Coherent Graded Rings and the Non-Existence of Spaces of Finite Stable Homotopy Type.

Coherent Graded Rings and the Non-Existence of Spaces of Finite Stable Homotopy Type

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As is indicated by the title, this paper will contain discussions of both algebra and algebraic topology. The main results are topological but the main work is algebra.

The main theorem concerns the stable homotopy groups $\pi_n^s(X)$ of a space X and localizes to the p-primary part for some prime p, ${}_p\pi_n^s(X)$, i.e. the group modulo the elements of finite order prime to p.

THEOREM II. 3.1 *If* $\tilde{H}^n(X; Z_p) \neq 0$ *for any n then* ${}_p\pi_n^s(X) \neq 0$ *for infinitely many n.*

The most interesting part of the algebra stems from the concept of coherence (a generalization of Noetherian). This condition with its beautiful properties is well-known to algebraic geometers (e.g. [6]) but is little known to algebraic topologists and to ring theorists.

This paper lists all the standard results on coherence but there are no new results here that are likely to be useful to algebraists. We define a variant of coherence which is topologically useful (it is non-trivial only in the graded case) and for which all the results on coherence pass over. We urge those algebraists who have come this far to read at least the definitions and known results on coherence (Section I. 1).

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The paper consists of two chapters, the first algebraic and the second topological, each with three sections. Within a chapter, theorems will be referred to as a.b. From one chapter to the other, they will be referred to as I.a.b. or II.a.b.

I. Coherent and Strongly Coherent Graded Rings

1. *Definitions*

The purpose of this chapter is to investigate some of the purely algebraic properties of graded rings which occur frequently in topology. We are interested in preserving to some degree the properties associated with Noetherian rings, but Noetherian is much too strong a condition especially in the case of graded rings. A useful concept is that of the *coherent* ring and a more restricted type the *strongly coherent ring.*

To make up for considering these weaker rings we must restrict the category of modules. Instead of considering finitely generated modules as one does for Noetherian,

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we shall consider finitely presented modules (and a weaker version which makes sense only in the graded case).

First let us recall some definitions.

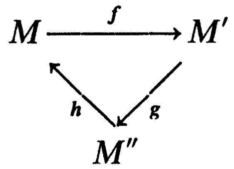
DEFINITIONS. A graded ring R is a collection of abelian groups R_n for all integers n and homomorphisms $R_n \otimes R_m \to R_{n+m}$ (which we write as juxtaposition) satisfying associativity and distributivity. We always assume the existence of a multiplicative identity element in R_0.

A graded (left) R-module M is a sequence of abelian groups M_n for all integers n and homomorphisms $R_n \otimes M_m \to R_{n+m}$ (which we write as juxtaposition) satisfying the usual module identities.

From now on all modules, ideals, and properties will be assumed to be left – unless they are specifically called otherwise. All modules, rings, and ideals will be graded.

An R-module map $f: M \to N$ of degree d is a sequence of maps $f: M_n \to N_{n+d}$ such that for $r \in R_n$ and $m \in M$, $f(rm) = (-1)^{dn} rf(m)$.

An exact triangle



is a sequence of R-module maps f, g, h such that the kernel of each equals the image of the previous.

An exact diagram is a commutative diagram in which the rows and columns form exact sequences. In topology the exact triangle appears in the guise of the long exact sequence; for example, those associated to the homotopy and homology of a pair of spaces. (cf. [7]).

We recall the following definitions (cf. [2] and [3] where they are used for ungraded rings):

DEFINITION. An R-module M is finitely presented if there is a short exact sequence

$$0 \to K \to F \to M \to 0$$

with K and F finitely generated and F free.

An R-module M is *coherent* if it and its finitely generated submodules are finitely presented.

A ring R is *coherent* if it is coherent as an R-module.

We recall the standard theorems about coherent rings (they are fairly straightforward to prove; most are the trivializations of SERRE's theorems on coherent sheaves [6], the rest are exercises in BOURBAKI [2] pp. 62–63). These propositions hold for the graded case as well as the ungraded with no change in the proofs.

PROPOSITION 1.1. *A finitely generated submodule of a coherent module is coherent.*

PROPOSITION 1.2. *If A, B, C are R-modules and*

$$0 \to A \to B \to C \to 0$$

is an exact sequence and any two are coherent, so is the third.

