ is a minimal prime component of $\mathfrak{p}B$. Furthermore if B_0 is integrally closed, then any minimal prime component $\mathfrak{q}$ of $\mathfrak{p}B$ lies over $\mathfrak{p}$.*

The first assertion is an immediate consequence of [5], III-3, Proposition 2. As for the second, let $\mathfrak{q}'$ be a minimal prime component of $\mathfrak{p}B$ and let $\mathfrak{p}'=\mathfrak{q}'\cap B_0$. Then by [5] (III-5) there exists a prime ideal $\mathfrak{q}$ of B such that $\mathfrak{q}\subset\mathfrak{q}'$ and $\mathfrak{q}\cap B_0=\mathfrak{p}$. Then $\mathfrak{q}=\mathfrak{q}'$ and $\mathfrak{p}=\mathfrak{p}'$, completing the proof of the lemma.

COROLLARY. *Let $\mathfrak{p}_0\subset\mathfrak{p}_1\subset\mathfrak{p}_2$ be prime ideals of a ring A, let B be an integral extension of A and let $\mathfrak{q}_0, \mathfrak{q}_2$ be prime ideals of B such that $\mathfrak{q}_i\cap A=\mathfrak{p}_i$, $i=0, 2$. Then if $A/\mathfrak{p}_0$ is integrally closed, there exists a prime ideal $\mathfrak{q}_1$ of B such that $\mathfrak{q}_0\subset\mathfrak{q}_1\subset\mathfrak{q}_2$, $\mathfrak{q}_1\cap A=\mathfrak{p}_1$ and $\mathfrak{q}_1$ is a component of $\mathfrak{p}_1B$.*

Let θ be an element of Θ which for the moment is arbitrary and set $\zeta_i=\theta\eta_i$ (where $\eta_1, \ldots, \eta_d$ is the differential transcendance basis we have chosen for R over $\mathfrak{F}$). In $\mathfrak{F}\{\zeta_1, \ldots, \zeta_d\}$ let $\mathfrak{A}$ be the family of all those differential ideals which are generated by the finite subsets of $\Theta\zeta_1 \cup \cdots \cup \Theta\zeta_d$. If the order of θ is sufficiently large, then for $\mathfrak{a} \in \mathfrak{A}$, $D \notin \mathfrak{a} R_0$. It will now be assumed that θ is so chosen.

Let $\mathfrak{B}$ be the family of all prime differential ideals $\mathfrak{b}$ of $R[1/D]$ such that $\mathfrak{b} \cap \mathfrak{F}\{\zeta_1, \ldots, \zeta_d\}$ is in $\mathfrak{A}$. One can canonically identify $\mathfrak{B}$ with the family of all prime differential ideals $\mathfrak{b}$ of R such that $D \notin \mathfrak{b}$ and $\mathfrak{b} \cap \mathfrak{F}\{\zeta_1, \ldots, \zeta_d\}$ is in $\mathfrak{A}$. To complete the proof of the theorem, it will be enough to show that $(m, d) \leqslant (\text{type}_{\mathfrak{B}} R, \dim_{\mathfrak{B}} R)$. Referring to the first lemma of this section and the first lemma of § 2 it is seen that it will suffice to show that a) and b) hold true when we take $A = \mathfrak{F}\{\zeta_1, \ldots, \zeta_d\}$ and $B = R$ and use the families $\mathfrak{A}$ and $\mathfrak{B}$ that have just been defined.

It is important to remark that R_0 being a polynomial ring over $\mathfrak{F}\{\eta_1, \ldots, \eta_d\}$ is also such over $\mathfrak{F}\{\zeta_1, \ldots, \zeta_d\}$. Also if $\mathfrak{a} \in \mathfrak{A}$, $R_0/\mathfrak{a} R_0$ is a polynomial ring over $\mathfrak{F}$ since the ideal $\mathfrak{a} R_0$ is generated by homogeneous linear elements of R_0. Consequently $R_0[1/D]$ is integrally closed and so is $R_0[1/D]/\mathfrak{a} R_0[1/D]$ for every $\mathfrak{a} \in \mathfrak{A}$. The verification of a) and b) will now follow quickly.

First notice that if $\mathfrak{a} \in \mathfrak{A}$ and if $\mathfrak{b}$ is a prime ideal of $R[1/D]$ lying over $\mathfrak{a} R_0[1/D]$, then $\mathfrak{b}$ is a minimal prime component of $\mathfrak{a} R[1/D]$ and hence is differential. (It is a known fact that if a differential ring A contains $\mathbf{Q}$, any minimal prime component of a differential ideal of A is itself differential.) If $\mathfrak{a} \subset \mathfrak{a}'$, $\mathfrak{a}' \in \mathfrak{A}$, then by integrality there exist prime ideals $\mathfrak{b}$, $\mathfrak{b}'$ of $R[1/D]$ such that $\mathfrak{b} \subset \mathfrak{b}'$ and $\mathfrak{b} \cap R_0[1/D] = \mathfrak{a} R_0[1/D]$, $\mathfrak{b}' \cap R_0[1/D] = \mathfrak{a}' R[1/D]$. Then $\mathfrak{b}$, $\mathfrak{b}' \in \mathfrak{B}$ and lie over $\mathfrak{a}$, $\mathfrak{a}'$ respectively; this proves a). To show b), let $\mathfrak{b}_0$, $\mathfrak{b}_2 \in \mathfrak{B}$ and $\mathfrak{a}_1 \in \mathfrak{A}$ with $\mathfrak{b}_0 \subset \mathfrak{b}_2$ and $\mathfrak{b}_0 \cap \mathfrak{F}\{\zeta_1, \ldots, \zeta_d\} \subset a_1 \subset \mathfrak{b}_2 \cap \cap \mathfrak{F}\{\zeta_1, \ldots, \zeta_d\}$. Let $\mathfrak{a}_0 = \mathfrak{b}_0 \cap \mathfrak{F}\{\zeta_1, \ldots, \zeta_d\}$. By the corollary there exists a prime ideal $\mathfrak{b}_1$ of $R[1/D]$ such that $\mathfrak{b}_0 \subset \mathfrak{b}_1 \subset \mathfrak{b}_2$ and $\mathfrak{b}_1 \cap R_0[1/D] = \mathfrak{a}_1 R_0[1/D]$ and this implies b). The proof of the main theorem is now complete.

BIBLIOGRAPHY

[1] JOHNSON, JOSEPH, *Differential dimension polynomials and a fundamental theorem on differential modules,* to appear in Amer. J. Math.

[2] JOHNSON, JOSEPH, *Kähler differentials and differential algebra,* to appear in Ann. of Math.

[3] MUMFORD, DAVID, *Introduction to algebraic geometry* (Preliminary version of first three chapters), Harvard University lecture notes.

[4] KOLCHIN, ELLIS, *The notion of dimension in the theory of algebraic differential equations,* Bull. Am. Math. Soc. *70* (1964), 570–573.

[5] SERRE, Jean-Pierre and GABRIEL, PIERRE, *Algèbre locale-multiplicites* (Springer-Verlag, 1965).

Received April 29, 1968