PROPOSITION 1.3. *If $f: M \to N$ is an R-module map and M and N are coherent then so are* $\ker f$, $\operatorname{im} f$ *and* $\operatorname{cok} f$.

PROPOSITION 1.4. *Let R be a ring. Then the following are equivalent:*
(i) *R is coherent.*
(ii) *An R-module M is coherent if and only if it is finitely presented.*

PROPOSITION 1.5. *Let $\{R_\alpha\}$ be a direct system of left coherent rings such that for all α, R is a right flat R_α-module. Then $\varinjlim R_\alpha = R$ is left coherent.*

An immediate consequence of Proposition 1.5 is the following:

COROLLARY 1.6. *Let $R = \varinjlim R_\alpha$ be a Hopf algebra where the $R_\alpha \subset R$ are coherent subHopf algebras. Then R is coherent.*

Proof. By MILNOR–MOORE [5] a Hopf algebra is free over a subHopf algebra.

We now introduce a concept that arises frequently as one studies properties of long exact sequences.

DEFINITION. If M is an R-module we say that M has size n over R
a) for $n=0$ if $M=0$
b) for $n>0$ if there is an exact triangle

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & {}_{h}\nwarrow \quad \swarrow_{g} & \\ & R & \end{array}$$

where N has size $n-1$ and n is minimal with respect to this.

In general, finite size does not imply finitely generated but we shall show in Theorem 3.1. that coherent rings are characterized by the property that finite size implies finitely generated or equivalently that finite size implies finitely presented.

In order to prove some theorems in II of a strictly topological nature we shall introduce a weakening of the concept of coherence for the case of a graded ring and show that most properties still hold.

DEFINITION. A *weakly finitely generated* module is one which has generators in only finitely many degrees. A *weakly finitely presented* module M is one for which there is an exact sequence

$$0 \to K \to F \to M \to 0$$

where K and F are weakly finitely generated and F is free.

A module is *weakly coherent* if it and all its weakly finitely generated submodules are weakly finitely presented.

A ring R is *strongly coherent* if it is coherent and if every weakly finitely generated free module is weakly coherent.

We shall prove in § 3 that most of the propositions true for coherence are true in this variant.

We observe that coherent does not imply strongly coherent:

Example: Let R be a k-algebra (k some Noetherian ring) on generators x_n of degree $1, n=1, 2, 3, \ldots$ with the relations $x_n^{n+1}=0$. R is easily seen to be coherent. But the ideal $(x_1, x_2, \ldots)$ is weakly finitely generated but not weakly finitely presented.

2. *Properties of Strongly Coherent Rings*

We shall first prove the following useful lemma the non-weak case of which is in [2] p. 37.

LEMMA 2.1. *Let M be (weakly) finitely presented and N (weakly) finitely generated. Then if*

$$0 \to A \to N \to M \to 0$$

is exact, A is (weakly) finitely generated.

Proof. Let $0 \to K \to F \to M \to 0$ be a (weakly) finite presentation. If N is projective, then SHANUEL's lemma [4] says $K \oplus N \simeq F \oplus A$, hence A is (weakly) finitely generated since F and K are. For general N, there exists a (weakly) finitely generated projective P and an epimorphism $f: P \to N$. Then $f^{-1}(A)$ is the kernel of $P \to N \to M$ hence by the part already proved $f^{-1}(A)$ is (weakly) finitely generated. But $f \mid f^{-1}(A): f^{-1}(A) \to A$ is onto hence A is (weakly) finitely generated.

We can now prove the analogues of Propositions 1.1–1.5.

PROPOSITION 2.2. *A weakly finitely generated submodule of a weakly coherent module is weakly coherent.*

The proof is obvious.

THEOREM 2.3. *If $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is an exact sequence of R-modules and any two are weakly coherent, then so the third.*

Proof. 1) Assume B and C weakly coherent. Then by Lemma 2.1, A is a weakly finitely generated submodule of B hence is weakly coherent by Proposition 2.2.

2) Assume A and B are weakly coherent. Then C is weakly finitely generated. Let F and G be weakly finitely generated free modules with epimorphisms $f:F\to A$, $g:G\to C$. Then there is an exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker f & \longrightarrow & \ker h & \longrightarrow & \ker g & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & F\oplus G & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{j} & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By Lemma 2.1, $\ker h$ is weakly finitely generated, hence so is $\ker g$ so C is weakly finitely presented.

If $C'\subset C$ is a weakly finitely generated submodule of C, let $B'=j^{-1}(C')$. Then the following is exact: $0\to A\to B'\to C'\to 0$.

Since A and C' are weakly finitely generated so is B'. Then since $B'\subset B$, B' is weakly coherent. Since A is weakly coherent using the above, C' is weakly finitely presented. So C is weakly coherent.

3) Assume A and C are weakly coherent. Choose F and G as in 2) yielding the exact diagram there. Now $\ker f$ and $\ker g$ are weakly finitely generated, hence $\ker h$ is also so B is weakly finitely presented. If $B'\subset B$ is a weakly finitely generated submodule let $A'=i(A)\cap B'$, $C'=j(B')$. Then

$$0\to A'\to B'\to C'\to 0$$

is exact. $C'\subset C$ is weakly finitely generated hence weakly coherent. Thus by Lemma 2.1, $A'\subset A$ is weakly finitely generated hence is weakly coherent. Thus by the above B' is weakly coherent.

PROPOSITION 2.4. *If f: $M\to N$ is an R-module map and M and N are weakly coherent then so are* $\ker f$, $\operatorname{cok} f$ *and* $\operatorname{im} f$.

Proof.
$$0\to \operatorname{im} f\to N\to \operatorname{cok} f\to 0$$
$$0\to \ker f\to M\to \operatorname{im} f\to 0$$

are exact. $\operatorname{im} f$ is a weakly finitely generated submodule of N hence is weakly coherent. Thus by the previous proposition, $\operatorname{cok} f$ and $\ker f$ are weakly coherent.

THEOREM 2.5. *The following are equivalent statements about a ring R*

(i) *R is strongly coherent*

(ii) *An R-module M is (weakly) finitely presented if and only if it is (weakly) coherent.*

Proof. (ii) implies (i): Eliminating "weakly", Proposition 1.4 says that R is coherent. Secondly, any weakly finitely generated free module is weakly finitely presented hence weakly coherent. Thus R is strongly coherent.

(i) implies (ii): By definition weakly coherent implies weakly finitely presented. Conversely assume

$$0 \to K \to F \to M \to 0$$

is a weakly finite presentation of M. Since R is strongly coherent F and hence K are weakly coherent so by Proposition 2.3, so is M.

Finally, we prove

THEOREM 2.6. *Let* $R = \varinjlim R_\alpha$ *where the* R_α *are strongly left coherent rings with R right flat over each* R_α *and such that for all integers m there exists* α *such that* $R_\alpha \to R$ *is an isomorphism up to degree m. Then R is strongly left coherent.*

Proof. First note that R is coherent from Proposition 1.5. Next let $M \subset F$ be a weakly finitely generated submodule of a free module. Choose m so that all the generators of M occur in degrees $\leqslant m$ and pick α as in the hypothesis. Let $F' \subset F$ be the weakly finitely generated R_α module on the same generators as F. Let $M' = F' \cap M$. Then $M \simeq R \otimes_{R_\alpha} M'$. Since R_α is strongly coherent there is a weakly finite presentation of R_α-modules

$$0 \to K \to H \to M' \to 0.$$

Since R is right R_α-flat

$$0 \to R \otimes_{R_\alpha} K \to R \otimes_{R_\alpha} H \to M \to 0$$

is a weakly finite presentation of M. Thus R is strongly coherent.

We prove two trivial but useful results.

PROPOSITION 2.7. *If R is finite then it is strongly coherent.*

Proof. If R is finite, it is trivially coherent. Let M be *any* weakly finitely generated module. Then if $f: F \to M$ is an epimorphism of a weakly finitely generated free module, then F vanishes in almost all degrees, hence so does $\ker f$ so $\ker f$ is trivially weakly finitely generated.

PROPOSITION 2.8. *If*

$$\begin{array}{ccc} M & \xrightarrow{g} & N \\ & {\scriptstyle f}\nwarrow \quad \swarrow{\scriptstyle h} & \\ & P & \end{array}$$

is an exact triangle and any two of the modules are (weakly) coherent then so is the third.

Proof. Assume M and N are (weakly) coherent. Then

$$0 \to \operatorname{im} h \to P \to \ker g \to 0$$

is exact. But by Proposition 2.4, $\operatorname{im} h$ and $\ker g$ are (weakly) coherent. So by Theorem 2.3, so is P.

PROPOSITION 2.9. *If M is a weakly coherent R-module and R is strongly coherent then for all $0 \neq x \in M$, $rx \neq 0$ for all indecomposables r except those in finitely many degrees.*

Proof. Let $f: R \to M$ be $f(r) = rx$. Then $\ker f$ is weakly coherent by Proposition 2.4. Thus $\ker f$ is weakly finitely generated. Since $f(1) = x \neq 0$, $1 \notin \ker f$ so indecomposables $r \in \ker f$ occur in only finitely many degrees and for $r \notin \ker f$, $0 \neq f(r) = rx$.

3. *A Characterization of Coherent Rings*

THEOREM 3.1. *The following are equivalent statements about a ring R.*

(i) *R is coherent,*

(ii) *Every R-module of finite size is coherent,*

(iii) *Every R-module of finite size is finitely generated.*

Proof. (i) implies (ii): By induction on the size of the R-module M: If size $M = 0$, then $M = 0$ so M is coherent. Assume the result holds for R-modules of size $n-1$. Let M be an R-module of size n. Then there is an R-module N of size $n-1$ and an exact triangle

$$\begin{array}{ccc} M & \xrightarrow{\quad g \quad} & N \\ & {\scriptstyle f}\nwarrow \quad \swarrow{\scriptstyle h} & \\ & R & \end{array}$$

Since R and N are coherent, Proposition 2.8 says that M is also.

(ii) implies (iii) trivially.

(iii) implies (i): Assume R is not coherent. Let $I \subset R$ be a finitely generated ideal which is not finitely presented. Let F be a free finitely generated module with

$$0 \to K \xrightarrow{f} F \xrightarrow{g} I \to 0$$

exact (whence K is not finitely generated)

Let $h: F \to R$ be the composite $F \xrightarrow{g} I \subset R$. Set $N = \ker h \oplus \operatorname{cok} h = K \oplus \operatorname{cok} h$. So N is not finitely generated. But

$$\begin{array}{ccc} N & \xrightarrow{\;l\;} & F \\ & {\scriptstyle k}\nwarrow\!\!\swarrow{\scriptstyle h} & \\ & R & \end{array}$$

is exact where k and l are the obvious maps. Inductively it is easy to see that F is of finite size. Thus N is of finite size but not finitely generated.

II. The Topological Theorems

1. *Topological Premilinaries.*

All spaces will be given a basepoint $* \in X$. Maps preserve this basepoint. We shall assume familiarity with spectra. Since the ones we shall be concerned with are of a simple type almost anybody's definition will do. For example this (due to ADAMS [1]):

DEFINITION. A spectrum $\mathbf{X}$ is a sequence of CW complexes X_n of dimension $(2n-1)$ such that $SX_n = X_{n+1}^{(2n)}$.

Example. 1) If G is an abelian group then $\mathbf{K}(G)$ is the spectrum whose n-th space is the $(2n-1)$-skeleton of a suitably chosen Eilenberg-MacLane space of type (G, n). Let $\mathbf{K}_p = \mathbf{K}(Z_p)$, $\mathbf{K} = \mathbf{K}(Z)$.

2) If X is any space, $\mathbf{S}X$ is the spectrum whose n-th space is $S^n X^{(n-1)}$ where by $X^{(n-1)}$ we mean the $(n-1)$-skeleton of a CW complex weakly homotopy equivalent to X. If we set $X = S^0$ we write $\mathbf{S} = \mathbf{S}X$, the sphere spectrum whose n-th space in S^n, the n sphere.

3) If $\mathbf{X}$ is any spectrum then $\mathbf{X}^d$ is the spectrum whose n-th space is $X_{n+d}^{(2n-1)}$. This is the d-th suspension of $\mathbf{X}$.

DEFINITION. A map of degree $r, f: \mathbf{X} \to \mathbf{Y}$ is a sequence of maps $f_n: X_n \to Y_{n+r}$ such that $Sf_n = f_{n+1} \mid X_{n+1}^{(2n)}$. The set of homotopy classes of such maps is $[\mathbf{X}, \mathbf{Y}]^r$. $[\mathbf{X}, \mathbf{Y}]^*$ is the total graded group.

We have the following properties of $\mathbf{X}$.

PROPOSITION 1.1. *If $f: \mathbf{X} \to \mathbf{Y}$ is of degree r then there is a canonically defined spectrum $\mathbf{E}_f$ called the fibre of f and maps $\sigma_f: \mathbf{E}_f \to \mathbf{X}$ of degree 0 and $j_f: \mathbf{Y} \to \mathbf{E}_f$ of degree $(1-r)$ such that for any spectrum $\mathbf{W}$ there are long exact sequences*

$$[\mathbf{W}, \mathbf{Y}]^{n+r-1} \xrightarrow{j_{f*}} [\mathbf{W}, \mathbf{E}_f]^n \xrightarrow{\sigma_{f*}} [\mathbf{W}, \mathbf{X}]^n \xrightarrow{f_*} [\mathbf{W}, \mathbf{Y}]^{n+r}$$

$$[\mathbf{Y}, \mathbf{W}]^{n-r+1} \xleftarrow{j_f^*} [\mathbf{E}_f, \mathbf{W}]^n \xleftarrow{\sigma_f^*} [\mathbf{X}, \mathbf{W}]^n \xleftarrow{f^*} [\mathbf{Y}, \mathbf{W}]^{n-r}$$

DEFINITION. If $\mathbf{X}$ is a spectrum we define $\Pi_n(\mathbf{X}) = [\mathbf{S}, \mathbf{X}]^{-n}$.

$H_n(\mathbf{X}; G) = \lim\limits_{\rightarrow} H_{n+r}(X_r; G)$ where the directed system is

$$H_{n+r}(X_r; G) \xrightarrow{\sim} H_{n+r+1}(S X_r; G) \to H_{n+r+1}(X_{r+1}; G).$$

Similarly $H^n(X; G) = \lim\limits_{\leftarrow} H^{n+r}(X_r; G)$.

The following properties are well-known:

PROPOSITION 1.2:

a) $H^n(\mathbf{X}; G) \cong [\mathbf{X}, \mathbf{K}(G)]^n$.

b) $H^*(X; Z_p) \cong H^*(\mathrm{S}\, X; Z_p)$ *as modules over the Steenrod algebra* $\mathscr{A}^*$.

c) *For any space* X, $\Pi_n^S(X) \cong \Pi_n(\mathrm{S}\, X)$.

d) *The Steenrod algebra* $\mathscr{A}^* \cong H^*(\mathbf{K}_p; Z_p)$.

Putting Propositions 1.1 and 1.2 together with the definitions we get

PROPOSITION 1.3. *If* $f: \mathbf{X} \to \mathbf{Y}$ *there are exact triangles:*

$$\begin{array}{ccc} \Pi_*(\mathbf{X}) & \xrightarrow{f_*} & \Pi_*(\mathbf{Y}) \\ {\scriptstyle \sigma_{f_*}} \nwarrow & & \swarrow {\scriptstyle j_{f_*}} \\ & \Pi_*(\mathbf{E}_f) & \end{array} \qquad \begin{array}{ccc} H^*(\mathbf{X}; G) & \xrightarrow{\sigma_f^*} & H^*(\mathbf{E}_f; G) \\ {\scriptstyle f^*} \nwarrow & & \swarrow {\scriptstyle j_f^*} \\ & H^*(\mathbf{Y}; G) & \end{array}$$

2. *On the* $\mathscr{A}^*$*-Module Structure of* $H^*(\mathbf{X})$

We shall now relate the topology to the algebra discussed earlier. We fix a prime p throughout. $\mathscr{A}_*$ shall refer to the mod p Steenrod algebra [8] and $H^*(\ ) = H^*(\ ; Z_p)$.

PROPOSITION 2.1. *The Steenrod algebra is strongly coherent.*

Proof. From [8] we know that $\mathscr{A}^*$ is a Hopf algebra and is the union of finite subHopf algebras $\mathscr{A}_n$ where $\mathscr{A}_n = \mathscr{A}^*$ in degrees less than $m = 2(p^n - 1)$ ($m = 2^n - 1$ for $p = 2$). Then apply Propositions I.2.6. and I.2.7 to get the desired result.

We now observe the following:

THEOREM 2.2. *If* ${}_p\pi_n(\mathbf{X}) \neq 0$ *for only finitely many* n *then* $H^*(\mathbf{X})$ *is a weakly coherent* $\mathscr{A}^*$*-module.*

Proof. By induction on the number of non-trivial ${}_p\pi_n(\mathbf{X})$:

If ${}_p\pi_n(\mathbf{X}) = 0$ for $n \neq k$ and ${}_p\pi_k(\mathbf{X}) = G$ then there is a map $f: \mathbf{X} \to \mathbf{K}(G)$ which is a ${}_p\pi_k$-isomorphism. Thus ${}_p\pi_*(E_f) = 0$ by Proposition 1.3 thus by the Hurewicz Theorem $H^*(E_f) = 0$ and again by Proposition 1.3, $f^*: H^*(\mathbf{K}(G)) \xrightarrow{\sim} H^*(\mathbf{X})$.

First, if $G = Z$ then the exact sequence $0 \to Z \xrightarrow{p} Z \to Z_p \xrightarrow{\sim} 0$ yields the exact trivangle

$$\begin{array}{ccc} H^*(\mathbf{K}) & \longrightarrow & H^*(\mathbf{K}_p) \\ {\scriptstyle p^*} \nwarrow & & \swarrow \\ & H^*(\mathbf{K}) & \end{array}$$

But $p^*=0$ and $\mathscr{A}^* \cong H^*(\mathbf{K}_p)$ so

$$0 \to H^*(\mathbf{K}) \to \mathscr{A}^* \to H^*(\mathbf{K}) \to 0$$

is exact and gives a finite presentation of $H^*(\mathbf{K})$.

Thus if G is free abelian

$$0 \to G \otimes H^*(G) \to G \otimes \mathscr{A}^* \to H^*(\mathbf{K}(G)) \to 0$$

gives a weak finite presentation of $H^*(\mathbf{K}(G))$. Thus since $\mathscr{A}^*$ is strongly coherent $H^*(\mathbf{K}(G))$ is weakly coherent.

Finally, if G is arbitrary, there is an exact sequence of groups

$$0 \to H \to F \to G \to 0$$

with H, F free abelian.

Then

$$\begin{array}{ccc} H^*(\mathbf{K}(F)) & \longrightarrow & H^*(\mathbf{K}(H)) \\ \nwarrow & & \swarrow \\ & H^*(\mathbf{K}(G)) & \end{array}$$

is exact (this comes from the fact that the fibre of the induced map $\mathbf{K}(F) \to \mathbf{K}(G)$ must be homotopy equivalent to $\mathbf{K}(H)$). Since $H^*(\mathbf{K}(F))$ and $H^*(\mathbf{K}(G))$ are weakly coherent, so is $H^*(\mathbf{K}(G))$ by Proposition I.2.8.

Assume inductively that $H^*(\mathbf{X})$ is weakly coherent if ${}_p\pi_*(\mathbf{X})$ has $(k-1)$-non-trivial groups. Assume ${}_p\pi_n(\mathbf{X}) \neq 0$ implies $n = i_1, i_2, \ldots, i_k$ with i_1 the smallest. Let ${}_p\pi_{i_1}(\mathbf{X}) = G$. Then there is a map $f: \mathbf{X} \to \mathbf{K}(G)$ such that ${}_p\pi_{i_1}(\mathbf{X}) \xrightarrow{f_*} \pi_0(\mathbf{K}(G))$ is an isomorphism. Then ${}_p\pi_*(\mathbf{E}_f)$ has $(k-1)$ non-trivial groups so $H^*(\mathbf{E}_f)$ is weakly coherent.

But

$$\begin{array}{ccc} H^*(\mathbf{K}(G)) & \longrightarrow & H^*(\mathbf{X}) \\ \nwarrow & & \swarrow \\ & H^*(\mathbf{E}_f) & \end{array}$$

is exact and $H^*(\mathbf{E}_f)$ and $H^*(\mathbf{K}(G))$ are weakly coherent. Thus $H^*(\mathbf{X})$ is also by Proposition I.2.8.

A special case of this is the following. First recall that a reduced group is one with no divisible subgroup and that if Q is divisible then $Q \otimes Z_p \simeq \mathrm{Tor}_z^1(Q, Z_p) = 0$.

Theorem 2.3. *If* $\sum_{n=-\infty}^{\infty} {}_p\pi_n(\mathbf{X})$ *has its reduced part finite then* $H^*(\mathbf{X})$ *is of finite size over* $\mathscr{A}^*$.

Proof. Since $Q \otimes Z_p \cong \mathrm{Tor}^1_z(Q, Z_p) = 0$ for Q divisible we may as well assume $\sum_{n=-\infty}^{\infty} {}_p\pi_n(\mathbf{X})$ itself is finite. Let $m \geqslant 0$ be such that $\sum_{n=-\infty}^{\infty} {}_p\pi_n(\mathbf{X})$ is of order p^m. If $m=0$ then ${}_p\pi_*(\mathbf{X}) = 0$ so $H^*(\mathbf{X}) = 0$ is of finite size. Assume that for $m<k$ the result holds. Let $\mathbf{X}$ be such that $\sum_{n=-\infty}^{\infty} {}_p\pi_n(\mathbf{X})$ is of order p^k. Assume ${}_p\pi_i(\mathbf{X}) = 0$ for $i<r$ and ${}_p\pi_r(\mathbf{X}) \neq 0$. Then $\pi_r(\mathbf{X})$ contains an element of order p so then does $H_r(\mathbf{X})$. Thus $H^r(\mathbf{X})$ contains some element θ of order p. Let $f : \mathbf{X} \to \mathbf{K}_p$ represent θ. Then $f_* : \pi_n(\mathbf{X}) \to \pi_0(\mathbf{K}_p)$ is onto. Thus from Proposition 1.3 $\sum_{n=-\infty}^{\infty} {}_p\pi_n(\mathbf{E}_f)$ contains p^{k-1} elements. Thus $H^*(\mathbf{E}_f)$ is of finite size. But

$$\begin{array}{ccc} H^*(\mathbf{X}) & \longrightarrow & H^*(E_f) \\ & \nwarrow \quad \swarrow & \\ & H^*(\mathbf{K}_p) \cong \mathscr{A}^* & \end{array}$$

is exact hence $H^*(\mathbf{X})$ is of finite size.

3. *The Main Theorem*

THEOREM 3.1. *If $\tilde{H}^n(X) \neq 0$ for any n then ${}_p\pi_n^s(X) \neq 0$ for infinitely many n, for any space X.*

Proof. Assume to the contrary that ${}_p\pi_n^s(X) \neq 0$ for almost all n. Then the same is true of ${}_p\pi_n(\mathrm{S}X)$ so $H^*(\mathrm{S}X)$ is a weakly coherent $\mathscr{A}^*$-module ty Theorem 2.2. Since $H^*(X) \cong H^*(\mathrm{S}X)$ as $\mathscr{A}^*$-modules and since from Proposition 2.1, $\mathscr{A}^*$ is strongly coherent, Proposition I.2.9 says that for every $0 \neq x \in H^*(X)$, $\mathscr{P}^{p^n} x \neq 0$ for almost all n, since the $\mathscr{P}^{p^n}$ and β are the indecomposables of $\mathscr{A}^*$ (read Sq^{2^n} for $p=2$). From [8], however, we have that $\mathscr{P}^{p^n} x = 0$ if degree $x < 2p^n$. This yields a contradiction unless $\tilde{H}^*(X) = 0$.

This proves the main theorem. In finishing we point out two further facts – ${}_p\pi_*(\mathrm{S})$ the p-primary part of the stable homotopy ring of the sphere is not coherent (hence the same is true of $\pi_*(\mathrm{S})$): There is a sequence of indecomposable elements $\alpha_1, \alpha_2, \ldots$ such that $\alpha_i \alpha_j = 0$ for all i and j. Thus the map $\alpha_1 : {}_p\pi_*(\mathrm{S}) \to {}_p\pi_*(\mathrm{S})$ has infinitely generated kernel.

Secondly we observe that there are non-trivial spaces with only a finite number of stable homotopy groups: Let Q be a divisible group, then the Eilenberg-MacLane space $K(Q, n)$ has its integral homology and stable homotopy isomorphic to its homotopy. Of course here $0 = \tilde{H}^*(K(Q, n); Z_p)$. (This is for n add.)

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